From Display to Labelled Proofs for Tense Logics

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Abstract. We introduce an effective translation from proofs in the display calculus to proofs in the labelled calculus in the context of tense logics. We identify the labelled calculus proofs in the image of this translation as those built from labelled sequents whose underlying directed graph possesses certain properties. For the minimal normal tense logic Kt, the image is shown to be the set of all proofs in the labelled calculus G3Kt.

1 Introduction

The widespread application of logical methods in several areas of computer science, epistemology and artificial intelligence has resulted in an explosion of new logics — each requiring an analytic calculus to facilitate study and applications. Analytic calculi, whose rules (de)compose in a stepwise manner the formulae to be proved, can be exploited to prove important metalogical properties of the formalized logics and are central to developing automated reasoning methods. Being relatively simple and not requiring much technical machinery ('bureaucracy'), the sequent calculus has always been the most popular formalism to use and try to construct analytic calculi. However, its simplicity means that it is also limited in its expressive power, and is hence unable to support analytic calculi for the many logics of interest. This has motivated the search for other, more expressive formalisms. Many proof formalisms generalizing the sequent calculus have been introduced in the last 30 years; each of them incorporates the bureaucracy in a distinct way and hence possesses distinct strengths, weaknesses, and expressive power. In particular, certain formalisms are more helpful than others for proving certain computational or metalogical properties. For this reason, it is fruitful to study logics in a number of different formalisms. For example, a large class of extensions of the minimal tense logic Kt have been presented as instances of the labelled calculus (e.g., [19, 15]) and of the display calculus [12, 20, 9]. The former is a extension of the sequent calculus in which the relational semantics of the formalized logics is made an explicit part of the syntax; the latter extends Gentzen's language of sequents with new structural connectives that allow each formula in a sequent to be "displayed" as the whole of the antecedent or the whole of the succedent.

Labelled and display calculi substantially differ in their nature. Display calculi are typically *internal* in the sense that each step in a proof can be read as a formula of the logic. In general, labelled calculi appear to manipulate formulae from a more expressive language which partially encodes the logic's semantics, and are hence termed *external*. Internal and external calculi have been introduced and studied within two essentially independent—and sometimes competing—streams in proof theory. These calculi possess different properties and lead to distinct proofs.

An effective way to relate calculi is by defining *embeddings*, i.e. functions that stepwise transform any proof in a calculus into a proof of the same formula in another calculus. A crucial feature of such a function is that the structure properties of the derivation are preserved in the translation. Such embeddings permit the transfer of certain proof theoretic results, thus alleviating the need for independent proofs in each system (see [8, 10, 16]). Moreover they shed light on the role of bureaucracy in proof calculi, and on the general problem of characterizing the relationships between different syntactic and semantic presentations of a logic.

In this paper we investigate the relationships between display and labelled proofs for a well known class of tense logics obtained by extending Kt with Scott-Lemmon axioms [13] $\Diamond^h \Box^i p \rightarrow \Box^j \Diamond^k p$. Due to their distinct foundational origins—the algebraic semantics for display calculi [12] and Kripke semantics for labelled calculi [15]— the relationship between their proofs is *prima facie* unclear; this is particularly true for the direction from labelled to display proofs (e.g., [17] contains a translation of display sequents into labelled sequents). Exploiting the work of Goré *et al.* [9] who present the display calculus for tense logic as a nested sequent with two types of nesting constructors, we show the equivalence of the display calculus to a calculus on labelled directed graphs whose underlying undirected graph is a tree. These structures – *labelled UT graphs* – are a natural generalization of the labelled trees shown in [10] to correspond to nested sequents [11, 3].

In particular, we give a bi-directional embedding between proofs in the display calculus and the labelled UT graph calculus. The latter are then mapped into Negri's [15] labelled sequent proofs. In the reverse direction, we then consider specifically Negri's labelled calculus for Kt and show that every derivation there is a derivation in the labelled UT graph calculus.

2 Display and Labelled Calculi for Tense Logics

The tense logic Kt extends the normal modal logic K by the addition of the tense operators $\blacklozenge$ and $\blacksquare$ with the following axioms and inference rule (see, e.g. [2, 4]):

$$\blacksquare(p \rightarrow q) \rightarrow (\blacksquare p \rightarrow \blacksquare q) \qquad \blacklozenge p \leftrightarrow \neg \blacksquare \neg p$$

$$p \rightarrow \Box \blacklozenge p \qquad p \rightarrow \blacksquare \Diamond p \qquad \frac{A}{\blacksquare A}\ (\mathsf{nec})$$

Intuitively, we interpret $\Box A$ as claiming that the formula A holds at every point in the future, whereas $\blacksquare A$ is interpreted as claiming that A holds at every point in the past. Similarly, the formula $\Diamond A$ is interpreted as A holding at some point as the future, while $\blacklozenge A$ intuitively means that A holds at some point in the past.

We assume that our language consists of formulae in negation normal form, where all negation signs are pushed inward onto the propositional atoms. In particular, formulae are built from literals p and $\overline{p}$ using the $\wedge$, $\vee$, $\Diamond$, $\Box$, $\blacklozenge$, and $\blacksquare$ operators. Note that all results still hold for the full language where the $\neg$, $\rightarrow$, and $\leftrightarrow$ operators as taken as primitive as well. Nevertheless, we restrict ourselves to negation normal form for matters of convenience.

The logics we consider in this paper are extensions of Kt with the Scott-Lemmon axioms $\Diamond^h \Box^i p \rightarrow \Box^j \Diamond^k p$ (or equivalently, $\blacklozenge^h \Diamond^j p \rightarrow \Diamond^i \blacklozenge^k p$), for $h, j, i, k \geq 0$. In negation normal form and in the absence of implication, the axioms become $\Box^h \Diamond^i \bar{p} \vee \Box^j \Diamond^k p$ (equivalently, $\blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p$).

Display Calculi for Tense Logics Introduced under the name Display Logic, Belnap's Display Calculus [1] generalises Gentzen's sequent calculus by supplementing the structural connective (comma) with new structural connectives. A (display) sequent $X \vdash Y$ is a tuple (X, Y) where X and Y are *structures* which are built from formulae and structure constants using the structural connectives of the calculus. The defining feature of a display calculus is that it satisfies the display property.

Definition 1 (Display property, display rules). *Let Z be an occurrence of a substructure occurring in a sequent $X \vdash Y$. Using invertible structural rules (the 'display rules') a sequent of the form $Z \vdash U$ or $U \vdash Z$ can be derived for suitable U.*

The beauty of the display calculus lies in a general cut-elimination theorem for all calculi obeying eight easily verifiable syntactic conditions [1, 20]; this makes the display calculus a good candidate for capturing large classes of logics in a unified way, irrespective of their semantics or connectives.

We recall below Goré *et al.* [9] one-sided display calculus SKT for Kt, which is a variant of Kashima's calculus [11]. SKT is referred to as a shallow nested sequent calculus because (i) the $\circ\{\ \}$ and $\bullet\{\ \}$ provide (two types of) nesting and (ii) all the rules are shallow in the sense that they operate at the *root* of the sequent (when the sequent is viewed in terms of its grammar tree). Although the rules in SKT are shallow, successive applications of the display rules (rf) and (rp) enable nested formulae to be brought to the root without the use of additional rules.

The display sequents of SKT are generated by the following grammar, where A is a tense formula in negation normal form: $X := A | X, X | \circ \{X\} | \bullet \{X\}$.

Definition 2 (The Calculus SKT[9]).

$$\frac{}{\Gamma, p, \overline{p}}\ (id) \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ (\vee) \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ (\wedge)$$

$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ (ctr) \qquad \frac{\Gamma}{\Gamma, \Delta}\ (wk) \qquad \frac{\Gamma, \circ\{\Delta\}}{\bullet\{\Gamma\}, \Delta}\ (rf) \qquad \frac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta}\ (rp)$$

$$\frac{\Gamma, \bullet\{A\}}{\Gamma, \blacksquare A}\ (\blacksquare) \qquad \frac{\Gamma, \circ\{A\}}{\Gamma, \Box A}\ (\Box) \qquad \frac{\Gamma, \bullet\{\Delta, A\}, \blacklozenge A}{\Gamma, \bullet\{\Delta\}, \blacklozenge A}\ (\blacklozenge) \qquad \frac{\Gamma, \circ\{\Delta, A\}, \Diamond A}{\Gamma, \circ\{\Delta\}, \Diamond A}\ (\Diamond)$$

A modular method of extending a base display calculus for Kt by a large class of axioms inclusive of the Scott-Lemmon axioms was introduced in [12]. Following [12], Goré *et al.* [9] present the rule $d(h,i,j,k)$ corresponding to the Scott-Lemmon axiom $\blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p$.

$$\frac{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}}{\Gamma, \bullet^h\{\circ^j\{\Delta\}\}}\ d(h,i,j,k)$$

The interpretation $\mathcal{I}$ of a display sequent as a tense formula is defined as follows.

$$\mathcal{I}(A) = A \text{ for every formula } A \qquad \mathcal{I}(\circ X) = \Box\mathcal{I}(X)$$
$$\mathcal{I}(X,Y) = \mathcal{I}(X) \vee \mathcal{I}(Y) \qquad \mathcal{I}(\bullet X) = \blacksquare\mathcal{I}(X)$$

Theorem 1 ([12, 9]). *Let S be any finite set of Scott-Lemmon axioms. $A \in \mathsf{Kt}+S$ iff A is derivable in $\mathsf{SKT}+S'$, where $S' = \{d(h,i,j,k) | \blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p \in S\}$.*

Labelled Calculi for Tense Logics Labelled sequents [7, 14] generalise Gentzen sequents by the prefixing of *state variables* to formulae occurring in the sequent and by making the relational semantics explicit in the syntax. A labelled sequent has the form $\mathcal{R}, \Gamma$ where the *relation mset* (multiset) $\mathcal{R}$ consists of terms of the form Rxy. Meanwhile Γ is a multiset of labelled formulae (e.g. $x : A \rightarrow B$, $y : p$). A labelled sequent can be viewed as a directed graph (defined using the set $\mathcal{R}$) with formulae decorating each node [17, 18].

Negri [15] has presented a method for generating cut-free and contraction-free labelled sequent calculi for the large family of modal logics whose Kripke semantics are defined by geometric (first-order) formulae. The proof of cut-elimination is general in the sense that it applies uniformly to every modal logic defined by geometric formulae. This result has been extended to labelled sequent calculi for intermediate and other non-classical logics [5] and indeed to arbitrary first-order formulae [6]. See also Viganò [19] where non-classical logics with semantics defined by Horn formulae are investigated using cut-free labelled calculi introduced therein.

We begin by extending in the natural way the usual labelled sequent calculus for K to a labelled sequent calculus for Kt.

Definition 3 (The labelled sequent calculus G3Kt[15]).

$$\frac{}{\mathcal{R}, x : p, x : \bar{p}, \Gamma}\ (id)$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma}{\mathcal{R}, x : A \vee B, \Gamma}\ (\vee) \qquad \frac{\mathcal{R}, x : A, \Gamma \qquad \mathcal{R}, x : B, \Gamma}{\mathcal{R}, x : A \wedge B, \Gamma}\ (\wedge)$$

$$\frac{\mathcal{R}, Ryx, y : A, \Gamma}{\mathcal{R}, x : \blacksquare A, \Gamma}\ (\blacksquare)^* \qquad \frac{\mathcal{R}, Rxy, y : A, \Gamma}{\mathcal{R}, x : \Box A, \Gamma}\ (\Box)^*$$

$$\frac{\mathcal{R}, Ryx, y : A, x : \blacklozenge A, \Gamma}{\mathcal{R}, Ryx, x : \blacklozenge A, \Gamma}\ (\blacklozenge) \qquad \frac{\mathcal{R}, Rxy, y : A, x : \Diamond A, \Gamma}{\mathcal{R}, Rxy, x : \Diamond A, \Gamma}\ (\Diamond)$$

The ($\Box$) and ($\blacksquare$) rules have a side condition: the variable y does not occur in the conclusion. When a variable is not allowed to occur in the conclusion of an inference, we refer to it as an *eigenvariable*.

Following the method in [15], the rule $l(h,i,j,k)$ corresponding to the Scott-Lemmon axiom $\blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p$ is given below. We use the notation $R^n xz$ to represent a relational sequence Rxy_1, Ry_1y_2, ..., $Ry_{n-1}z$ of length n.

$$\frac{\mathcal{R}, R^i vx, R^k ux, R^h wv, R^j wu, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, R^h wv, R^j wu, v : \Delta, u : \Delta', \Gamma}\ l(h,i,j,k)^*$$

All variables occurring in the relational atoms $R^i vx, R^k ux$ with the exception of v and u are eigenvariables.

Remark 1. In the rule above, some care is needed when some of the parameters h, i, j, and k are equal to zero. There are sixteen possible cases to consider, though we only give six of the cases below since all others can be obtained by switching h and j, or i and k. The table below specifies the instances of the rule depending on if the parameter is greater than zero (marked with a greater than $>$ sign in the table), or equal to zero (marked with an equality symbol $=$ in the table):

h	j	i	k	Premise	Conclusion
$=$	$>$	$>$	$>$	$\mathcal{R}, R^i vx, R^k ux, R^j vu, v : \Delta, u : \Delta', \Gamma$	$\mathcal{R}, R^j vu, v : \Delta, u : \Delta', \Gamma$
$=$	$>$	$>$	$=$	$\mathcal{R}, R^i vu, R^j wu, v : \Delta, u : \Delta', \Gamma$	$\mathcal{R}, R^j vu, v : \Delta, u : \Delta', \Gamma$
$>$	$=$	$=$	$>$	$\mathcal{R}, R^k uv, R^h uv, v : \Delta, u : \Delta', \Gamma$	$\mathcal{R}, R^h uv, v : \Delta, u : \Delta', \Gamma$
$=$	$=$	$>$	$>$	$\mathcal{R}, R^i vx, R^k vx, v : \Delta, v : \Delta', \Gamma$	$\mathcal{R}, v : \Delta, v : \Delta', \Gamma$
$=$	$=$	$>$	$=$	$\mathcal{R}, R^i vv, v : \Delta, v : \Delta', \Gamma$	$\mathcal{R}, v : \Delta, v : \Delta', \Gamma$
$>$	$>$	$>$	$=$	$\mathcal{R}, R^i vu, R^h wv, R^j wu, v : \Delta, u : \Delta', \Gamma$	$\mathcal{R}, R^h wv, R^j wu, v : \Delta, u : \Delta', \Gamma$

The cases when all parameters are greater than zero gives the rule in its full form as presented above, and when all parameters are equal to zero all relational atoms are removed from the rule instance, and the premise is equal to the conclusion.

The rule instances when $i = k = 0$ appear to necessitate the addition of equality symbols to the language of $\mathsf{G3Kt}$, along with the structural, equality rules specified in Negri [15]. However, for matters of simplicity, we omit these additional rules and note that all following results are preserved even in the addition of such rules.

The following contraction and weakening rules are admissible [15] in $\mathsf{G3Kt} + l(h,i,j,k)$.

$$\frac{\mathcal{R}, \mathcal{Q}, \mathcal{Q}, \Delta, \Delta, \Gamma}{\mathcal{R}, \mathcal{Q}, \Delta, \Gamma}\ (\mathsf{ctr}) \qquad \frac{\mathcal{R}, \Gamma}{\mathcal{R}, \mathcal{Q}, \Gamma, \Delta}\ (\mathsf{wk})$$

Theorem 2 ([15]). *Let S be any finite set of Scott-Lemmon axioms. $A \in \mathsf{Kt}+S$ iff $x : A$ is derivable in $\mathsf{SKT}+S'$, where $S' = \{l(h,i,j,k) | \blacksquare^h \Box^i \bar{p} \vee \Diamond^j \blacklozenge^k p \in S\}$.*

3 Interpreting a display sequent as a labelled UT

In this section we show how to translate (back and forth) a display sequent into a labelled directed graph whose underlying undirected graph is a tree.

We write $V = V_1 \sqcup V_2$ to mean that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. The multiset union of multisets M_1 and M_2 is denoted $M_1 \uplus M_2$. A *labelling function* L is a map from a set V to a multiset of tense formulae. For labelling functions L_1 and L_2 on the set V_1 and V_2 respectively, let $L_1 \cup L_2$ be the labelling function on $V_1 \cup V_2$ defined as follows:

$$L_1 \cup L_2(x) = \begin{cases} L_1(x) & x \in V_1, x \notin V_2 \\ L_2(x) & x \notin V_1, x \in V_2 \\ L_1(x) \uplus L_2(x) & x \in V_1, x \in V_2 \end{cases}$$

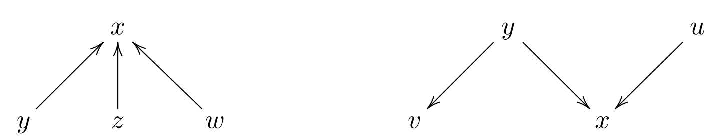
A *labelled graph* (V, E, L) is a directed graph (V, E) $(V \neq \emptyset)$ equipped with a labelling function L on V.

Definition 4 (Labelled graph isomorphism). *We say that two labelled graphs $u_1 = (V_1, E_1, L_1)$ and $u_2 = (V_2, E_2, L_2)$ are isomorphic (written $u_1 \cong u_2$) if and only if there is an isomorphism $f : V_1 \to V_2$ such that:*

(i) for every $x, y \in V_1$, $(x, y) \in E_1$ iff $(fx, fy) \in E_1$
(ii) for every $x \in V$, $L(x) = L(fx)$.

Definition 5 (Labelled UT). *A labelled graph whose underlying (undirected) graph is a tree is termed a UT (underlying tree).*

Example 1. Assuming that the nodes are decorated with multisets of formulae, the following two graphs represent labelled UTs:

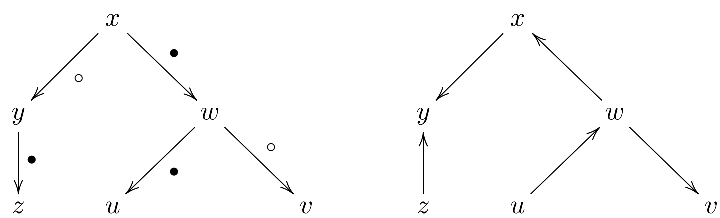


Interpreting a display sequent Γ as a labelled UT. Starting from the display sequent $\Gamma = A_1, ..., A_n, \circ\{\Delta_1\}, ..., \circ\{\Delta_k\}, \bullet\{\Sigma_1\}, ..., \bullet\{\Sigma_m\}$ we define the labelled UT $\delta\mathcal{U}(\Gamma) = (V, E, L)$ as follows:

1. Construct a labelled tree for Γ by parsing $\circ\{\Delta_i\}$ by adding a $\circ$-edge from the current node to a new node containing Δ_i and then construct the labelled tree for Δ_i; parse $\bullet\{\Sigma_i\}$ by adding a $\bullet$-edge from the current node to a new node containing Σ_i and then construct a labelled tree for Σ_i.
2. Now read each $\bullet$-edge (x, y) as a $\circ$-edge (y, x).
3. For each multiset of formulas associated with a node x, define $L(x)$ to be equal to that multiset.

The resulting graph is no longer a tree, but it consists solely of $\circ$-edges since every $\bullet$-edge has been replaced. In particular, this graph can be viewed naturally as a labelled UT.

Example 2. Take the display sequent $A, \circ\{B, \bullet\{\}\}, \bullet\{D, E, \bullet\{F\}, \circ\{G\}\}$ to interpret. Below left is the grammar tree of the sequent after step one of the above algorithm, and the graph below right is the complete UT:

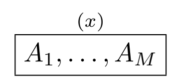


Note that $L(x) = \{A\}$, $L(y) = \{B\}$, $L(z) = \emptyset$, $L(w) = \{D, E\}$, $L(u) = \{F\}$, and $L(v) = \{G\}$.

For concreteness, we give the formal details of the interpretation below. Let $\mathbb{N}^{<\mathbb{N}}$ denote the set of finite sequences on $\mathbb{N}$. We recursively define a function $\delta\mathcal{U}_{(x)}(\Gamma)$ (for $(x) \in \mathbb{N}^{<\mathbb{N}}$) mapping a display sequent Γ to a labelled UT (on the depth of Γ):

1. Base case. A pictorial representation is given below right.

$$\delta\mathcal{U}_{(x)}(A_1, \ldots, A_M) = (\{(x)\}, \emptyset, x \mapsto \{A_1, \ldots, A_n\})$$



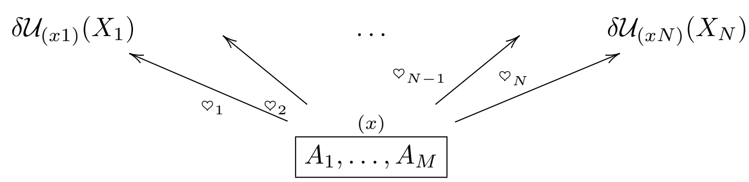
2. Inductive case. Let Γ be the display sequent below where $\heartsuit_j \in \{\circ, \bullet\}$.

$$A_1, \ldots, A_M, \heartsuit_1\{X_1\}, \ldots, \heartsuit_N\{X_N\}$$

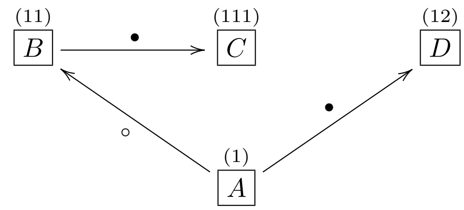
Suppose that $\delta\mathcal{U}_{(x,j)}(\heartsuit_j\{X_j\}) = (V_j, E_j, L_j)$ for $1 \leq j \leq N$. Then we define $\delta\mathcal{U}_{(x)}(\Gamma) = (V, E, L)$ where

$$V = \{(x)\} \cup V_1 \cup \ldots \cup V_N$$
$$E = \{((x), (xj)) \mid \heartsuit_j = \circ\} \cup \{((xj), (x)) \mid \heartsuit_j = \bullet\}$$
$$L = L_{(x)} \cup L_1 \cup \ldots \cup L_N$$

A pictorial representation is given below:



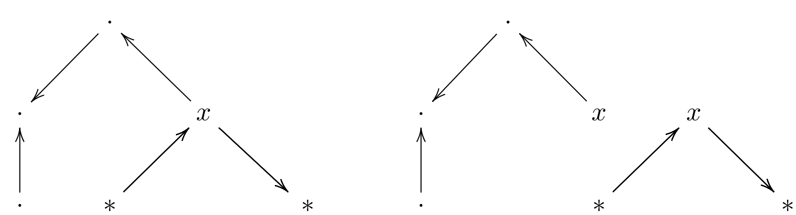
Example 3. Under the translation $\delta\mathcal{U}$, the sequent $\Gamma = A, \circ\{B, \bullet\{C\}\}, \bullet\{D\}$ becomes the UT $\delta\mathcal{U}_{(1)}(\Gamma) = \langle V, E, L\rangle$ below:



We have that $V = \{(1), (11), (111), (12)\}$ and the labelling function L maps (1) to the multiset $\{A\}$, (11) to $\{B\}$, (111) to $\{C\}$, and (12) to $\{D\}$. Note that in practice we use the more familiar symbols x, y, z, ... to denote labels. The numerical labels presented here are just a matter of technical convenience.

Definition 6 ($u[v]$ notation). *We write $u[v]$ to mean the labelled graph containing labelled subgraphs $u[\,]$ and v which have a single vertex x in common such that the label of x in $u[v]$ is the union of $L(x)$ from $u[\,]$ and v.*

Example 4. Suppose that the graph (bottom right) is the labelled graph $u[v]$ where x is the common vertex between $u[]$ and v. The labelled graph $u[]$ is shown below middle, with the labelled graph v shown below right.



If $u[v] = (V, E, L)$, then there exist partitions $V = V_1 \sqcup \{x\} \sqcup V_2$, $E = E_1 \sqcup E_2$, and L_1 and L_2 such that $L = L_1 \cup L_2$, where $u[\,] = (V_1 \sqcup \{x\}, E_1, L_1)$ and $v = (V_2 \sqcup \{x\}, E_2, L_2)$. In particular, $L(x) = L_1(x) \uplus L_2(x)$. Note that when $u[v]$ is a labelled UT, then $u[\,]$ and v must necessarily be labelled UTs.

We have seen that every display sequent defines (up to isomorphism) a labelled UT. With a slight abuse of notation, we will use the display sequent notation to denote a labelled UT. For example, we will write $u[X]$ to mean the labelled graph such that the labelled graph $u[\,]$ and the labelled UT $\delta\mathcal{U}(X)$ are subgraphs with a single common vertex. The context will make it clear if we are referring to a display sequent or a labelled UT.

The translation from a display sequent to a labelled UT extends naturally to a translation from a display sequent rule to a labelled UT rule. This leads us to the definition of the following calculus.

Definition 7 (UT calculus). *Every sequent in this calculus is a labelled UT.*

$$\frac{}{u[p, \bar{p}]}\ (\mathit{id})_u \quad \frac{u[A] \qquad u[B]}{u[A \wedge B]}\ (\wedge)_u \qquad \frac{u[A, B]}{u[A \vee B]}\ (\vee)_u$$

$$\frac{A, \circ\{X\}}{\blacksquare A, X}\ (\blacksquare)_u \quad \frac{u[\circ\{\Delta, A\}, \Diamond A]}{u[\circ\{\Delta\}, \Diamond A]}\ (\Diamond)_u \quad \frac{u[\circ\{\Delta, \blacklozenge A\}, A]}{u[\circ\{\Delta, \blacklozenge A\}]}\ (\blacklozenge)_u$$

$$\frac{u[\circ\{A\}]}{u[\Box A]}\ (\Box)_u \qquad \frac{u[\Gamma]}{u[\Gamma, \Delta]}\ (w)_u \qquad \frac{u[\Delta, \Delta]}{u[\Delta]}\ (c)_u$$

For convenience, we drop the subscript (x) and write $\delta\mathcal{U}$ for $\delta\mathcal{U}_{(x)}$.

Recall that $\mathsf{SKT} + d(h, i, j, k)$ (see below left) is a calculus for the extension of Kt with the Scott-Lemmon axiom $\blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p$. We define the UT rule $u(h, i, j, k)$ as below right.

$$\frac{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}}{\Gamma, \bullet^h\{\circ^j\{\Delta\}\}}\ d(h, i, j, k) \qquad \frac{u[\circ^i\{\bullet^k\{\Delta\}\}]}{u[\bullet^h\{\circ^j\{\Delta\}\}]}\ u(h, i, j, k)$$

Since display sequents may be interpreted as trees with two types of edges ($\circ$-edges and $\bullet$-edges), they possess a root node, whereas UTs do not possess a root in general. Nevertheless, the underlying tree structure of a UT permits us to view any node as the root, and the lemma below ensures that we obtain deductively equivalent labelled UTs via the residuation rules regardless of the node where we begin the translation.

Lemma 1. *For every Γ and Δ, $\delta\mathcal{U}(\Gamma, \circ\{\Delta\}) \cong \delta\mathcal{U}(\bullet\{\Gamma\}, \Delta)$*

Proof. Let (V, E, L) be the labelled UT corresponding to $\Gamma, \circ\{\Delta\}$. Then there exists $x, y \in V$ and $(x, y) \in E$ such that $V = V_1 \sqcup \{x\} \sqcup V_2 \sqcup \{y\}$ and $E = E_1 \sqcup E_2 \sqcup \{(x, y)\}$ and $\delta\mathcal{U}(\Gamma) = (V_1 \sqcup \{x\}, E_1)$ and $\delta\mathcal{U}(\Delta) = (V_2 \sqcup \{y\}, E_2)$.

Now consider the interpretation (V', E', L') of $\bullet\{\Gamma\}, \Delta$. Then there exists $u, v \in V'$ and $(u, v) \in E$ such that $V' = V_1' \sqcup \{u\} \sqcup V_2' \sqcup \{y\}$ and $E' = E_1' \sqcup E_2' \sqcup \{(u, v)\}$ and $\delta\mathcal{U}(\Gamma) = (V_1' \sqcup \{u\}, E_1')$ and $\delta\mathcal{U}(\Delta) = (V_2' \sqcup \{v\}, E_2')$. By inspection, $(V_1, E_1, L_1) \cong (V_1', E_1', L_1')$ and $(V_2, E_2, L_2) \cong (V_2', E_2', L_2')$. It follows that $(V, E, L) \cong (V', E', L')$.

Interpreting a labelled UT as a display sequent. Given a UT $u = \langle V, E, L\rangle$ we first pick a vertex $x \in V$ to compute the display sequent $\mathcal{U}\delta_x(u)$. If $E = \emptyset$, then $\mathcal{U}\delta(u) = L(x)$ is the desired display sequent. Otherwise, for all n forward looking edges $(x, y_i) \in E$ (with $1 \leq i \leq n$) where y_i is the common label of $u = u[v_i]$ and v_i, and for all k backward looking edges $(z_j, x) \in E$ (with $1 \leq j \leq k$) where z_j is the common label of $u = u[w_j]$ and w_j, we define the image of $\mathcal{U}\delta_x(u)$ as the display sequent

$$L(x), \circ\{\mathcal{U}\delta_{y_1}(v_1)\}, \ldots, \circ\{\mathcal{U}\delta_{y_n}(v_n)\}, \bullet\{\mathcal{U}\delta_{z_1}(w_1)\}, \ldots, \bullet\{\mathcal{U}\delta_{y_k}(v_k)\}$$

Since the UTs $v_1, \ldots, v_n, w_1, \ldots, w_k$ are smaller than u, the recursive definition of $\mathcal{U}\delta$ is well-founded.

Lemma 2. *For any UT $u = \langle V, E, L\rangle$, and for any vertices $x, y \in V$, the display sequent $\mathcal{U}\delta_x(u)$ is derivable from $\mathcal{U}\delta_y(u)$ via the residuation rules (*rf*) and (*rp*).*

Proof. Follows by lemma 1.

Lemma 3. *(i) For every Γ and Δ, $\delta\mathcal{U}(\Gamma, \Delta)$ is the UT $u[v]$, where v is the UT $\delta\mathcal{U}(\Delta)$ and $u[\,]$ is the UT $\delta\mathcal{U}(\Gamma)$.*
(ii) For every UT $u[v]$, $\mathcal{U}\delta(u[v])$ is the display sequent Γ, Δ (up to display equivalence) where $\Gamma = \mathcal{U}\delta(u[\,])$ and $\Delta = \mathcal{U}\delta(v)$.

Proof. By construction of $\delta\mathcal{U}$ and $\mathcal{U}\delta$.

Theorem 3 (Translating derivations: $\mathsf{SKT}+S$ and UT calculus$+S'$). *Let S be any finite set of $d(h,i,j,k)$ rules and S' be the set $\{u(h,i,j,k) | d(h,i,j,k) \in S\}$. Then:*

(i) Let δ be a derivation of Γ in $\mathsf{SKT}+S$. Then there is an effective translation of δ to a derivation δ' of $\delta\mathcal{U}(\Gamma)$ in the UT calculus with S'.
(ii) Let δ be a derivation of the labelled UT u in the UT calculus with S'. Then there is an effective translation of δ to a derivation of $\mathcal{U}\delta(g)$ in $\mathsf{SKT}+S$.

Proof. (i) Induction on the height of δ.

Base case. $\delta\mathcal{U}(\Gamma, p, \bar{p})$ is a UT of the form $u[p, \bar{p}]$ (Lemma 3(i)) and is hence an initial sequent in the UT calculus.

Inductive case. It suffices to simulate each rule instance of SKT in the UT calculus. Every rule in SKT other than (rf), (rp), ($\blacksquare$) and ($\blacklozenge$) has the form below left for suitable Y_1 and Y_0; moreover, there is a corresponding rule in the UT calculus as shown below right.

$$\dfrac{\Gamma, Y_1}{\Gamma, Y_0}\,(\mathsf{r}) \qquad\qquad \dfrac{u[\Gamma, Y_1]}{u[\Gamma, Y_0]}\,(\mathsf{r})_u$$

The induction hypothesis gives us a derivation of $\delta\mathcal{U}(\Gamma, Y_1) = u[\Gamma, Y_1]$. Applying $(\mathsf{r})_u$ we get $u[\Gamma, Y_0] = \delta\mathcal{U}(\Gamma, Y_0)$ as required.

We consider the remaining rules below.

$$\dfrac{\Gamma, \circ\{\Delta\}}{\bullet\{\Gamma\}, \Delta}\,(\mathsf{rf}) \qquad\qquad \begin{array}{c} \delta\mathcal{U}_x(\Gamma, \circ\{\Delta\}) \\ \hdashline \cong \delta\mathcal{U}_x(\bullet\{\Gamma\}, \Delta) \end{array}\ \text{Lem. 1}$$

$$\dfrac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta}\,(\mathsf{rp}) \qquad\qquad \begin{array}{c} \delta\mathcal{U}_x(\Gamma, \bullet\{\Delta\}) \\ \hdashline \cong \delta\mathcal{U}_x(\circ\{\Gamma\}, \Delta) \end{array}\ \text{Lem. 1}$$

$$\dfrac{\Gamma, \bullet\{A\}}{\Gamma, \blacksquare A}\,(\blacksquare) \qquad\qquad \begin{array}{c} \delta\mathcal{U}_x(\Gamma, \bullet\{A\}) \\ \hdashline \dfrac{\circ\{\Gamma\}, A}{\Gamma, \blacksquare A}\,(\blacksquare) \end{array}$$

$$\dfrac{\Gamma, \bullet\{\Delta, A\}, \blacklozenge A}{\Gamma, \bullet\{\Delta\}, \blacklozenge A}\,\blacklozenge \qquad\qquad \begin{array}{c} \delta\mathcal{U}_x(\Gamma, \bullet\{\Delta, A\}, \blacklozenge A) \\ \hdashline \dfrac{\Delta, A, \circ\{\Gamma, \blacklozenge A\}}{\Delta, \circ\{\Gamma, \blacklozenge A\}}\,\blacklozenge \\ \hdashline \delta\mathcal{U}(\Gamma, \bullet\{\Delta\}, \blacklozenge A) \end{array}$$

(ii) Induction on the height of δ. The argument is similar to the above case and uses Lemma 3(ii).

4 From labelled UTs to labelled sequents

We identify a subclass of labelled sequents which we call *G3Kt*(UT) *sequents*, and prove that they correspond to labelled UT graphs. Due to the relations of the latter with the display calculi shown in the previous section, it follows that every derivation in the $\mathsf{SKT} + u(h, i, j, k)$ calculus corresponds to a derivation in the labelled calculus restricted to $\mathsf{G3Kt}(UT)$ sequents.

Transforming a labelled UT $u = (V, E, L)$ into a labelled sequent $\mathcal{R}, \Gamma$. Define $\mathcal{R} = \{Rxy | (x, y) \in E\}$ and

$$\Gamma = \biguplus_{x \in V, L(x) \neq \emptyset} x : L(x)$$

where $x : L(x)$ represents the multiset $L(x)$ with each formula prepended with a label x.

Example 5. The UT $u = \langle V, E, L\rangle$ where $V = \{x, y, z\}$, $E = \{(x, y), (z, x)\}$, $L(x) = \{A\}$, $L(y) = \{B\}$, and $L(z) = \{C\}$ corresponds to the labelled sequent $Rxy, Rzx, x : A, y : B, z : C$.

Transforming a labelled sequent $\mathcal{R}, \Gamma$ into a labelled graph (V, E, L). Let V be the set of all labels occurring in $\mathcal{R}, \Gamma$. Define

$$E = \{(x, y) | Rxy \in \mathcal{R}\} \qquad L(x) = \{\text{multiset of formulae with label } x \text{ in } \Gamma\}$$

Example 6. The labelled sequent $Rxy, Ryz, Rux, x : A, z : B, z : C, u : D$ becomes the UT $u = \langle V, E, L\rangle$ where $V = \{x, y, z, u\}$, $E = \{(x, y), (y, z), (u, x)\}$, $L(x) = \{A\}$, $L(y) = \emptyset$, $L(z) = \{B, C\}$ and $L(u) = \{D\}$.

The reader will observe that the translations are obtained rather directly. This is because the main difference between a labelled graph and a labelled sequent is notation. The main step of the translation was already established in the previous section. Our interest in this work is the image of a display sequent in the labelled calculus. This motivates the following definitions.

Definition 8 (G3Kt(UT) sequent). *A labelled sequent whose image (under the above translation) is a labelled UT is called a* $\mathsf{G3Kt}(UT)$ sequent.

Definition 9 (G3Kt(UT) calculus). *Define the calculus G3Kt(UT) to be the labelled calculus restricted to G3Kt(UT) sequents and with weakening and contraction defined as follows:*

$$\frac{\mathcal{R}, \Gamma}{\mathcal{R}, \mathcal{Q}, \Delta, \Gamma}\ (\mathit{wk})^*_{ul} \qquad\qquad \frac{\mathcal{R}, \mathcal{Q}, \hat{\mathcal{Q}}, \Delta, \hat{\Delta}, \Gamma}{\mathcal{R}, \mathcal{Q}, \Delta, \Gamma}\ (\mathit{ctr})^*_{ul}$$

Weakening has the side condition that the conclusion must be a $\mathsf{G3Kt}(UT)$*-sequent. Contraction possesses side conditions that ensure it behaves just as the* $(\mathsf{ctr})_u$ *rule:*

1. *The labelled graph of* $\hat{\mathcal{Q}}, \hat{\Delta}$ *must be isomorphic to the labelled graph of* $\mathcal{Q}, \Delta$.
2. *The conclusion must be a* $\mathsf{G3Kt}(UT)$*-sequent.*
3. *Both* $\mathcal{Q}, \Delta$ *and* $\hat{\mathcal{Q}}, \hat{\Delta}$ *form labelled UTs that share a root, and all other variables in* $\hat{\mathcal{Q}}, \hat{\Delta}$ *do not appear in the conclusion of the inference, i.e. they are eigenvariables.*

We use the notation $(\mathsf{r})_{ul}$ *to indicate the remaining inference rules of* $\mathsf{G3Kt}(UT)$.

For $h, i, j, k \in \mathbb{N}$, define $ul(h,i,j,k)$ as follows:

$$\frac{\mathcal{R}, R^i vx, R^k ux, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, R^h wv, R^j wu, v : \Delta, u : \Delta', \Gamma}\ ul(h,i,j,k)^*$$

The asterisk indicates the following side conditions: (i) all variables occurring in $R^i vx, R^k ux$ with the exception of v and u are eigenvariables and (ii) all variables occurring in $R^h wv, R^j wu$ with the exception of v and u are fresh.

Remark 2. Similar to the presentation of the $l(h,i,j,k)$ rules (cf. Remark 1), we provide the table below showing the different instances of the rule depending on the values of the parameters h, i, j, and k. The reduction in cases is due to the fact that we allow the $ul(h,i,j,k)$ rules to relabel formulae from premise to conclusion–an action which is not allowed for the $l(h,i,j,k)$ rules.

i	k	Premise
$>$	$>$	$\mathcal{R}, R^i vx, R^k ux, v : \Delta, u : \Delta', \Gamma$
$=$	$>$	$\mathcal{R}, R^k uv, v : \Delta, u : \Delta', \Gamma$
$>$	$=$	$\mathcal{R}, R^i vu, v : \Delta, u : \Delta', \Gamma$
$=$	$=$	$\mathcal{R}, v : \Delta, v : \Delta', \Gamma$

h	j	Conclusion
$>$	$>$	$\mathcal{R}, R^h wv, R^k wu, v : \Delta, u : \Delta', \Gamma$
$=$	$>$	$\mathcal{R}, R^j wu, w : \Delta, u : \Delta', \Gamma$
$>$	$=$	$\mathcal{R}, R^h wv, v : \Delta, w : \Delta', \Gamma$
$=$	$=$	$\mathcal{R}, w : \Delta, w : \Delta', \Gamma$

To see that the $\mathsf{G3Kt}(UT) + ul(h,i,j,k)$ calculus is well-defined, it suffices to observe that the conclusion of every $\mathsf{G3Kt}$ rule is a $\mathsf{G3Kt}(UT)$ sequent given that the premise(s) is (are) $\mathsf{G3Kt}(UT)$ sequents.

Lemma 4. *If the premise of a* $\mathsf{G3Kt}(UT)+ul(h,i,j,k)$ *inference is a* $\mathsf{G3Kt}(UT)$*-sequent, then the conclusion is an* $\mathsf{G3Kt}(UT)$*-sequent.*

Proof. We argue the result for the $(\mathsf{wk})_{ul}$, $(\mathsf{ctr})_{ul}$, $(\blacksquare)_{ul}$, and $ul(h,i,j,k)$ rules since all other cases are similar or trivial.

Case 1 and 2. These cases follow from the side conditions on the $(\mathsf{wk})_{ul}$ and $(\mathsf{ctr})_{ul}$ rules, which only allow application of the rule when the result is a $\mathsf{G3Kt}(UT)$ sequent.

Case 3. Assume that $\mathcal{R}, Ryx, y : A, \Gamma$ is a $\mathsf{G3Kt}(UT)$-sequent and that $u = \langle V, E, L\rangle$ is the corresponding UT. Since y is an eigenvariable, the conclusion

$\mathcal{R}, x : \blacksquare A, \Gamma$ gives a labelled graph $u' = \langle V', E', L' \rangle$ where $V' = V - \{y\}$, $E' = E - \{(y, x)\}$, $L'(y)$ is undefined, $L'(x)$ is equal to $L(x)$ extended with $x \mapsto \{\blacksquare A\}$, and L' is equal to L for all other labels in V'.

Case 4. We prove the claim for when $h, i, j, k > 0$ since other cases are similar. Assume that the premise $\mathcal{R}, R^i xy, R^k zy, \Gamma$ is a $\mathsf{G3Kt}(UT)$-sequent with all variables y_m strictly between x and z eigenvariables. Observe that in $u = \langle V, E, L \rangle$ there is a path of length $i + k$ from the node x to z where the first i edges are forward looking, and the last k edges are backwards looking. Observe that the UT $u' = \langle V', E', L' \rangle$ of the conclusion $\mathcal{R}, R^h wx, R^j wz, \Gamma$ will contain a path of length $h + j$ from the node x to z where the first h edges are backwards looking, and the last j edges are forwards looking. Due to the eigenvariable condition on all nodes y_m strictly between x and z, it cannot be the case that an edge given by $\mathcal{R}$ contains a label y_m, and it must be the case that $L(y_m) = \emptyset$ (thus ensuring u' is connected). Also, all new nodes along the $h + j$-path strictly between x and z will be fresh (thus ensuring u' is free of cycles). Hence, u' will be a UT.

Lemma 5 (Translating derivations: $\mathsf{G3Kt}(UT)+S$ and UT calculus$+S'$). *Let S be any finite set of $ul(h, i, j, k)$ rules and $S' = \{u(h, i, j, k) | ul(h, i, j, k) \in S\}$. Then*

- *(i) Let δ be a derivation of $x : A$ in $\mathsf{G3Kt}(UT) + S$. Then there is an effective translation of δ to a derivation δ' of A in the UT calculus$+S'$.*
- *(ii) Let δ be a derivation of A in the UT calculus$+S'$. Then there is an effective translation of δ to a derivation δ' of $x : A$ in $\mathsf{G3Kt}(UT) + S$.*

Proof. Follows from the observation that the translation of every rule instance in $\mathsf{G3Kt}(UT) + S$ is a rule instance in the UT calculus$+S'$ and *vice versa*.

Combining the previous results we obtain:

Theorem 4 (Translating derivations: $\mathsf{SKT} + S$ and $\mathsf{G3Kt}(UT) + S'$). *Let S be any finite set of $d(h, i, j, k)$ rules and $S' = \{ul(h, i, j, k) | d(h, i, j, k) \in S\}$. Then*

1. *Let δ be a derivation of A in $\mathsf{SKT} + S$. Then there is an effective translation of δ to a derivation δ' of $x : A$ in $\mathsf{G3Kt}(UT) + S'$.*
2. *Let δ be a derivation of $x : A$ in $\mathsf{G3Kt}(UT) + S'$. Then there is an effective translation δ to a derivation δ' of A in $\mathsf{SKT} + S$.*

Proof. Immediate from theorem 3 and lemma 4.

5 Labelled UTs vs labelled sequents

In the previous sections, we observed how to embed the display calculus $\mathsf{SKT} + S$ (for a finite set S of $d(h, i, j, k)$ rules) in the labelled calculus formalism, in particular, as a proper fragment, which we called $\mathsf{G3Kt}(UT) + S'$ ($S' =$

$\{ul(h,i,j,k)|d(h,i,j,k) \in S\}$). Indeed, an $\mathsf{G3Kt}(UT)$-sequent is a severe restriction of a labelled sequent since the underlying graph in the former is restricted to a tree. As a result we have two distinct labelled calculi for Scott-Lemmon extensions of Kt. In this section we investigate the natural question that arises: what is the relationship between these calculi? As seen below, the labelled calculus simulates $\mathsf{G3Kt}(UT)+S'$, despite the slightly different rules (i.e. $ul(h,i,j,k)$) used by the latter to capture the Scott-Lemmon axioms. The next question is therefore whether the converse also holds, that is, whether the two calculi can represent the same proofs. In the case of the normal minimal tense logic Kt the answer is affirmative.

From $\mathsf{G3Kt}(UT) + ul(h,i,j,k)$ to $\mathsf{G3Kt} + l(h,i,j,k)$. As stated in Remark 1, when $i = k = 0$ it appears that the language of sequents must be extended to include equality atoms, and the calculus extended to include equality structural rules, in order to capture all Scott-Lemmon extensions with the $l(h,i,j,k)$ rules. For reasons of simplicity, we only give the effective translation from $\mathsf{G3Kt}(UT)+ul(h,i,j,k)$ derivations to $\mathsf{G3Kt}+l(h,i,j,k)$ derivations when $i > 0$ or $k > 0$. We remark that the effective translation also goes through for $i = k = 0$ in the presence of the equality structural rules given in Negri [15].

Lemma 6. *The calculus G3Kt+$l(h,i,j,k)$ admits height-preserving substitution of variables [15].*

Theorem 5. *Let δ be a derivation of $x : A$ in G3Kt$(UT) + ul(h,i,j,k)$, with $i > 0$ or $k > 0$. Then there is an effective translation of δ to a derivation δ' of $x : A$ in G3Kt$+ l(h,i,j,k)$.*

Proof. We prove the result by induction on the height of the derivation δ.

Base case. It is easy to see that initial sequents of $\mathsf{G3Kt}(UT)$ are initial sequents of G3Kt.

Inductive step. We show the inductive step for two of the $ul(h,i,j,k)$ rules and the $(\mathsf{ctr})_{ul}$ rule, since all other rules and instances are easily confirmed.

$$\frac{\mathcal{R}, R^i vx, R^k ux, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, v : \Delta, v : \Delta', \Gamma} \qquad \dfrac{\dfrac{\mathcal{R}, R^i vx, R^k ux, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, R^i vx, R^k vx, v : \Delta, v : \Delta', \Gamma}\text{ lem. 6}}{\mathcal{R}, v : \Delta, v : \Delta', \Gamma}\, l(h,i,j,k)$$

$$\frac{\mathcal{R}, R^i vu, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, v : \Delta, v : \Delta', \Gamma} \qquad \dfrac{\dfrac{\mathcal{R}, R^i vu, v : \Delta, u : \Delta', \Gamma}{\mathcal{R}, R^i vv, v : \Delta, v : \Delta', \Gamma}\text{ lem. 6}}{\mathcal{R}, v : \Delta, v : \Delta', \Gamma}\, l(h,i,j,k)$$

$$\frac{\mathcal{R}, \mathcal{Q}, \hat{\mathcal{Q}}, \Delta, \hat{\Delta}, \Gamma}{\mathcal{R}, \mathcal{Q}, \Delta, \Gamma} \qquad \dfrac{\dfrac{\mathcal{R}, \mathcal{Q}, \hat{\mathcal{Q}}, \Delta, \hat{\Delta}, \Gamma}{\mathcal{R}, \mathcal{Q}, \mathcal{Q}, \Delta, \Delta, \Gamma}\text{ lem. 6}}{\mathcal{R}, \mathcal{Q}, \Delta, \Gamma}\, (\mathsf{ctr})$$

From $\mathsf{G3Kt} + l(h,i,j,k)$ to $\mathsf{G3Kt}(UT) + ul(h,i,j,k)$. Consider now the converse direction. Let S be a finite set of Scott-Lemmon axioms and define

$$S_{ul} = \{ul(h,i,j,k) | \blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p \in S\}$$
$$S_l = \{l(h,i,j,k) | \blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p \in S\}$$

Given a derivation δ in $\mathsf{G3Kt} + S_l$, in general δ will not be a derivation in $\mathsf{G3Kt}(UT) + S_{ul}$ because some sequents in δ (possibly even the endsequent) may not be a $\mathsf{G3Kt}(UT)$-sequent. A more meaningful question is: given a derivation of $x : A$ in $\mathsf{G3Kt} + S_l$, is there a derivation of $x : A$ in $\mathsf{G3Kt}(UT) + S_{tu}$ that is *effectively related to* δ? The constraint that the new derivation is "effectively related" is crucial, for otherwise one could trivially relate δ with the derivation δ' obtained from the following equivalence:

$$\vdash^{\delta}_{\mathsf{G3Kt}+S_l} x : A \text{ iff } A \in \mathsf{Kt} + \blacksquare^h \Box^j \bar{p} \vee \Diamond^i \blacklozenge^k p \text{ iff } \exists \delta'. \vdash^{\delta'}_{\mathsf{G3Kt}(UT)+S_{ul}} x : A$$

Although the phrase 'effectively related' has not been explicitly defined, what we envisage is a local (i.e. rule by rule) transformation on δ, which is sensitive to its structure, that ultimately yields a $\mathsf{G3Kt}(UT) + S_{ul}$ derivation of $x : A$. Notice that the $\mathsf{G3Kt}(UT) + S_{ul}$ derivation obtained via the above argument is not sensitive to the input in the sense that any two $\mathsf{G3Kt}+S_l$ derivations of $x : A$ would be mapped to the same $\mathsf{G3Kt}(UT) + S_{ul}$ derivation.

In the boundary case when $S = S_l = S_{ul} = \emptyset$ we have the following result.

Proposition 1. *Every labelled derivation in G3Kt of $x : A$ is also a derivation in G3Kt(UT).*

Proof. We argue by contradiction. Let δ be a derivation of $x : A$ in $\mathsf{G3Kt}$ and suppose there is a labelled sequent $\mathcal{R}, \Gamma$ in δ that is not a $\mathsf{G3Kt}(UT)$-sequent. This means that the underlying graph of $\mathcal{R}$ is not a tree. If $\mathcal{R}$ is not connected, then by inspection of the rules of $\mathsf{G3Kt}$, the underlying graph of every sequent below it (and hence $x : A$) would not be connected and this is a contradiction. On the other hand, if $\mathcal{R}$ is connected and its underlying graph is not a tree, then the underlying graph must contain a cycle. This follows from the fact that $\mathcal{R}$ is assumed connected, and the fact that any acyclic connected graph forms a tree. This means that there exist x, y, w such that $\{Rxw, Ryw\} \subseteq \mathcal{R}$. By inspection of the rules of $\mathsf{G3Kt}$, every sequent below $\mathcal{R}, \Gamma$ will contain this cycle contradicting the assumption that $x : A$ is the end sequent.

This argument does not work for extensions of $\mathsf{G3Kt}$ because the additional strutural rules can remove underlying cycles from the premise. Indeed, consider the rule for transitivity:

$$\frac{\mathcal{R}, Rxy, Ryz, Rxz, \Gamma}{\mathcal{R}, Ryz, Rxz, \Gamma} \ (\mathsf{Trans})$$

The underlying graph of a sequent satisfying the premise of (Trans) necessarily contains a cycle. However it need not be the case that the conclusion contains an underlying cycle.

In summary: embedding the display calculus into the labelled calculus has yielded two seemingly distinct labelled calculi for the tense logics: G3Kt$+l(h,i,j,k)$ and G3Kt$(UT) + ul(h,i,j,k)$. Investigating the (im)possibility of a pointwise translation from the derivations in the former to the latter is an interesting problem which we defer to future work.

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