Exercise 5: Tree width and Hypertree width

Database Theory

2023-05-09

Maximilian Marx, Markus Krötzsch

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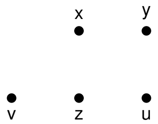

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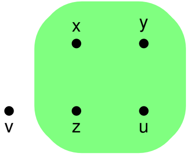

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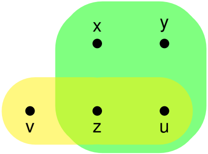

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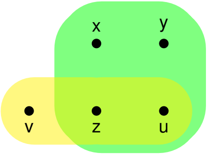

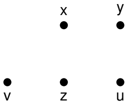

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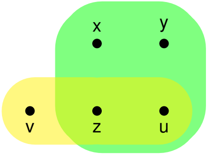

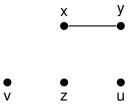

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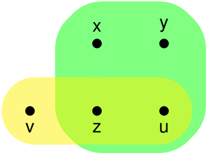

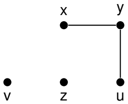

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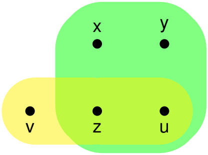

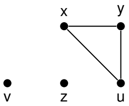

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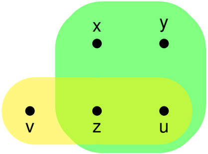

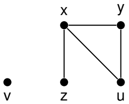

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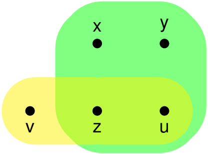

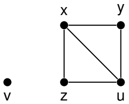

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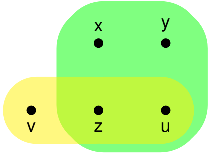

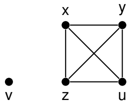

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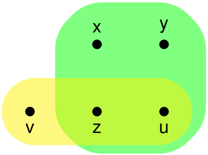

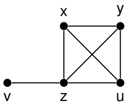

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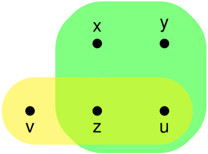

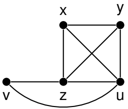

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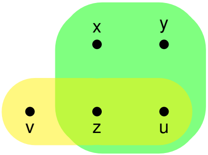

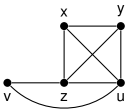

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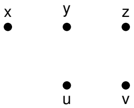

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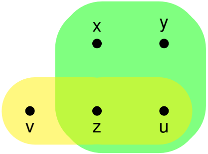

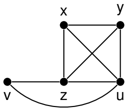

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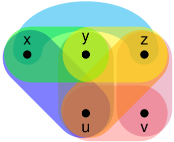

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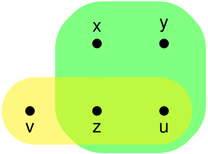

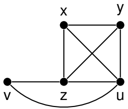

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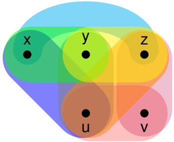

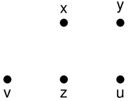

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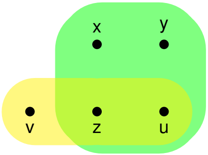

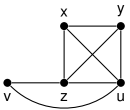

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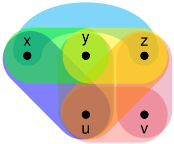

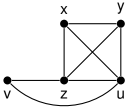

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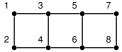

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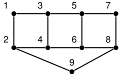

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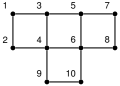

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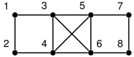

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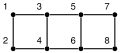

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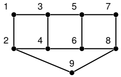

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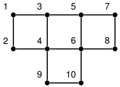

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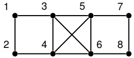

Solution.

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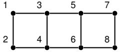

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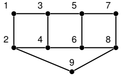

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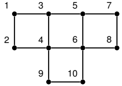

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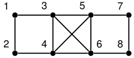

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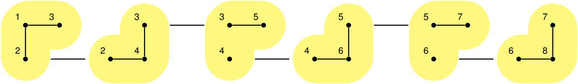
1.



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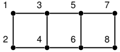

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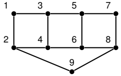

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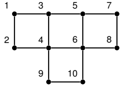

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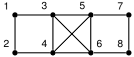

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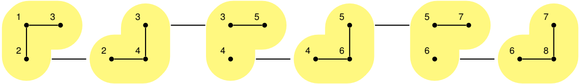

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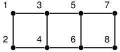

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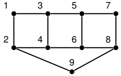

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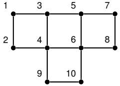

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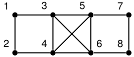

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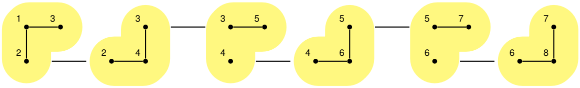

⇝ tree width 2.

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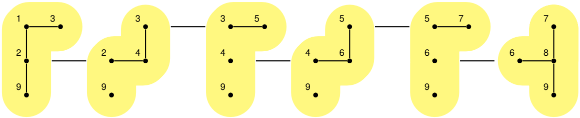

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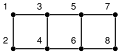

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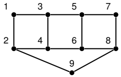

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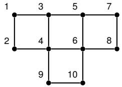

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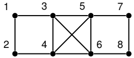

Solution.

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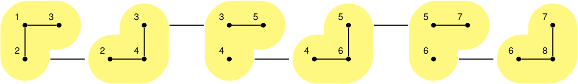

⇝ tree width 2.

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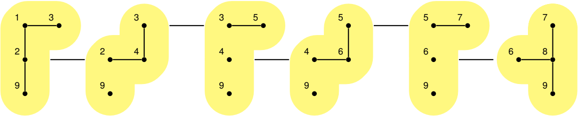

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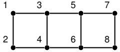

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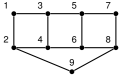

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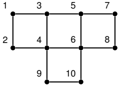

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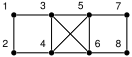

Solution.

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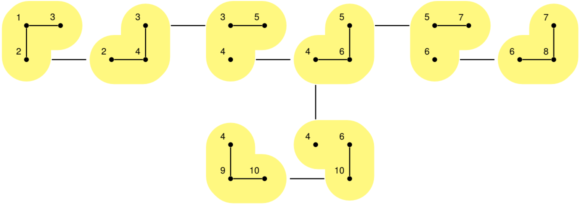

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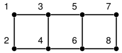

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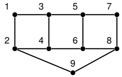

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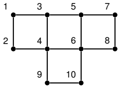

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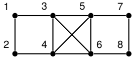

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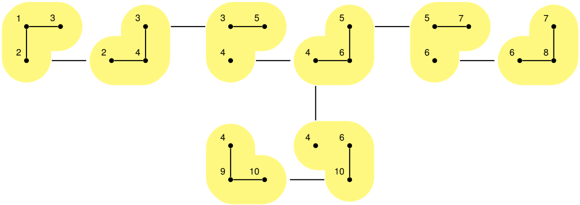

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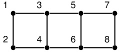

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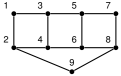

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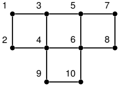

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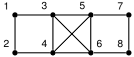

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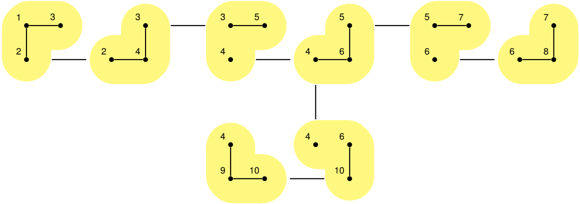

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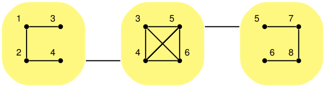

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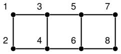

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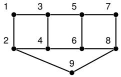

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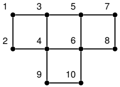

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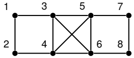

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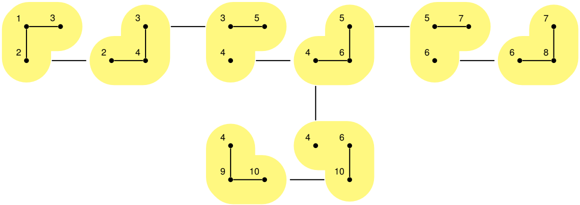

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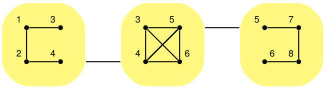

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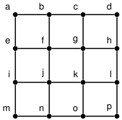

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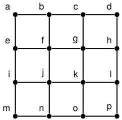

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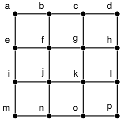

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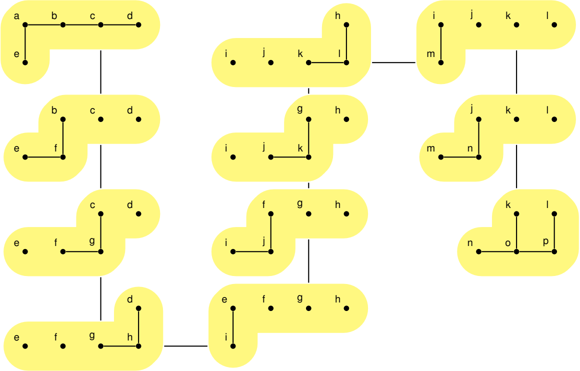

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Theorem (Seymour and Thomas; Lecture 7, Slide 15)

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- Thus, the robber wins if there are at most $n-1$ cops.

Exercise 4

Exercise. Show that a clique (fully connected graph) of size n has tree width $n - 1$.

Theorem (Seymour and Thomas; Lecture 7, Slide 15)

A graph G is of tree width $\leq k - 1$ iff k cops have a winning strategy in the cops & robber game on G.

Solution.

- Consider an n-clique.
- Clearly, n cops have a winning strategy: they can occupy every vertex.
- Thus, an n-clique has tree width at most $n - 1$.
- Consider the cops & robber game with $n - 1$ cops.
- Every vertex has $n - 1$ neighbours.
- While the cops can occupy all neighbouring vertices, they cannot catch the robber: if they move to the robbers position, one of the neighbouring vertices becomes free.
- Thus, the robber wins if there are at most $n - 1$ cops.
- Hence the n-clique cannot have tree width $\leq n - 2$.

Exercise 5

Exercise. Recall that a graph is 3-colourable if one can assign three colours to its vertices in such a way that neighbouring vertices never share the same colour. Let C_3 be the set of all 3-colourable graphs. Are the graphs in C_3 of bounded or unbounded tree width? Explain your answer.

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Solution.

- Any $n \times n$ grid is 2-colourable.

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- Hence, C_3 contains all grids.

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Solution.

- Any $n \times n$ grid is 2-colourable.
- Hence, C_3 contains all grids.
- Grids have unbounded tree width.
- Thus, C_3 contains graphs of unbounded tree width.

Exercise 6

Exercise. Decide whether the following claims are true or false. Explain your answer.

1. Deleting an edge from a graph may make the tree width smaller but never larger.
2. Deleting a vertex from a graph (and removing all of its edges) may make the tree width smaller but never larger.
3. Deleting a hyperedge from a hypergraph may make the hypertree width smaller but never larger.
4. Deleting a vertex from a hypergraph (and removing empty edges) may make the hypertree width smaller but never larger.

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Solution.

1. True: cops don't walk along edges, so deleting edges does not invalidate winning strategies.

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2. True: analogous.

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Solution.

1. True: cops don't walk along edges, so deleting edges does not invalidate winning strategies.
2. True: analogous.
3. False: Consider a hypergraph that has a hyperedge containing all vertices. Then the hypergraph is acyclic (i.e., has hypertree width 1), but removing the hyperedge may result in a cyclic hypergraph (i.e., hypertree width > 1).

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3. False: Consider a hypergraph that has a hyperedge containing all vertices. Then the hypergraph is acyclic (i.e., has hypertree width 1), but removing the hyperedge may result in a cyclic hypergraph (i.e., hypertree width > 1).
4. True: marshals don't occupy vertices, but hyperedges, so deleting vertices does not invalidate winning strategies.

Exercise 7

Exercise. The following BCQ corresponds to graph (a) in Exercise 2:

$$\begin{aligned}\exists x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8.\ & \mathrm{r}(x_1, x_2) \wedge \mathrm{r}(x_1, x_3) \wedge \mathrm{r}(x_2, x_4) \wedge \mathrm{r}(x_3, x_4) \wedge \mathrm{r}(x_3, x_5) \wedge \\ & \mathrm{r}(x_4, x_6) \wedge \mathrm{r}(x_5, x_6) \wedge \mathrm{r}(x_5, x_7) \wedge \mathrm{r}(x_6, x_8) \wedge \mathrm{r}(x_7, x_8)\end{aligned}$$

According to the logical characterisation from the lecture, this query can be expressed in the $\exists$-$\wedge$-fragment of FO using only tree width$+1$ variables. Find such a formula.

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According to the logical characterisation from the lecture, this query can be expressed in the $\exists$-$\land$-fragment of FO using only tree width$+1$ variables. Find such a formula.

Solution.

$$\begin{aligned}
&\exists x, y, z.\ \mathsf{r}(x, y) \land \mathsf{r}(x, z) \land \\
&\quad \big(\exists x.\ \mathsf{r}(y, x) \land \mathsf{r}(z, x) \land \\
&\quad \big(\exists y.\ \mathsf{r}(z, y) \land \\
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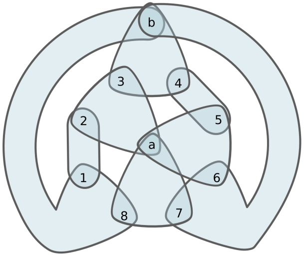
Exercise 8

Exercise. Consider *Adler's Hypergraph*:



Play the marshals & robber game on this graph.

1. Can one marshal catch the robber?
2. Can two marshals catch the robber?
3. Can three marshals catch the robber?
4. Adler et al. [Eur. J. Comb., 2007] proposed this graph as an example where fewer marshals can win if they are allowed to play non-monotonically, that is, if they are not required to shrink the remaining space in each turn. Can you confirm her findings?

(∗) Can you explain why non-monotone play is unavoidable in one of the above cases if the marshals want to win?

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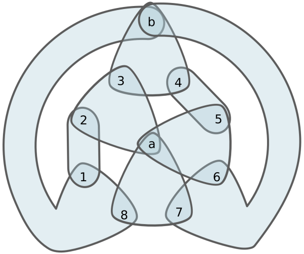

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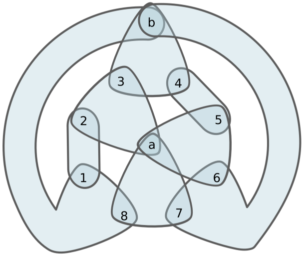

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Solution.

1. No.

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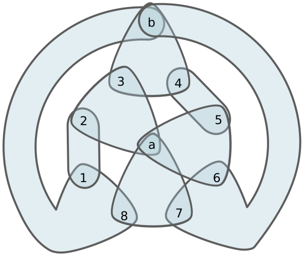

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Solution.

1. No.
2. Yes, but only non-monotonically.

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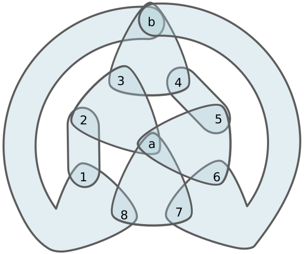

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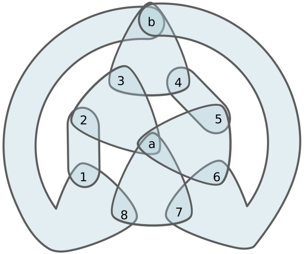

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Solution.

1. No.
2. Yes, but only non-monotonically.
3. Yes, even when playing monotonically.

(∗) The graph has hypertree width 3, but generalised hypertree width 2.