

Coordination Control of Discrete-Event Systems Revisited*

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Abstract. In this paper, we revise and further investigate the coordination control approach proposed for supervisory control of distributed discrete-event systems with synchronous communication based on the Ramadge-Wonham automata framework. The notions of conditional decomposability, conditional controllability, and conditional closedness ensuring the existence of a solution are carefully revised and simplified. The paper is generalized to non-prefix-closed languages, that is, supremal conditionally controllable sublanguages of not necessary prefix-closed languages are discussed. Non-prefix-closed languages introduce the blocking issue into coordination control, hence a procedure to compute a coordinator for nonblockingness is included. The optimization problem concerning the size of a coordinator is under investigation. We prove that to find the minimal extension of the coordinator event set for which a given specification language is conditionally decomposable is NP-hard. In other words, unless $P=NP$, it is not possible to find a polynomial algorithm to compute the minimal coordinator with respect to the number of events.

Keywords: Discrete-event systems, distributed systems with synchronous communication, supervisory control, coordination control, conditional decomposability.

1 Introduction

In this paper, we revise and further investigate the coordination control approach proposed for supervisory control of distributed discrete-event systems with synchronous communication based on the Ramadge-Wonham automata framework. A distributed discrete-event system with synchronous communication is modeled as a parallel composition of two or more subsystems, each of which has its own observation channel. The local control synthesis consists in synthesizing local nonblocking supervisors for each of the subsystems. It is well-known that such a purely decentralized (often referred

* A preliminary version was presented at the 11th International Workshop on Discrete Event Systems (WODES 2012) held in Guadalajara, Mexico [10].

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to as modular) approach does not work in general. Recently, Komenda and Van Schuppen [13] have proposed a coordination control architecture as a trade-off between the purely local control synthesis, which is not effective in general because the composition of local supervisors may violate the specification, and the global control synthesis, which is not always possible because of the complexity reasons since the composition of all subsystems can result in an exponential blow-up of states in the monolithic plant. The coordination control approach has been developed for prefix-closed languages in [12] and extended to systems with partial observations in [9]. The case of non-prefix-closed languages has partially been discussed in [8]. Most of these approaches for prefix-closed languages have already been implemented in the software library libFAUDES [17].

In the last two decades several alternative approaches have been proposed for supervisory control of large discrete-event systems. Among the different control architectures are such as hierarchical control based on abstraction [25,29,33], modular approaches [4,7,14,19], decentralized control [22,31] also with inferencing (conditional decisions) [15,32] or with communicating supervisors [21], and the so-called interface-based approach [16]. Nowadays, these approaches are combined to achieve even better results, cf. [23,24]. Our coordination control approach can be seen as a combination of the horizontal and vertical modularity. The coordinator level corresponds to the abstraction (i.e., the higher level) of hierarchical control, while the local control synthesis is a generalization of the modular control synthesis. Moreover, coordination control is closely related to decentralized control with communication, because local supervisors communicate indirectly via a coordinator, cf. [1].

In this paper, the notions of conditional decomposability, conditional controllability, and conditional closedness, which are the central notions to characterize the solvability of the coordination control problem, are carefully revised and simplified. The paper is generalized to non-prefix-closed languages, hence supremal conditionally controllable sublanguages of not necessary prefix-closed languages are discussed. This generality, however, introduces the problem of nonblockingness into the coordination control approach, therefore a part with a procedure to compute a coordinator for nonblockingness is included in the paper. The optimization problem concerning the size of a coordinator is nowadays the main problem under investigation. The construction of a coordinator described in this paper depends mainly on a set of events, including the set of all shared events. We prove that to construct the coordinator so that its event set is minimal with respect to the number of events or, in other words, to find the minimal extension of the coordinator event set for which a given specification language is conditionally decomposable, is NP-hard.

The main contributions and the organization of the paper are as follows. Section 2 recalls the basics of supervisory control theory and revises the fundamental concepts. Section 3 gives the computational complexity analysis of the minimal extension problem for conditional decomposability and proves that it is NP-hard to find the minimal extension with respect to set inclusion (Corollary 1). Section 4 formulates the problem of coordination supervisory control. The notion of conditional controllability (Definition 3) is revised and simplified, however still equivalent to the previous definition in, e.g., [12]. Section 5 provides results concerning non-prefix-closed languages. Theorem 6 shows that in a special case the parallel composition of local supervisors results

in the supremal conditionally controllable languages. However, the problem how to compute the supremal conditionally controllable sublanguage in general is open. Section 6 discusses the construction of a coordinator for nonblockingness (Theorem 8) and presents an algorithm. Section 7 revises the prefix-closed case, where a less restrictive condition, LCC, is used instead of OCC. The possibility to use LCC instead of OCC has already been mentioned in [12] without proofs, therefore the proofs are provided here. Finally, Section 8 concludes the paper.

2 Preliminaries and definitions

We assume that the reader is familiar with the basic notions and concepts of supervisory control of discrete-event systems modeled by deterministic finite automata with partial transition functions. For unexplained notions, the reader is referred to the monograph [3].

Let Σ be a finite nonempty set whose elements are called *events*, and let Σ^* denote the set of all finite words (finite sequences of events) over Σ ; the *empty word* is denoted by ε . Let $|\Sigma|$ denote the cardinality of Σ .

A *generator* is a quintuple $G = (Q, \Sigma, f, q_0, Q_m)$, where Q is a finite nonempty set of *states*, Σ is a finite set of events (an *event set*), $f : Q \times \Sigma \rightarrow Q$ is a *partial transition function*, $q_0 \in Q$ is the *initial state*, and $Q_m \subseteq Q$ is a set of *marked states*. In the usual way, the transition function f can be extended to the domain $Q \times \Sigma^*$ by induction. The behavior of generator G is described in terms of languages. The language *generated* by G is the set $L(G) = \{s \in \Sigma^* \mid f(q_0, s) \in Q\}$, and the language *marked* by G is the set $L_m(G) = \{s \in \Sigma^* \mid f(q_0, s) \in Q_m\}$. Obviously, $L_m(G) \subseteq L(G)$.

A (*regular*) *language* L over an event set Σ is a set $L \subseteq \Sigma^*$ such that there exists a generator G with $L_m(G) = L$. The prefix closure of a language L over Σ is the set $\bar{L} = \{w \in \Sigma^* \mid \text{there exists } u \in \Sigma^* \text{ such that } wu \in L\}$ of all prefixes of words of the language L . A language L is *prefix-closed* if $L = \bar{L}$.

A *controlled generator* over an event set Σ is a triple (G, Σ_c, Γ) , where G is a generator over Σ , $\Sigma_c \subseteq \Sigma$ is a set of *controllable events*, $\Sigma_u = \Sigma \setminus \Sigma_c$ is the set of *uncontrollable events*, and $\Gamma = \{\gamma \subseteq \Sigma \mid \Sigma_u \subseteq \gamma\}$ is the *set of control patterns*. A *supervisor* for the controlled generator (G, Σ_c, Γ) is a map $S : L(G) \rightarrow \Gamma$. The *closed-loop system* associated with the controlled generator (G, Σ_c, Γ) and the supervisor S is defined as the minimal language $L(S/G)$ such that the empty word ε belongs to $L(S/G)$, and for any word s in $L(S/G)$ such that sa is in $L(G)$ and a in $S(s)$, the word sa also belongs to $L(S/G)$. We define the marked language of the closed-loop system as the intersection $L_m(S/G) = L(S/G) \cap L_m(G)$. The intuition is that the supervisor disables some of the transitions of the generator G , but it can never disable any transition under an uncontrollable event. If the closed-loop system is nonblocking, which means that $\bar{L}_m(S/G) = L(S/G)$, then the supervisor S is called *nonblocking*.

Given a specification language K and a plant (generator) G , the control objective of supervisory control is to find a nonblocking supervisor S such that $L_m(S/G) = K$. For the monolithic case, such a supervisor exists if and only if the specification K is both *controllable* with respect to the plant language $L(G)$ and uncontrollable event set Σ_u , that is the inclusion $\bar{K}\Sigma_u \cap L \subseteq \bar{K}$ is satisfied, and $L_m(G)$ -*closed*, that is the equality

$K = \overline{K} \cap L_m(G)$ is satisfied. For uncontrollable specifications, controllable sublanguages of the specification are considered instead. The notation $\text{supC}(K, L(G), \Sigma_u)$ denotes the supremal controllable sublanguage of the specification K with respect to the plant language $L(G)$ and uncontrollable event set Σ_u , which always exists and is equal to the union of all controllable sublanguages of the specification K , see [30].

A (natural) projection $P : \Sigma^* \rightarrow \Sigma_0^*$, where Σ_0 is a subset of Σ , is a homomorphism defined so that $P(a) = \varepsilon$ for a in $\Sigma \setminus \Sigma_0$, and $P(a) = a$ for a in Σ_0 . The projection of a word is thus uniquely determined by projections of its letters. The *inverse image* of P is denoted by $P^{-1} : \Sigma_0^* \rightarrow 2^{\Sigma^*}$. For three event sets $\Sigma_i, \Sigma_j, \Sigma_\ell$, subsets of Σ , we use the notation P_ℓ^{i+j} to denote the projection from $(\Sigma_i \cup \Sigma_j)^*$ to Σ_ℓ^* . If $\Sigma_i \cup \Sigma_j = \Sigma$, we simplify the notation to P_ℓ . Similarly, the notation P_{i+k} stands for the projection from Σ^* to $(\Sigma_i \cup \Sigma_k)^*$. The projection of a generator G , denoted by $P(G)$, is a generator whose behavior satisfies $L(P(G)) = P(L(G))$ and $L_m(P(G)) = P(L_m(G))$.

The synchronous product of languages L_1 over Σ_1 and L_2 over Σ_2 is defined as the language $L_1 \parallel L_2 = P_1^{-1}(L_1) \cap P_2^{-1}(L_2)$, where $P_i : (\Sigma_1 \cup \Sigma_2)^* \rightarrow \Sigma_i^*$ is a projection, for $i = 1, 2$. A similar definition for generators can be found in [3]. The relation between the language definition and the generator definition is specified by the following equations. For generators G_1 and G_2 , $L(G_1 \parallel G_2) = L(G_1) \parallel L(G_2)$ and $L_m(G_1 \parallel G_2) = L_m(G_1) \parallel L_m(G_2)$. In the automata framework, where a supervisor S has a finite representation as a generator, the closed-loop system is a synchronous product of the supervisor and the plant. Thus, we can write the closed-loop system as $L(S/G) = L(S) \parallel L(G)$.

For a generator G over an event set Σ , let $\Sigma_r(G) = \{a \in \Sigma \mid \text{there are words } u, v \in \Sigma^* \text{ such that } uav \in L(G)\}$ denote the set of all events appearing in words of the language $L(G)$. Generators G_1 and G_2 are *conditionally independent* with respect to a generator G_k if all events shared by the subsystems appear in the generator G_k , that is, if the inclusion $\Sigma_r(G_1) \cap \Sigma_r(G_2) \subseteq \Sigma_r(G_k)$ is satisfied. In other words, there is no simultaneous move in both generators G_1 and G_2 without the generator G_k being also involved.

Now, the notion of conditional decomposability is simplified compared to our previous work [12], but still equivalent.

Definition 1. A language K is conditionally decomposable with respect to event sets $\Sigma_1, \Sigma_2, \Sigma_k$, where $\Sigma_1 \cap \Sigma_2 \subseteq \Sigma_k \subseteq \Sigma_1 \cup \Sigma_2$, if

$$K = P_{1+k}(K) \parallel P_{2+k}(K),$$

where $P_{i+k} : (\Sigma_1 \cup \Sigma_2)^* \rightarrow (\Sigma_i \cup \Sigma_k)^*$ is a projection, for $i = 1, 2$.

Note that there always exists an extension of Σ_k which satisfies this condition; $\Sigma_k = \Sigma_1 \cup \Sigma_2$ is a trivial example. Here the index k is related to projection P_k used later in the paper. There exists a polynomial algorithm to check this condition, and to extend the event set to satisfy the condition, see [11]. However, the question which extension is the most appropriate requires further investigation. In Section 3, we show that to find the minimal extension is NP-hard.

Languages K and L are *synchronously nonconflicting* if $\overline{K \parallel L} = \overline{K} \parallel \overline{L}$.

Lemma 1. Let K be a language. If the language \overline{K} is conditionally decomposable, then the languages $P_{1+k}(K)$ and $P_{2+k}(K)$ are synchronously nonconflicting.

Proof. Assume that the language \overline{K} is conditionally decomposable. From a simple observation that $K \subseteq P_{i+k}^{-1}(P_{i+k}(K))$, for $i = 1, 2$, we immediately obtain that $K \subseteq P_{1+k}(K) \parallel P_{2+k}(K)$. As the prefix-closure is a monotone operation,

$$\overline{K} \subseteq \overline{P_{1+k}(K) \parallel P_{2+k}(K)} \subseteq \overline{P_{1+k}(K)} \parallel \overline{P_{2+k}(K)} = \overline{K},$$

which proves the lemma. \square

The following example shows that there exists, in general, no relation between the conditional decomposability of languages K and \overline{K} .

Example 1. Let $\Sigma_1 = \{a_1, b_1, a, b\}$, $\Sigma_2 = \{a_2, b_2, a, b\}$, and $\Sigma_k = \{a, b\}$ be event sets, and define the language $K = \{a_1 a_2 a, a_2 a_1 a, b_1 b_2 b, b_2 b_1 b\}$. Then $P_{1+k}(K) = \{a_1 a, b_1 b\}$, $P_{2+k}(K) = \{a_2 a, b_2 b\}$, and $K = P_{1+k}(K) \parallel P_{2+k}(K)$. Notice that whereas $a_1 b_2$ is in $P_{1+k}(K) \parallel P_{2+k}(K)$, $a_1 b_2$ is not in \overline{K} , which means that the language \overline{K} is not conditionally decomposable.

On the other hand, consider the language $L = \{\varepsilon, ab, ba, abc, bac\}$ over the event set $\{a, b, c\}$ with $\Sigma_1 = \{a, c\}$, $\Sigma_2 = \{b, c\}$, $\Sigma_k = \{c\}$. Then $\overline{L} = P_{1+k}(L) \parallel P_{2+k}(L) = P_{1+k}(L) \parallel P_{2+k}(L)$, and it is obvious that $L \neq \overline{L}$. \triangleleft

3 Conditional decomposability minimal extension problem

We have defined conditional decomposability only for two event sets, but the definition can be extended to more event sets as follows. A language K is *conditionally decomposable* with respect to event sets $(\Sigma_i)_{i=1}^n$, for some $n \geq 2$, and an event set Σ_k , where $\Sigma_k \subseteq \cup_{i=1}^n \Sigma_i$ contains all shared events, that is, it satisfies

$$\Sigma_s := \bigcup_{i \neq j} (\Sigma_i \cap \Sigma_j) \subseteq \Sigma_k,$$

if

$$K = \bigparallel_{i=1}^n P_{i+k}(K).$$

The conditional decomposability minimal extension problem is to find a minimal extension (with respect to set inclusion) of the event set Σ_s of all shared events so that the language is conditionally decomposable with respect to given event sets and the extension of Σ_s . The optimization problem can be reformulated to a decision version as follows.

Problem 1 (CD MIN EXTENSION).

INSTANCE: A language K over an event set $\Sigma = \cup_{i=1}^n \Sigma_i$, where $n \geq 2$, and a positive integer $r \leq |\Sigma|$.

QUESTION: Is the language K conditionally decomposable with respect to event sets $(\Sigma_i)_{i=1}^n$ and $\Sigma_s \cup \Sigma_r$, where $|\Sigma_r| \leq r$?

We now prove that the CD MIN EXTENSION problem is NP-complete. This then immediately implies that the optimization problem of finding the minimal extension of the event set Σ_s is NP-hard. On the other hand, it is not hard to see that the optimization problem is in PSPACE. Indeed, we can check all subsets generated one by one using the polynomial algorithm described in [11].

To prove NP-completeness, we reduce the MINIMUM SET COVER problem to the CD MIN EXTENSION problem; the MINIMUM SET COVER problem is NP-complete [6].

Problem 2 (MINIMUM SET COVER).

INSTANCE: A collection C of subsets of a finite set S , and a positive integer $t \leq |C|$.

QUESTION: Does the collection C contain a cover for the set S of cardinality t or less, that is, a subset C' with $|C'| \leq t$ such that every element of the set S belongs to at least one member of C' ?

Theorem 1. *The CD MIN EXTENSION problem is NP-complete.*

Proof. First, we show that CD MIN EXTENSION is in NP. To do this, a Turing machine guesses a set Σ_r of cardinality at most r and uses Algorithm 1 of [11] to verify in polynomial time whether the given language is conditionally decomposable with respect to the given event sets.

To prove the NP-hardness, consider an instance (S, C) of the MINIMUM SET COVER problem as defined in Problem 2 such that the union of all elements of the collection C covers the set S (otherwise it is trivial to solve the problem). Denote

$$S = \{b_1, b_2, \dots, b_n\} \text{ and } C = \{c_1, c_2, \dots, c_m\}.$$

We now construct a language K over the event set $S \cup \{a_i \mid i = 1, 2, \dots, n\} \cup C \cup \{a\}$ as follows. For each b_i in S , let $C_{b_i} = \{c_j \mid b_i \in c_j\}$ be the set of all elements of the collection C containing the element b_i . Then, for $C_{b_i} = \{c_{i_1}, c_{i_2}, \dots, c_{i_{b_i}}\}$, where we assume without loss of generality that $i_1 < i_2 < \dots < i_{b_i}$, add the two words $a_i b_i$ and $a_i c_{i_1} c_{i_2} \dots c_{i_{b_i}} a$ to the language K . Then the language K is

$$K = \sum_{i=1}^n (a_i b_i + a_i c_{i_1} c_{i_2} \dots c_{i_{b_i}} a).$$

To demonstrate the construction, let $S = \{b_1, b_2, b_3, b_4, b_5\}$ and $C = \{c_1 = \{b_1, b_2, b_3\}, c_2 = \{b_2, b_4\}, c_3 = \{b_3, b_4\}, c_4 = \{b_4, b_5\}\}$. The generator for language K is depicted in Fig. 1. Note that $\{c_1, c_4\}$ is the minimum set cover. Next, we define two event sets

$$\Sigma_1 = S \cup \{a\} \cup \{a_i \mid i = 1, 2, \dots, n\}$$

and

$$\Sigma_2 = C \cup \{a\} \cup \{a_i \mid i = 1, 2, \dots, n\}.$$

As the intersection $S \cap C$ is empty, it gives that the event set $\Sigma_s = \{a\} \cup \{a_i \mid i = 1, 2, \dots, n\}$. We now prove that there exists a minimum set cover of cardinality at most r if and only if there exists an extension of the event set Σ_s of cardinality at most r making the language K conditionally decomposable.

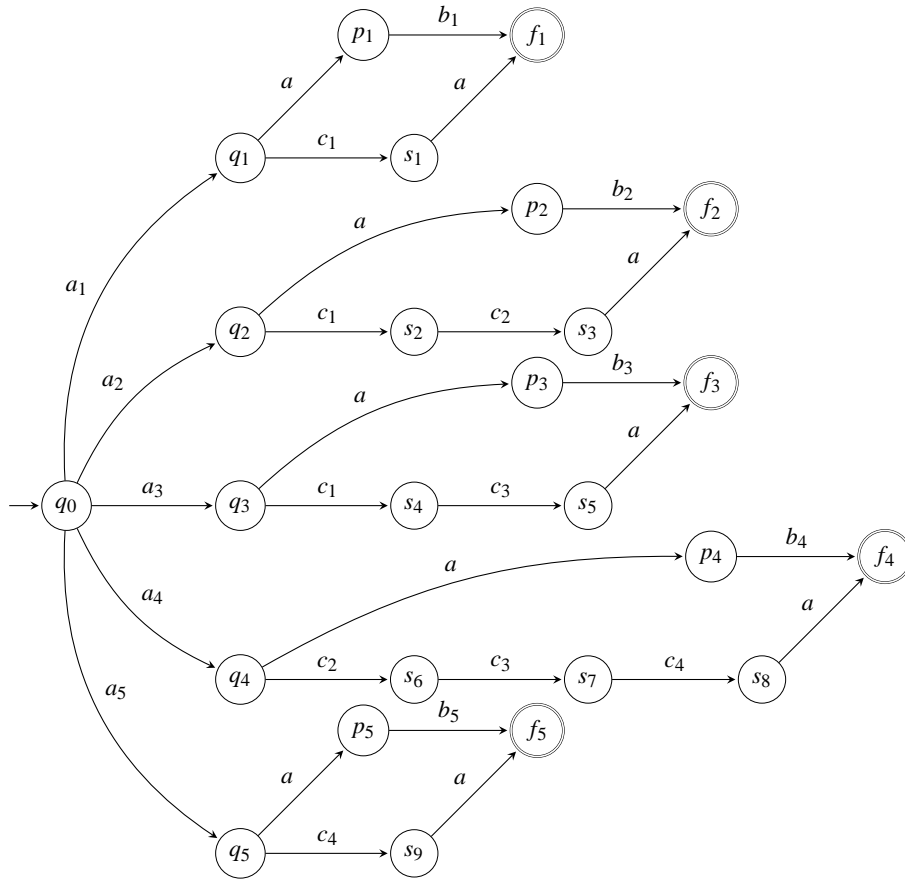


Fig. 1. The generator for language K corresponding to the MINIMUM SET COVER instance (S, C) , where $S = \{b_1, b_2, b_3, b_4, b_5\}$ and $C = \{c_1 = \{b_1, b_2, b_3\}, c_2 = \{b_2, b_4\}, c_3 = \{b_3, b_4\}, c_4 = \{b_4, b_5\}\}$.

Assume that there exists a minimum set cover $C' = \{c_{i_1}, c_{i_2}, \dots, c_{i_r}\} \subseteq C$ of cardinality r . We prove that the language K is conditionally decomposable with respect to Σ_1 , Σ_2 , and $\Sigma_k = \Sigma_s \cup \{c_{i_1}, c_{i_2}, \dots, c_{i_r}\}$. The application of projection P_{1+k} to language K results in the language

$$P_{1+k}(K) = \sum_{i=1}^n (a_i a b_i + a_i P_{1+k}(c_{i_1} c_{i_2} \dots c_{i_{b_i}}) a),$$

and the application of projection P_{2+k} to language K results in the language

$$P_{2+k}(K) = \sum_{i=1}^n (a_i a + a_i c_{i_1} c_{i_2} \dots c_{i_{b_i}} a).$$

Note that the word $P_{1+k}(c_{i_1}c_{i_2}\dots c_{i_{b_i}}) \in C'^*$ is nonempty because at least one set of the collection C' covers the element b_i , for all $i = 1, 2, \dots, n$. Let

$$X = C \setminus C'$$

denote the complement of the collection C' , then the intersection $X \cap S$ is empty. As $C_{b_i} \cap C \neq \emptyset$, for each element b_i of the set S , the language $P_{1+k}^{-1}P_{1+k}(c_{i_1}c_{i_2}\dots c_{i_{b_i}})$ is not a subset of the language X^* . It can be seen that the intersection $S^*c_{i_1}S^*c_{i_2}S^*\dots S^*c_{i_{b_i}}S^* \cap X^* = \emptyset$ is empty, that the intersection $P_{1+k}^{-1}P_{1+k}(c_{i_1}c_{i_2}\dots c_{i_{b_i}}) \cap S^* = \emptyset$ is empty, and that the intersection $P_{1+k}^{-1}P_{1+k}(c_{i_1}c_{i_2}\dots c_{i_{b_i}}) \cap S^*c_{i_1}S^*c_{i_2}S^*\dots S^*c_{i_{b_i}}S^* = \{c_{i_1}c_{i_2}\dots c_{i_{b_i}}\}$. Then the parallel composition of both projections of the language K ,

$$\begin{aligned} & P_{1+k}(K) \parallel P_{2+k}(K) \\ &= \sum_{i=1}^n (X^*a_iX^*aX^*b_iX^* + X^*a_iP_{1+k}^{-1}(P_{1+k}(c_{i_1}c_{i_2}\dots c_{i_{b_i}}))aX^*) \\ & \cap \sum_{i=1}^n (S^*a_iS^*aS^* + S^*a_iS^*c_{i_1}S^*c_{i_2}S^*\dots S^*c_{i_{b_i}}S^*aS^*) \\ &= \sum_{i=1}^n (a_iab_i + a_i c_{i_1}c_{i_2}\dots c_{i_{b_i}}a) = K, \end{aligned}$$

is equal to K .

On the other hand, let $\Sigma_r \subseteq S \cup C$ be an extension of the event set Σ_s of cardinality r such that the language K is conditionally decomposable with respect to event sets Σ_1 , Σ_2 , and $\Sigma_k = \Sigma_s \cup \Sigma_r$. Consider a symbol b_i and two corresponding words a_iab_i and $a_i c_{i_1}c_{i_2}\dots c_{i_{b_i}}a$ from the language K . If $\Sigma_r \cap \{b_i, c_{i_1}, c_{i_2}, \dots, c_{i_{b_i}}\} = \emptyset$, then the projections of these words to event sets $\Sigma_2 \cup \Sigma_k$ and $\Sigma_1 \cup \Sigma_k$ are, respectively, $P_{2+k}(a_iab_i) = a_i a$ and $P_{1+k}(a_i c_{i_1}c_{i_2}\dots c_{i_{b_i}}a) = a_i a$. But then the word $a_i c_{i_1}c_{i_2}\dots c_{i_{b_i}}ab_i \notin K$ belongs to $P_{1+k}(K) \parallel P_{2+k}(K)$, which is a contradiction. Hence, at least one of the symbols $b_i, c_{i_1}, c_{i_2}, \dots, c_{i_{b_i}}$ must belong to the set Σ_r . In other words, at least one of these symbols covers the symbol b_i . We can now construct a covering $C' \subseteq C$ of cardinality at most r as follows. For each c in Σ_r , add the set c to the covering C' , and for each b in Σ_s , add any set c from the set C_b to the covering C' . It is then easy to see that the collection C' covers the set S . \square

Note that an immediate consequence of the construction is that the minimal extension problem is NP-hard even for finite languages and two event sets.

Corollary 1. *The minimal extension problem is NP-hard.*

Similar minimal extension problems have been shown to be NP-hard in the literature, e.g., the minimal extension of observable event sets that guarantees observability of a language. However, unlike coobservability of decentralized control, conditional decomposability has an important property for large systems composed of many concurrent components—it can be checked in polynomial time in the number of components as shown in [11]. In addition, an algorithm is presented there to compute an extension

(but not necessarily the minimal one) of the shared event set such that the language under consideration becomes conditionally decomposable with respect to the original event sets Σ_1 and Σ_2 and the new (coordinator) event set Σ_k .

4 Coordination control synthesis

In this section, we recall the coordination control problem and revise the necessary and sufficient conditions established in [8,9,12] under which the problem is solvable. This revision leads to a simplification of existing notions and proofs, e.g., compare Definition 3 with [8, Definition 9] or the proof of Proposition 1 with the proof of [8, Proposition 10].

We now summarize the results of this section compared to the existing results. The coordination control problem for non-prefix-closed languages was formulated in [8, Problem 7]. The contribution of this paper is a simplification of the problem statement, namely, the prefix-closed part of the closed-loop system with a coordinator is shown to be a consequence of the non-prefix-closed case (see the note below the problem statement). The original definition of conditional controllability is simplified in Definition 3. A simplified proof of Proposition 1 is presented. Proposition 2 is new. Theorem 4 is a simplified version of Theorem 18 stated in [8] without proof.

Problem 3 (Coordination control problem). Consider generators G_1 and G_2 over Σ_1 and Σ_2 , respectively, and a generator G_k (called a *coordinator*) over Σ_k . Assume that generators G_1 and G_2 are conditionally independent with respect to coordinator G_k , and that a specification $K \subseteq L_m(G_1 \parallel G_2 \parallel G_k)$ and its prefix-closure \bar{K} are conditionally decomposable with respect to event sets Σ_1 , Σ_2 , and Σ_k . The aim of the coordination control synthesis is to determine nonblocking supervisors S_1 , S_2 , and S_k for respective generators such that

$$L_m(S_k/G_k) \subseteq P_k(K) \quad \text{and} \quad L_m(S_i/[G_i \parallel (S_k/G_k)]) \subseteq P_{i+k}(K), \quad i = 1, 2,$$

and the closed-loop system with the coordinator satisfies

$$L_m(S_1/[G_1 \parallel (S_k/G_k)]) \parallel L_m(S_2/[G_2 \parallel (S_k/G_k)]) = K.$$

◇

One could expect that the equality $L(S_1/[G_1 \parallel (S_k/G_k)]) \parallel L(S_2/[G_2 \parallel (S_k/G_k)]) = \bar{K}$ for prefix-closed languages should also be required in the statement of the problem. However, it is really sufficient to require only the equality for marked languages since it then implies that the equality $L(S_1/[G_1 \parallel (S_k/G_k)]) \parallel L(S_2/[G_2 \parallel (S_k/G_k)]) = \bar{K}$ holds true because

$$\begin{aligned} \bar{K} &= \overline{L_m(S_1/[G_1 \parallel (S_k/G_k)]) \parallel L_m(S_2/[G_2 \parallel (S_k/G_k)])} \\ &\subseteq \overline{L_m(S_1/[G_1 \parallel (S_k/G_k)])} \parallel \overline{L_m(S_2/[G_2 \parallel (S_k/G_k)])} \\ &\subseteq \overline{P_{1+k}(K)} \parallel \overline{P_{2+k}(K)} \\ &= \bar{K}. \end{aligned}$$

Moreover, if such supervisors exist, their synchronous product is a nonblocking supervisor for the global plant, cf. [8].

Note that several conditions are required in the statement of the problem, namely, (i) the generators are conditionally independent with respect to the coordinator and (ii) the specification and its prefix-closure are conditionally decomposable with respect to event sets Σ_1 , Σ_2 , and Σ_k . These conditions can easily be fulfilled by the choice of an appropriate coordinator event set Σ_k . The reader is referred to [11] for a polynomial algorithm extending a given event set so that the language becomes conditionally decomposable.

In the statement of the problem, we have mentioned the notion of a coordinator. The fundamental question is the construction of such a coordinator. We now discuss one of the possible constructions of a suitable coordinator, which has already been discussed in the literature [8,9,12]. We recall it here for the completeness.

Algorithm 2 (Construction of a coordinator) Consider generators G_1 and G_2 over Σ_1 and Σ_2 , respectively, and let K be a specification. Construct an event set Σ_k and a coordinator G_k as follows:

1. Set $\Sigma_k = \Sigma_1 \cap \Sigma_2$ to be the set of all shared events.
2. Extend Σ_k so that K and \bar{K} are conditional decomposable, for instance using a method described in [11].
3. Let the coordinator $G_k = P_k(G_1) \parallel P_k(G_2)$.

So far, the only known condition ensuring that the projected generator is smaller than the original one is the observer property. Therefore, we might need to add step (2b) to extend the event set Σ_k so that the projection P_k is an $L(G_i)$ -observer, for $i = 1, 2$, cf. Definition 2 below.

Note that if we generalize this approach to more than two subsystems, the set Σ_k of step 1 is replaced with the set Σ_s of all shared events defined in Section 3 above.

Definition 2 (Observer). The projection $P_k : \Sigma^* \rightarrow \Sigma_k^*$, where Σ_k is a subset of Σ , is an L -observer for a language L over Σ if, for all words t in $P_k(L)$ and s in \bar{L} , the word $P_k(s)$ is a prefix of t implies that there exists a word u in Σ^* such that su is in L and $P_k(su) = t$.

For a generator G with n states, the time and space complexity of the verification whether a projection P is an $L(G)$ -observer is $O(n^2)$, see [18,2]. An algorithm extending the event set to satisfy the property runs in time $O(n^3)$ and linear space. The most significant consequence of the observer property is the following theorem.

Theorem 3 ([28]). If a projection P is an $L(G)$ -observer, for a generator G , then the minimal generator for the language $P(L(G))$ has no more states than the generator G .

This is an important result because it guarantees that the coordinator computed in Algorithm 2 is smaller than the plant whenever the projection P_k is an $L(G_1) \parallel L(G_2)$ -observer.

4.1 Conditional controllability

The concept of conditional controllability introduced in [13] and later studied in [8,9,12] plays the central role in the coordination control approach. In this paper, we revise and simplify this notion. In what follows, we use the notation $\Sigma_{i,u} = \Sigma_i \cap \Sigma_u$ to denote the set of locally uncontrollable events of the event set Σ_i .

Definition 3. Let G_1 and G_2 be generators over Σ_1 and Σ_2 , respectively, and let G_k be a coordinator over Σ_k . A language $K \subseteq L(G_1 \| G_2 \| G_k)$ is conditionally controllable for generators G_1 , G_2 , G_k and uncontrollable event sets $\Sigma_{1,u}$, $\Sigma_{2,u}$, $\Sigma_{k,u}$ if

1. $P_k(K)$ is controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$,
2. $P_{1+k}(K)$ is controllable with respect to $L(G_1) \parallel \overline{P_k(K)}$ and $\Sigma_{1+k,u}$,
3. $P_{2+k}(K)$ is controllable with respect to $L(G_2) \parallel \overline{P_k(K)}$ and $\Sigma_{2+k,u}$,

where $\Sigma_{i+k,u} = (\Sigma_i \cup \Sigma_k) \cap \Sigma_u$, for $i = 1, 2$.

The difference between Definition 3 and the definition in previous papers is that in item 2 we write $L(G_1) \parallel \overline{P_k(K)}$ instead of $L(G_1) \parallel \overline{P_k(K)} \parallel P_k^{2+k}(L(G_2) \parallel \overline{P_k(K)})$. This is possible because the assumption $K \subseteq L(G_1 \| G_2 \| G_k)$ implies the inclusion $\overline{P_k(K)} \subseteq (P_{k \cap 2}^k)^{-1} P_{k \cap 2}^2(L(G_2))$, which results in the equality

$$\begin{aligned} \overline{P_k(K)} \parallel P_k^{2+k}(L(G_2) \parallel \overline{P_k(K)}) &= \overline{P_k(K)} \parallel P_{k \cap 2}^2(L(G_2)) \\ &= \overline{P_k(K)} \cap (P_{k \cap 2}^k)^{-1} P_{k \cap 2}^2(L(G_2)) \\ &= \overline{P_k(K)} \end{aligned}$$

by Lemma 9 (see the Appendix). Hence we have the following.

Lemma 2. Definition 3 and [8, Definition 9] of conditional controllability are equivalent.

The following proposition demonstrates that every conditionally controllable and conditionally decomposable language is controllable.

Proposition 1. Let G_i be a generator over Σ_i , for $i = 1, 2, k$, and let $G = G_1 \| G_2 \| G_k$. Let $K \subseteq L_m(G)$ be such a specification that the language \overline{K} is conditionally decomposable with respect to event sets Σ_1 , Σ_2 , Σ_k , and conditionally controllable for generators G_1 , G_2 , G_k and uncontrollable event sets $\Sigma_{1,u}$, $\Sigma_{2,u}$, $\Sigma_{k,u}$. Then the language K is controllable with respect to the plant language $L(G)$ and uncontrollable event set $\Sigma_u = \Sigma_{1,u} \cup \Sigma_{2,u}$.

Proof. Since the language $\overline{P_{1+k}(K)}$ is controllable with respect to $L(G_1) \parallel \overline{P_k(K)}$ and $\Sigma_{1+k,u}$, and $\overline{P_{2+k}(K)}$ is controllable with respect to $L(G_2) \parallel \overline{P_k(K)}$ and $\Sigma_{2+k,u}$, Lemma 7 implies that the language $\overline{K} = \overline{P_{1+k}(K)} \parallel \overline{P_{2+k}(K)}$ is controllable with respect to $L(G_1) \parallel \overline{P_k(K)} \parallel L(G_2) \parallel \overline{P_k(K)} = L(G) \parallel \overline{P_k(K)}$ and Σ_u , where the equality is by commutativity of the synchronous product and by the fact that $\overline{P_k(K)} \subseteq L(G_k)$. As the language $\overline{P_k(K)}$ is controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$, by Definition 3, the language $L(G) \parallel \overline{P_k(K)}$ is controllable with respect to $L(G) \parallel L(G_k) = L(G)$ by Lemma 7. Finally, by Lemma 8, \overline{K} is controllable with respect to $L(G)$ and Σ_u , which means that K is controllable with respect to $L(G)$ and Σ_u . \square

On the other hand, controllability does not imply conditional controllability.

Example 2. Let G be a generator such that $L(G) = \overline{\{au\}} \parallel \overline{\{bu\}} = \overline{\{abu, bau\}}$. Then the language $K = \{a\}$ is controllable with respect to $L(G)$ and $\Sigma_u = \{u\}$. Moreover, both languages K and \bar{K} are conditionally decomposable with respect to event sets $\{a, u\}$, $\{b, u\}$, and $\Sigma_k = \{u\}$, but the language $P_k(K) = \{\varepsilon\}$ is not controllable with respect to $L(G_k) = P_k(L(G)) = \{u\}$ and $\Sigma_{k,u} = \{u\}$. \triangleleft

However, we show below that if the observer property and local control consistency (LCC) are satisfied, the previous implication holds. To prove this, we need the following definition of LCC. Note that unlike our previous papers, we use a weaker notion of local control consistency (LCC) presented in [24] instead of output control consistency (OCC).

Definition 4 (LCC). Let L be a prefix-closed language over Σ , and let Σ_0 be a subset of Σ . The projection $P_0 : \Sigma^* \rightarrow \Sigma_0^*$ is locally control consistent (LCC) with respect to a word $s \in L$ if for all events $\sigma_u \in \Sigma_0 \cap \Sigma_u$ such that $P_0(s)\sigma_u \in P_0(L)$, it holds that either there does not exist any word $u \in (\Sigma \setminus \Sigma_0)^*$ such that $su\sigma_u \in L$, or there exists a word $u \in (\Sigma_u \setminus \Sigma_0)^*$ such that $su\sigma_u \in L$. The projection P_0 is LCC with respect to a language L if P_0 is LCC for all words of L .

Now the opposite implication to the one proven in Proposition 1 can be stated.

Proposition 2. Let L be a prefix-closed language over Σ , and let $K \subseteq L$ be a language that is controllable with respect to L and Σ_u . If, for $i \in \{k, 1+k, 2+k\}$, the projection P_i is an L -observer and LCC for L , then the language K is conditionally controllable.

Proof. Let $s \in \overline{P_k(K)}$, $a \in \Sigma_{k,u}$, and $sa \in P_k(L)$. Then there exists a word w in \bar{K} such that $P_k(w) = s$. By the observer property, there exists a word u in $(\Sigma \setminus \Sigma_k)^*$ such that $wua \in L$ and $P_k(wua) = sa$. By LCC, there exists another word u' in $(\Sigma_u \setminus \Sigma_k)^*$ such that $wu'a \in L$, that is, $wu'a$ is in \bar{K} by controllability. Hence, $sa \in \overline{P_k(K)}$.

Let $s \in \overline{P_{1+k}(K)}$, $a \in \Sigma_{1+k,u}$, and $sa \in L(G_1) \parallel \overline{P_k(K)}$. Then there exists a word w in \bar{K} such that $P_{1+k}(w) = s$. By the observer property, there exists a word u in $(\Sigma \setminus \Sigma_{1+k})^*$ such that $wua \in L$ and $P_{1+k}(wua) = sa$. By LCC, there exists another word u' in $(\Sigma_u \setminus \Sigma_{1+k})^*$ such that $wu'a \in L$, that is, $wu'a$ is in \bar{K} by controllability. Hence, $sa \in \overline{P_{1+k}(K)}$.

The proof for the case of $k+2$ is similar to that of $k+1$. \square

4.2 Conditionally closed languages

In this subsection we turn our attention to general specification languages that need not be prefix-closed. Analogously to the notion of $L_m(G)$ -closed languages, we recall the notion of conditionally-closed languages defined in [8].

Definition 5. A nonempty language K over Σ is conditionally closed for generators G_1, G_2, G_k if

1. $P_k(K)$ is $L_m(G_k)$ -closed,
2. $P_{1+k}(K)$ is $L_m(G_1) \parallel P_k(K)$ -closed,

3. $P_{2+k}(K)$ is $L_m(G_2) \parallel P_k(K)$ -closed.

If a language K is conditionally closed and conditionally controllable, then there exists a nonblocking supervisor S_k such that $L_m(S_k/G_k) = P_k(K)$, which follows from the basic theorem of supervisory control applied to languages $P_k(K)$ and $L(G_k)$, see [3].

As noted in [3, page 164], if $K \subseteq L_m(G)$ is $L_m(G)$ -closed, then so is the supremal controllable sublanguage of K . However, this does not imply that the language $P_k(K)$ is $L_m(G_k)$ -closed, for any generator $G = G_1 \parallel G_2 \parallel G_k$ such that the coordinator G_k makes generators G_1 and G_2 conditionally independent.

Example 3. Let the event sets be $\Sigma_1 = \{a_1, a\}$, $\Sigma_2 = \{a_2, a\}$, and $\Sigma_k = \{a\}$, respectively, and let the specification language be $K = \{a_1 a_2 a, a_2 a_1 a\}$. Then the application of projections results in languages $P_{1+k}(K) = \{a_1 a\}$, $P_{2+k}(K) = \{a_2 a\}$, and $P_k(K) = \{a\}$, and the language $K = P_{1+k}(K) \parallel P_{2+k}(K)$ is conditionally decomposable. Define generators G_1, G_2, G_k so that $L_m(G_1) = P_{1+k}(K)$, $L_m(G_2) = P_{2+k}(K)$, and $L_m(G_k) = P_k(K) = \{a\}$. Then $L_m(G) = K$ and the language K is $L_m(G)$ -closed. However, the language $P_k(K) \subseteq P_k(K)$ is not $L_m(G_k)$ -closed. \triangleleft

4.3 Existence of supervisors

The following theorem is a revised version (based on the simplification of conditional controllability, Definition 3) of a result presented without proof in [8].

Theorem 4. *Consider the setting of Problem 3. There exist nonblocking supervisors S_1, S_2, S_k such that*

$$L_m(S_1/[G_1 \parallel (S_k/G_k)]) \parallel L_m(S_2/[G_2 \parallel (S_k/G_k)]) = K \quad (1)$$

if and only if the specification language K is both conditionally controllable with respect to generators G_1, G_2, G_k and uncontrollable event sets $\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}$, and conditionally closed with respect to generators G_1, G_2, G_k .

Proof. Let K satisfy the assumptions, and let $G = G_1 \parallel G_2 \parallel G_k$ be the global plant. As the language K is a subset of $L_m(G)$, its projection $P_k(K)$ is a subset of $L_m(G_k)$. By the assumption, the language $P_k(K)$ is $L_m(G_k)$ -closed and controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$. By the basic theorem of supervisory control [20] there exists a nonblocking supervisor S_k such that $L_m(S_k/G_k) = P_k(K)$. As the language $P_{1+k}(K)$ is a subset of languages $L_m(G_1 \parallel G_k)$ and $(P_k^{1+k})^{-1} P_k(K)$, we have that $P_{1+k}(K)$ is included in $L_m(G_1) \parallel P_k(K)$. These relations and the assumption that the system is conditionally controllable and conditionally closed imply the existence of a nonblocking supervisor S_1 such that $L_m(S_1/[G_1 \parallel (S_k/G_k)]) = P_{1+k}(K)$. A similar argument shows that there exists a nonblocking supervisor S_2 such that $L_m(S_2/[G_2 \parallel (S_k/G_k)]) = P_{2+k}(K)$. Since K and \bar{K} are conditionally decomposable, it follows that $L_m(S_1/[G_1 \parallel (S_k/G_k)]) \parallel L_m(S_2/[G_2 \parallel (S_k/G_k)]) = P_{1+k}(K) \parallel P_{2+k}(K) = K$.

To prove the converse implication, the projections P_k, P_{1+k}, P_{2+k} are applied to (1), which can be rewritten as $K = L_m(S_1 \parallel G_1 \parallel S_2 \parallel G_2 \parallel S_k \parallel G_k)$. Thus, the projection $P_k(K) = P_k(L_m(S_1 \parallel G_1 \parallel S_2 \parallel G_2 \parallel S_k \parallel G_k))$ is a subset of $L_m(S_k \parallel G_k) = L_m(S_k/G_k)$. On

the other hand, $L_m(S_k/G_k) \subseteq P_k(K)$, cf. Problem 3. Hence, by the basic controllability theorem, the language $P_k(K)$ is both controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$, and $L_m(G_k)$ -closed. As $\Sigma_{1+k} \cap \Sigma_{2+k} = \Sigma_k$, the application of projection P_{1+k} to (1) and assumptions of Problem 3 give that $P_{1+k}(K) \subseteq L_m(S_1/[G_1 \parallel (S_k/G_k)]) \subseteq P_{1+k}(K)$. Taking $G_1 \parallel (S_k/G_k)$ as a new plant, we get from the basic supervisory control theorem that the language $P_{1+k}(K)$ is controllable with respect to $L(G_1 \parallel (S_k/G_k))$ and $\Sigma_{1+k,u}$, and that it is $L_m(G_1 \parallel (S_k/G_k))$ -closed. The case of the language $P_{2+k}(K)$ is analogous. \square

5 Supremal conditionally controllable sublanguages

Necessary and sufficient conditions for the existence of nonblocking supervisors S_1 , S_2 , and S_k that achieve a considered specification language using our coordination control architecture have been presented in Theorem 4. However, in many cases control specifications fail to be conditionally controllable and, similarly as in the monolithic supervisory control, supremal conditionally controllable sublanguages should be investigated.

Let $\text{supcC}(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}))$ denote the supremal conditionally controllable sublanguage of K with respect to $L = L(G_1 \parallel G_2 \parallel G_k)$ and sets of uncontrollable events $\Sigma_{1,u}$, $\Sigma_{2,u}$, $\Sigma_{k,u}$. The supremal conditionally controllable sublanguage always exists, cf. [9] for the case of prefix-closed languages.

Theorem 5. *The supremal conditionally controllable sublanguage of a given language K always exists and is equal to the union of all conditionally controllable sublanguages of the language K .*

Proof. Let I be an index set, and let K_i , for $i \in I$, be conditionally controllable sublanguages of $K \subseteq L(G_1 \parallel G_2 \parallel G_k)$. To prove that the language $P_k(\cup_{i \in I} K_i)$ is controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$, note that

$$\begin{aligned} P_k(\cup_{i \in I} \overline{K_i}) \Sigma_{k,u} \cap L(G_k) &= \cup_{i \in I} (P_k(\overline{K_i}) \Sigma_{k,u} \cap L(G_k)) \\ &\subseteq \cup_{i \in I} P_k(\overline{K_i}) \\ &= P_k(\cup_{i \in I} \overline{K_i}), \end{aligned}$$

where the inclusion is by controllability of the language $P_k(K_i)$ with respect to $L(G_k)$ and $\Sigma_{k,u}$. Next, to prove that

$$P_{1+k}(\cup_{i \in I} \overline{K_i}) \Sigma_{1+k,u} \cap L(G_1) \parallel P_k(\cup_{i \in I} \overline{K_i}) \subseteq P_{1+k}(\cup_{i \in I} \overline{K_i}),$$

note that

$$\begin{aligned} P_{1+k}(\cup_{i \in I} \overline{K_i}) \Sigma_{1+k,u} \cap L(G_1) \parallel P_k(\cup_{i \in I} \overline{K_i}) \\ &= \cup_{i \in I} (P_{1+k}(\overline{K_i}) \Sigma_{1+k,u}) \cap \cup_{i \in I} (L(G_1) \parallel P_k(\overline{K_i})) \\ &= \cup_{i \in I} \cup_{j \in I} (P_{1+k}(\overline{K_i}) \Sigma_{1+k,u} \cap L(G_1) \parallel P_k(\overline{K_j})). \end{aligned}$$

Consider two different indexes i and j from I such that

$$P_{1+k}(\overline{K_i}) \Sigma_{1+k,u} \cap L(G_1) \parallel P_k(\overline{K_j}) \not\subseteq P_{1+k}(\cup_{i \in I} \overline{K_i}).$$

Then there exist a word x in $P_{1+k}(\overline{K_i})$ and an uncontrollable event u in $\Sigma_{1+k,u}$ such that xu belongs to the language $L(G_1) \parallel P_k(\overline{K_j})$, and xu does not belong to $P_{1+k}(\cup_{i \in I} \overline{K_i})$. It follows that $P_k(x)$ belongs to $P_k(\overline{K_i})$ and $P_k(xu)$ belongs to $P_k(\overline{K_j})$. If $P_k(xu)$ belongs to $P_k(\overline{K_i})$, then xu belongs to $L(G_1) \parallel P_k(\overline{K_i})$, and controllability of the language $P_{1+k}(\overline{K_i})$ with respect to $L(G_1) \parallel P_k(\overline{K_i})$ implies that xu belongs to $P_{1+k}(\cup_{i \in I} \overline{K_i})$; hence, $P_k(xu)$ does not belong to $P_k(\overline{K_i})$. If the event u does not belong to $\Sigma_{k,u}$, then $P_k(xu) = P_k(x)$ belongs to $P_k(\overline{K_i})$, which is not the case. Thus, u belongs to $\Sigma_{k,u}$. As $P_k(\overline{K_i}) \cup P_k(\overline{K_j})$ is a subset of $L(G_k)$, we get that $P_k(xu) = P_k(x)u$ belongs to $L(G_k)$. However, controllability of the language $P_k(\overline{K_i})$ with respect to $L(G_k)$ and $\Sigma_{k,u}$ implies that the word $P_k(xu)$ belongs to $P_k(\overline{K_i})$. This is a contradiction.

As the case for the projection P_{2+k} is analogous, the proof is complete. \square

Still, it is a difficult problem to compute a supremal conditional controllable sublanguage. Consider the setting of Problem 3 and define the languages

$$\begin{array}{l} \sup C_k = \sup C(P_k(K), L(G_k), \Sigma_{k,u}) \\ \sup C_{1+k} = \sup C(P_{1+k}(K), L(G_1) \parallel \overline{\sup C_k}, \Sigma_{1+k,u}) \\ \sup C_{2+k} = \sup C(P_{2+k}(K), L(G_2) \parallel \overline{\sup C_k}, \Sigma_{2+k,u}) \end{array} \quad (*)$$

Interestingly, the following inclusion always holds.

Lemma 3. *Consider the setting of Problem 3, and languages defined in (*). Then the language $P_k(\sup C_{i+k})$ is a subset of the language $\sup C_k$, for $i = 1, 2$.*

Proof. By definition, the language $P_k(\sup C_{i+k})$ is a subset of languages $\overline{\sup C_k}$ and $P_k(K)$. To prove that $P_k(\sup C_{i+k})$ is a subset of $\sup C_k$, we prove that the language $\overline{\sup C_k} \cap P_k(K)$ is a subset of $\sup C_k$. To do this, it is sufficient to show that the language $\sup C_k \cap P_k(K)$ is controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$.

Thus, consider a word s in $\overline{\sup C_k} \cap P_k(K)$, an uncontrollable event u in $\Sigma_{k,u}$, and the word su in $L(G_k)$. By controllability of $\sup C_k$, the word su belongs to $\overline{\sup C_k}$, which is a subset of $\overline{P_k(K)}$. That is, there exists a word v such that su is in $\sup C_k$, which is a subset of $P_k(K)$. This means that the word su belongs to $\overline{\sup C_k} \cap P_k(K)$, which implies that the word su is in $\sup C_k \cap P_k(K)$. This completes the proof. \square

It turns out that if the converse inclusion also holds, then we immediately obtain the supremal conditionally-controllable sublanguage.

Theorem 6. *Consider the setting of Problem 3, and languages defined in (*). If $\sup C_k$ is a subset of $P_k(\sup C_{i+k})$, for $i = 1, 2$, then*

$$\sup C_{1+k} \parallel \sup C_{2+k} = \sup cC(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u})).$$

Proof. Let $\sup cC = \sup cC(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}))$ and $M = \sup C_{1+k} \parallel \sup C_{2+k}$. To prove that M is a subset of $\sup cC$, we show that (i) M is a subset of K and (ii) M is conditionally controllable with respect to generators G_1, G_2, G_k and uncontrollable event sets $\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}$. To this aim, notice that M is a subset of $P_{1+k}(K) \parallel P_{2+k}(K) = K$, because K is conditionally decomposable. Moreover, by Lemmas 9 and 3, the language

$P_k(M) = P_k(\text{sup}C_{1+k}) \cap P_k(\text{sup}C_{2+k}) = \text{sup}C_k$, which is controllable with respect to $L(G_k)$ and $\Sigma_{k,u}$. Similarly, $P_{i+k}(M) = \text{sup}C_{i+k} \parallel P_k(\text{sup}C_{j+k}) = \text{sup}C_{i+k} \parallel \text{sup}C_k = \text{sup}C_{i+k}$, for $j \neq i$, which is controllable with respect to $L(G_i) \parallel \overline{P_k(M)}$. Hence, M is a subset of $\text{sup}cC$.

To prove the opposite inclusion, it is sufficient, by Lemma 10, to show that the language $P_{i+k}(\text{sup}cC)$ is a subset of $\text{sup}C_{i+k}$, for $i = 1, 2$. To prove this note that the language $P_{1+k}(\text{sup}cC)$ is controllable with respect to $L(G_1) \parallel \overline{P_k(\text{sup}cC)}$ and $\Sigma_{1+k,u}$, and the language $L(G_1) \parallel \overline{P_k(\text{sup}cC)}$ is controllable with respect to $L(G_1) \parallel \overline{\text{sup}C_k}$ and $\Sigma_{1+k,u}$ by Lemma 7, because the language $P_k(\text{sup}cC)$ being controllable with respect to $L(G_k)$ implies that it is also controllable with respect to $\overline{\text{sup}C_k}$, which is a subset of $L(G_k)$. By Lemma 8, the language $P_{1+k}(\text{sup}cC)$ is controllable with respect to $L(G_1) \parallel \overline{\text{sup}C_k}$ and $\Sigma_{1+k,u}$, which implies that $P_{1+k}(\text{sup}cC)$ is a subset of $\text{sup}C_{1+k}$. The other case is analogous. Hence, the language $\text{sup}cC$ is a subset of M and the proof is complete. \square

Example 4. This example demonstrates that the language $\text{sup}C_k$ is not always included in the language $P_k(\text{sup}C_{i+k})$. Moreover, it does not hold even if projections are observers or satisfy the LCC property.

Consider systems G_1 and G_2 shown in Fig. 2, and the specification K as shown in Fig. 3. Controllable events are $\Sigma_c = \{a_1, a_2, c\}$, and coordinator events are $\Sigma_k =$

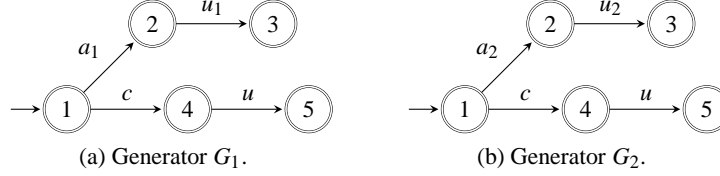


Fig. 2. Generators G_1 and G_2 .

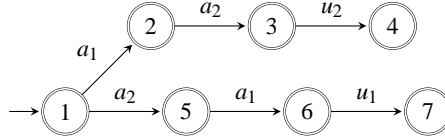


Fig. 3. Specification K .

$\{a_1, a_2, c, u\}$. Construct the coordinator $G_k = P_k(G_1) \parallel P_k(G_2)$. It can be verified that K is conditionally decomposable, $\text{sup}C_k = \{a_1 a_2, a_2 a_1\}$, $\text{sup}C_{1+k} = \{a_2 a_1 u_1\}$, and $\text{sup}C_{2+k} = \{a_1 a_2 u_2\}$. Hence, $\text{sup}C_k$ is not a subset of $P_k(\text{sup}C_{i+k})$.

It can be verified that projections P_k, P_{1+k}, P_{2+k} are $L(G_1 \parallel G_2)$ -observers and LCC for the language $L(G_1 \parallel G_2)$. \triangleleft

Recall that it is still open how to compute the supremal conditionally-controllable sublanguage for a general, non-prefix-closed language. Consider the example above and note that the words a_1a_2 and a_2a_1 from $\sup C_k$ do not appear in the projection of the supremal conditionally-controllable sublanguage, that is, no words with both letters a_1 and a_2 appear in the supremal conditionally-controllable sublanguage. Thus, we can remove these words from $\sup C_k$ (basically from the coordinator) and recompute the supremal controllable sublanguage (denoted by $\sup C'_k$), that is,

$$\sup C'_k = \sup C(\cap_{i=1,2} P_k(\sup C_{i+k}), L(G_k), \Sigma_{k,u}) = \{\varepsilon\}$$

and, similarly, recompute $\sup C_{i+k}$ using $\sup C'_k$ instead of $\sup C_k$. Note that the plant is changed because the coordinator restricts it more than before. An application of Theorem 6 could thus be as follows. If $\sup C_k \not\subseteq P_k(\sup C_{i+k})$, then the natural approach seems to be to remove from $\sup C_k$ all words violating the inclusion, and to recompute $\sup C_{i+k}$, for $i = 1, 2$, with respect to this new $\sup C'_k$, that is

$$\begin{array}{l} \sup C'_k = \sup C(P_k(\sup C_{1+k}) \cap P_k(\sup C_{2+k}), L(G_k), \Sigma_{k,u}) \\ \sup C'_{1+k} = \sup C(\sup C_{1+k}, L(G_1) \parallel \overline{\sup C'_k}, \Sigma_{1+k,u}) \\ \sup C'_{2+k} = \sup C(\sup C_{2+k}, L(G_2) \parallel \overline{\sup C'_k}, \Sigma_{2+k,u}) \end{array} \quad (**)$$

In our example, we get that $\sup C'_{1+k} = \{\varepsilon\}$ and $\sup C'_{2+k} = \{\varepsilon\}$ satisfy the assumption that $\sup C'_k \subseteq P_k(\sup C'_{i+k})$, for $i = 1, 2$, hence Theorem 6 applies. It is not yet clear whether this method can be used in general, namely whether it always terminates and the result is the supremal conditionally-controllable sublanguage. It is only known that if it terminates, the result is conditionally controllable (see the end of Section 6 for more discussion). Another problem is that it requires to compute the projection, which can be exponential in general, because the observer property is not ensured. One of the natural investigations of this problem is to work with nondeterministic representations. Several attempts in this direction were done in the literature although they usually handle the case where only the plant is nondeterministic, while the specification is deterministic, see, e.g., [26,27]. Even more, it is a question how to test the inclusion from Theorem 6.

Finally, if $\sup C_{i+k}$ and $\sup C'_k$ are nonconflicting, the language $\sup C_{i+k} \parallel \sup C'_k$ is controllable with respect to $L(G_i) \parallel \overline{\sup C'_k} \parallel \sup C'_k = L(G_i) \parallel \overline{\sup C'_k}$ by Lemma 7. This observation gives the following result for prefix-closed languages.

Lemma 4. *Let $K = \overline{K} \subseteq L = L(G_1 \parallel G_2 \parallel G_k)$, where G_i is a generator over Σ_i , for $i = 1, 2, k$. Assume that K is conditionally decomposable, and define the languages $\sup C_k$, $\sup C_{1+k}$ and $\sup C_{2+k}$ as in (*). If $\sup C_k \not\subseteq P_k(\sup C_{i+k})$, for $i \in \{1, 2\}$, define the language $\sup C'_k$ as in (**). Then the language*

$$\sup C_{1+k} \parallel \sup C_{2+k} \parallel \sup C'_k$$

is conditionally controllable with respect to G_1, G_2, G_k and $\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}$.

Note that if we have any specification K , which is conditionally decomposable, then the specification $K \parallel L$ is also conditionally decomposable. The opposite is not true.

Lemma 5. Let K be conditionally decomposable with respect to event sets $\Sigma_1, \Sigma_2, \Sigma_k$, and let $L = L_1 \parallel L_2 \parallel L_k$, where L_i is over Σ_i , for $i = 1, 2, k$. Then the language $K \parallel L$ is conditionally decomposable with respect to event sets $\Sigma_1, \Sigma_2, \Sigma_k$.

Proof. By the assumption we have that $K = P_{1+k}(K) \parallel P_{2+k}(K)$. Then

$$\begin{aligned} K \parallel L &= P_{1+k}(K) \parallel P_{2+k}(K) \parallel L_1 \parallel L_2 \parallel L_k \\ &= P_{1+k}(K) \parallel L_1 \parallel L_k \parallel P_{2+k}(K) \parallel L_2 \parallel L_k \\ &= P_{1+k}(K \parallel L_1 \parallel L_k) \parallel P_{2+k}(K \parallel L_2 \parallel L_k) \end{aligned}$$

where the last equality is by Lemma 9. By Lemma 12, $K \parallel L$ is conditionally decomposable with respect to event sets Σ_1, Σ_2 , and Σ_k . \square

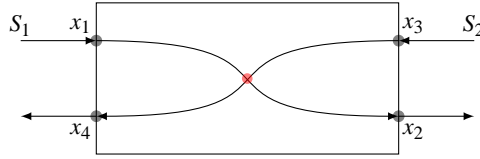


Fig. 4. A railway crossing

Example 5. Consider a situation at a railway station. There are several tracks that cross each other at some points. Obviously, the traffic has to be controlled at those points. For simplicity, we consider only two one-way tracks that cross at some point, that is, trains going from west to east use track one, while trains going from east to west use track two. The traffic is controlled by traffic lights.

Thus, consider the railway crossing with two traffic lights, S_1 and S_2 , and two entry points x_1, x_3 and two exit points x_2, x_4 , as depicted in Fig. 4. Each traffic light has values g_i (green) and r_i (red), for $i = 1, 2$. Colors of the traffic lights are controllable. The plant is then given as a parallel composition of two systems G_1 and G_2 depicted in Fig. 5. For safety reasons, each system is able to set the traffic light to red at any moment. It can set the traffic light to green and the trains are detected entering (x_1 or x_3) and leaving (x_2 or x_4) the crossing.



Fig. 5. Generators G_1 and G_2

To define the specification, it is natural that a train is allowed to enter the crossing only if its traffic light is green. The purpose of the entry and exit points x_i , $i = 1, 2, 3, 4$,

is to allow a limited number of trains in the crossroad area from the direction of the green light. The light can turn red at any moment, but the other traffic light can be set to green only if all the trains have left the crossroad area. In this example, we consider the case where at most three trains are allowed to enter the crossroad area on one green light. For this purpose, the entry points must also be controllable to protect another train to enter. This part of the specification is modeled by buffers depicted in Fig. 6. Another part of the specification governs the behavior of the traffic lights. First, both

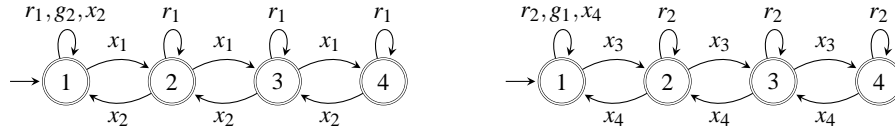


Fig. 6. The two buffers

lights must be red before one of the traffic lights is set to green, stay green for a while, and then must be set to red again. The traffic lights should take turns, so that no trains are waiting for ever, see Fig. 7. For simplicity, we do not model the mechanism (such as a clock) that sets the traffic lights to green for a specific amount of time units. The overall specification is then depicted in Fig. 8. The set of uncontrollable events is thus

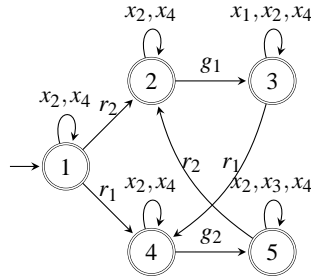


Fig. 7. The traffic lights' part of the specification

$\Sigma_u = \{x_2, x_4\}$; all other events are controllable.

To make the specification controllable with respect to Σ_1 , Σ_2 , and Σ_k (where Σ_k is initialized to the empty set), we need to take $\Sigma_k = \{g_1, g_2, r_1\}$. Now we can compute the coordinator as the projection $P_k(G_1) \parallel P_k(G_2)$, and the languages $\sup C_k$, $\sup C_{1+k}$ and $\sup C_{2+k}$ as defined in (*), see Figs. 9, 10, and 11. It can be verified that $\sup C_k \subseteq P_k(\sup C_{i+k})$, for $i = 1, 2$, hence Theorem 6 applies and the result (that is, in the monolithic notation, the language $\sup C_{1+k} \parallel \sup C_{2+k}$) is the supremal conditionally-controllable sublanguage of the specification, cf. Fig. 12. Note that the difference with the specification is the correct marking of the states.

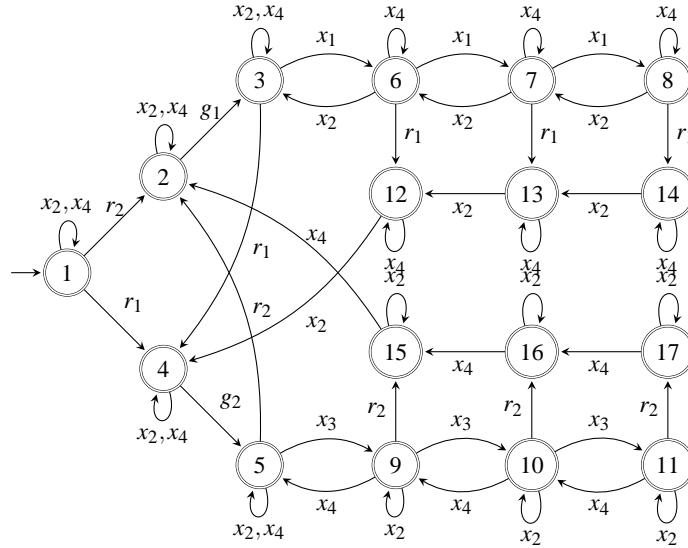


Fig. 8. The overall specification

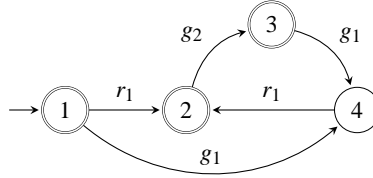


Fig. 9. Supervisor $\text{sup } C_k$

6 Coordinator for nonblockingness

So far, we have only considered a coordinator for safety. In this section, we discuss a coordinator for nonblockingness. To this end, we first prove a fundamental theoretical result and then give an algorithm to construct a coordinator for nonblockingness.

Recall that a generator G is nonblocking if $\overline{L_m(G)} = L(G)$.

Theorem 7. Consider languages L_1 over Σ_1 and L_2 over Σ_2 , and let the projection $P_0 : (\Sigma_1 \cup \Sigma_2)^* \rightarrow \Sigma_0^*$, with $\Sigma_1 \cap \Sigma_2 \subseteq \Sigma_0$, be an L_i -observer, for $i = 1, 2$. Let G_0 be a nonblocking generator with $L_m(G_0) = \overline{P_0(L_1)} \parallel \overline{P_0(L_2)}$. Then the composed language $L_1 \parallel L_2 \parallel L_m(G_0)$ is nonblocking, that is, $\overline{L_1 \parallel L_2 \parallel L_m(G_0)} = \overline{L_1} \parallel \overline{L_2} \parallel \overline{L_m(G_0)}$.

Proof. Let $L_0 = L_m(G_0)$. By Lemma 11, $\overline{L_1 \parallel L_2 \parallel L_0} = \overline{L_1} \parallel \overline{L_2} \parallel \overline{L_0}$ if and only if

$$\overline{P_0(L_1) \parallel P_0(L_2) \parallel L_0} = \overline{P_0(L_1)} \parallel \overline{P_0(L_2)} \parallel \overline{L_0}.$$

However, for our choice of the coordinator, this equality always holds because both sides of the later equation are $\overline{L_0}$. \square

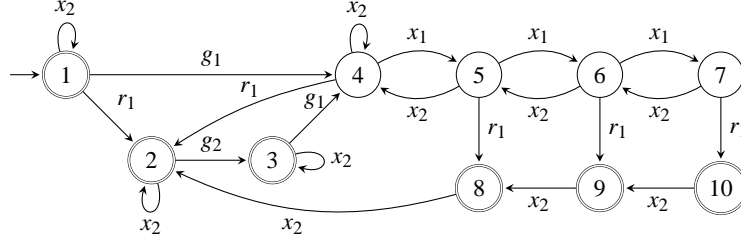


Fig. 10. Supervisor $\text{sup}C_{1+k}$

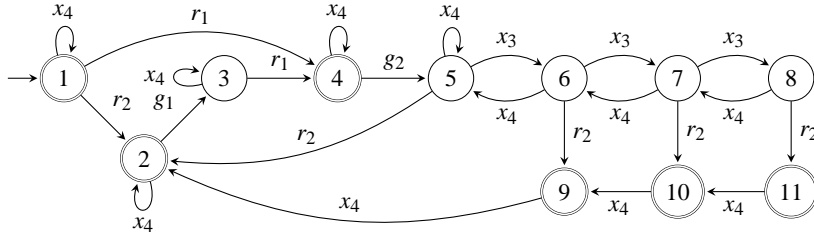


Fig. 11. Supervisor $\text{sup}C_{2+k}$

This result is demonstrated in the following example.

Example 6. Consider two nonblocking generators G_1 and G_2 depicted in Fig. 13. Their synchronous product is shown in Fig. 14. One can see that the generator $G_1 \parallel G_2$ is blocking because no marked state is reachable from state 3. It can be verified that the projection $P : \{a, b, c, d\}^* \rightarrow \{a, b, d\}^*$ is an $L(G_1)$ - and $L(G_2)$ -observer. The generator G_0 is then a nonblocking (trimmed) part of the synchronous product $P(G_1) \parallel P(G_2)$ of generators depicted in Fig. 15, that is $L_m(G_0) = \{a\}$, and the synchronous product of $G_1 \parallel G_2$ with G_0 is shown in Fig. 16. One can see that the result is nonblocking. It is important to notice that event b belongs to the event set of the generator G_0 .

The previous example shows that even though the result is nonblocking, it is disputable whether such a coordinator is acceptable. If we assume that event b is uncontrollable, then the coordinator prevents an uncontrollable event from happening and the result depicted in Fig. 16 is not controllable with respect to the plant depicted in Fig. 14. Although it is not explicitly stated that a coordinator is not allowed to do so, we further discuss this issue and suggest a solution useful in our coordination control framework.

In general, local supervisors $\text{sup}C_{1+k}$ and $\text{sup}C_{2+k}$ computed in Section 5 might be blocking. However, we can always choose the language

$$L_C = \text{sup}C(P_0(\text{sup}C_{1+k}) \parallel P_0(\text{sup}C_{2+k}), \overline{P_0(\text{sup}C_{1+k})} \parallel \overline{P_0(\text{sup}C_{2+k})}, \Sigma_{0,u}), \quad (2)$$

where the projection P_0 is a $\text{sup}C_{i+k}$ -observer, for $i = 1, 2$. The following result shows that the language $\text{sup}C_{1+k} \parallel \text{sup}C_{2+k} \parallel L_C$ is nonblocking and controllable.

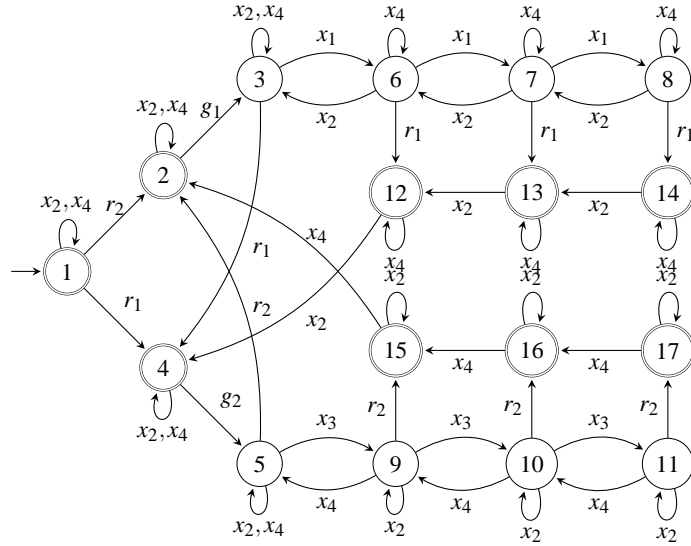


Fig. 12. The supremal conditionally-controllable sublanguage $\sup C_{1+k} \parallel \sup C_{2+k}$

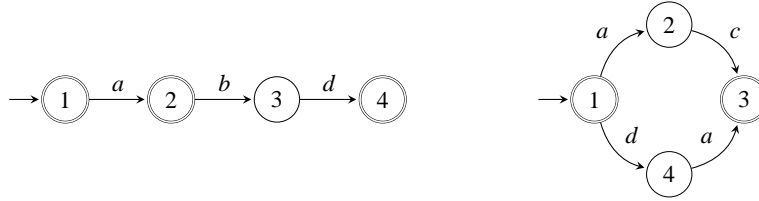


Fig. 13. Generators G_1 and G_2

Theorem 8. Consider the notation as defined in Problem 3, Algorithm 2, (*), and (2). Then the language

$$\overline{\sup C_{1+k} \parallel \sup C_{2+k} \parallel L_C} = \overline{\sup C_{1+k}} \parallel \overline{\sup C_{2+k}} \parallel \overline{L_C}$$

is controllable with respect to the plant language $L(G_1) \parallel L(G_2)$.

Proof. To prove nonblockingness, we use Lemma 11 in two steps. Namely, it holds that $\overline{\sup C_{i+k} \parallel L_C} = \overline{\sup C_{i+k}} \parallel \overline{L_C}$ if and only if $P_0(\sup C_{i+k} \parallel L_C) = P_0(\sup C_{i+k}) \parallel \overline{L_C}$, for $i = 1, 2$, which always holds because both sides of the later equation are equal to $\overline{L_C}$. Using Lemma 11 again,

$$\overline{\sup C_{1+k} \parallel L_C} \parallel \overline{\sup C_{2+k} \parallel L_C} = \overline{\sup C_{1+k} \parallel L_C} \parallel \overline{\sup C_{2+k} \parallel L_C}$$

if and only if

$$P_0(\sup C_{1+k} \parallel L_C) \parallel P_0(\sup C_{2+k} \parallel L_C) = P_0(\sup C_{1+k} \parallel L_C) \parallel P_0(\sup C_{2+k} \parallel L_C) \quad (3)$$

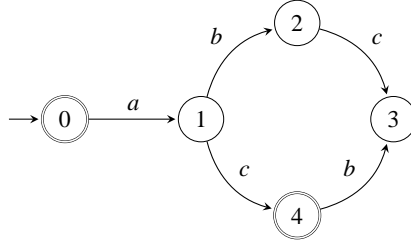


Fig. 14. Synchronous product $G_1 \parallel G_2$



Fig. 15. Generators $P(G_1)$ and $P(G_2)$

because if the projection P_0 is a $\text{sup}C_{i+k}$ -observer, for $i = 1, 2$, and an L_C -observer (since it is an identity), then the projection P_0 is also an $\text{sup}C_{i+k} \parallel L_C$ -observer by [19]. But (3) always holds because $P_0(\text{sup}C_{i+k} \parallel L_C) = P_0(\text{sup}C_{i+k}) \parallel L_C = L_C$, by Lemma 9, hence both sides are equal to $\overline{L_C}$. Thus, summarized, we have that

$$\begin{aligned} \overline{\text{sup}C_{1+k} \parallel \text{sup}C_{2+k} \parallel L_C} &= \overline{\text{sup}C_{1+k} \parallel L_C} \parallel \overline{\text{sup}C_{2+k} \parallel L_C} \\ &= \overline{\text{sup}C_{1+k} \parallel L_C} \parallel \overline{\text{sup}C_{2+k} \parallel L_C} \\ &= \overline{\text{sup}C_{1+k}} \parallel \overline{\text{sup}C_{2+k} \parallel L_C}. \end{aligned}$$

To prove controllability, note that $\text{sup}C_{i+k}$ is controllable with respect to $\overline{\text{sup}C_{i+k}}$, for $i = 1, 2$, and L_C is controllable with respect to $\overline{P_0(\text{sup}C_{1+k})} \parallel \overline{P_0(\text{sup}C_{2+k})}$. Now we use Lemma 7 several times, and the nonconflictiveness shown above, to obtain that

- $\text{sup}C_{i+k} \parallel L_C$ is controllable with respect to $(\overline{\text{sup}C_{i+k}}) \parallel (\overline{P_0(\text{sup}C_{1+k})} \parallel \overline{P_0(\text{sup}C_{2+k})})$, for $i = 1, 2$,
- $(\overline{\text{sup}C_{1+k} \parallel L_C}) \parallel (\overline{\text{sup}C_{2+k} \parallel L_C}) = \overline{\text{sup}C_{1+k} \parallel \text{sup}C_{2+k} \parallel L_C}$ is controllable with respect to $(\overline{\text{sup}C_{1+k}} \parallel \overline{P_0(\text{sup}C_{1+k})} \parallel \overline{P_0(\text{sup}C_{2+k})}) \parallel (\overline{\text{sup}C_{2+k}} \parallel \overline{P_0(\text{sup}C_{1+k})} \parallel \overline{P_0(\text{sup}C_{2+k})})$ that can be simplified to $\overline{\text{sup}C_{1+k}} \parallel \overline{\text{sup}C_{2+k}}$,
- $\overline{\text{sup}C_{1+k}} \parallel \overline{\text{sup}C_{2+k}}$ is controllable with respect to $(L(G_1) \parallel \overline{\text{sup}C_k}) \parallel (L(G_2) \parallel \overline{\text{sup}C_k}) = L(G_1) \parallel L(G_2) \parallel \overline{\text{sup}C_k}$, and
- $L(G_1) \parallel L(G_2) \parallel \overline{\text{sup}C_k}$ is controllable with respect to $L(G_1) \parallel L(G_2) \parallel L(G_k)$ because the language $\text{sup}C_k$ is controllable with respect to $L(G_k)$.

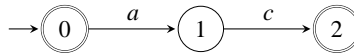


Fig. 16. Synchronous product $G_1 \parallel G_2 \parallel G_0$

Using transitivity of controllability, Lemma 8, we obtain that $\sup C_{1+k} \parallel \sup C_{2+k} \parallel L_C$ is controllable with respect to $L(G_1) \parallel L(G_2) \parallel L(G_k) = L(G_1) \parallel L(G_2)$, because the coordinator G_k is constructed in such a way that it does not change the plant. \square

To demonstrate this improvement, we consider Example 6.

Example 7. Consider the generators of Example 6. Note that $G_1 \parallel G_2 \parallel G_0$, Fig. 16, is not controllable with respect to the plant $G_1 \parallel G_2$, Fig. 14, if b is uncontrollable. The generator $G_0 = P(G_1) \parallel P(G_2)$ is depicted in Fig. 17. It is not hard to see that if b is

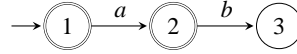


Fig. 17. Generator $P(G_1) \parallel P(G_2)$

not controllable, then the supremal controllable sublanguage of $L_m(G_0)$ with respect to $L(G_0)$ is $L_C = \{\varepsilon\}$, because event a must be prevented from happening. Therefore, the language of $L(G_1 \parallel G_2) \parallel L_C = \{\varepsilon\}$ as expected.

We can now summarize this method as an algorithm.

Algorithm 9 (Coordinator for nonblockingness) *Consider the notation above.*

1. Compute $\sup C_{1+k}$ and $\sup C_{2+k}$ as defined in (*).
2. Let $\Sigma_0 := \Sigma_k$ and $P_0 := P_k$.
3. Extend the event set Σ_0 so that the projection P_0 is both a $\sup C_{1+k}$ - and a $\sup C_{2+k}$ -observer.
4. Define the coordinator C as the minimal nonblocking generator such that $L_m(C) = \sup C(P_0(\sup C_{1+k}) \parallel P_0(\sup C_{2+k}), \overline{P_0(\sup C_{1+k})} \parallel \overline{P_0(\sup C_{2+k})}, \Sigma_{0,u})$.

This algorithm (Step 1) is based on the computation of the languages $\sup C_{1+k}$ and $\sup C_{2+k}$ defined in (*), which can be computed using a standard algorithm for the computation of supremal controllable sublanguages. If the assumption of Theorem 6 is satisfied, the computed languages are the languages of local supervisors that are the candidates to solve the problem. However, the composition $\sup C_{1+k} \parallel \sup C_{2+k}$ can be blocking, and a coordinator for nonblockingness is required.

In Step 2, we define a new event set Σ_0 (and the corresponding projection) that is initialized to be the event set Σ_k used in the computation in Step 1.

In Step 3 of the algorithm, the event set Σ_0 must be extended so that the projection P_0 is both a $\sup C_{1+k}$ - and $\sup C_{2+k}$ -observer. Thus, in consequence of the extension operation, Σ_k can become a proper subset of Σ_0 . Even though the computation of such a minimal extension is NP-hard, a polynomial algorithm computing a reasonable extension exists, cf. [5] for more details and the algorithm.

Finally, in Step 4, the coordinator generator C is defined as the minimal nonblocking generator accepting the supremal controllable sublanguage of the language $P_0(\sup C_{1+k}) \parallel P_0(\sup C_{2+k})$ with respect to the language $\overline{P_0(\sup C_{1+k})} \parallel \overline{P_0(\sup C_{2+k})}$. This idea has

been used by Feng in [4]. In other words, if S_1 and S_2 are generators for languages supC_{1+k} and supC_{2+k} , respectively, then the coordinator C is computed as the generator for the supremal controllable sublanguage of $P_0(S_1) \parallel P_0(S_2)$. Since P_0 is an observer, the computation can be done in polynomial time, cf. [30].

Remark 1. In the previous section we discussed the case when $\text{supC}_k \not\subseteq P_k(\text{supC}_{i+k})$, for $i = 1, 2$. Note that the coordinator L_C discussed in this section can also be used in that case because $\text{supC}_{1+k} \parallel L_C$ and $\text{supC}_{2+k} \parallel L_C$ then form synchronously nonconflicting local supervisors such that their overall behavior is controllable with respect to the global plant. Hence, although this solution may not be optimal, it presents a solution in the case of (non-prefix-closed) languages that do not satisfy the assumptions of Theorem 6, or of those of Section 7 in the case of prefix-closed languages.

7 Supremal prefix-closed languages

In this section, we revise the case of prefix-closed languages. We use the local control consistency property (LCC) instead of the output control consistency property (OCC), cf. [12]. The reason for this is that LCC is a less restrictive condition than OCC, as shown in [24, Lemma 4.4]. Moreover, the extension of our approach to an arbitrary number of local plants is sketched.

Theorem 10. *Let K be a prefix-closed sublanguage of the plant language L , where $L = L(G_1 \parallel G_2 \parallel G_k)$, and G_i is a generator over Σ_i , for $i = 1, 2, k$. Assume that the language K is conditionally decomposable, and define the languages supC_k , supC_{1+k} , supC_{2+k} as in (*). Let the projection P_k^{i+k} be an $(P_i^{i+k})^{-1}(L(G_i))$ -observer and LCC for the language $(P_i^{i+k})^{-1}(L(G_i))$, for $i = 1, 2$. Then*

$$\text{supC}_{1+k} \parallel \text{supC}_{2+k} = \text{supcC}(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u})).$$

Proof. In this proof, let supcC denote the supremal conditionally controllable language $\text{supcC}(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u}))$, and M the parallel composition $\text{supC}_{1+k} \parallel \text{supC}_{2+k}$. It is shown in [12, Theorem 11] that supcC is a subset of M and that M is a subset of K . To prove that $P_k(M) \Sigma_{k,u} \cap L(G_k)$ is a subset of $P_k(M)$, consider a word x in $P_k(M)$ and an uncontrollable event a in $\Sigma_{k,u}$ such that the word xa is in $L(G_k)$. To show that the word xa is in $P_k(M) = P_k^{1+k}(\text{supC}_{1+k}) \cap P_k^{2+k}(\text{supC}_{2+k})$, note that there exists a word w in M such that $P_k(w) = x$. It is shown in [12, Theorem 11] that there exists a word u in $(\Sigma_1 \setminus \Sigma_k)^*$ such that the word $P_{1+k}(w)ua$ is in $(P_1^{1+k})^{-1}(L(G_1))$ and the word $P_{1+k}(w)$ is in $L(G_1) \parallel \text{supC}_k$. As the projection P_k^{1+k} is LCC for the language $(P_1^{1+k})^{-1}(L(G_1))$, there exists a word u' in $(\Sigma_u \setminus \Sigma_k)^*$ such that $P_{1+k}(w)u'a$ is in $(P_1^{1+k})^{-1}(L(G_1))$. Then, controllability of supC_{1+k} implies that $P_{1+k}(w)u'a$ is in supC_{1+k} , that is, xa is in $P_k^{1+k}(\text{supC}_{1+k})$. Analogously, we can prove that xa is in $P_k^{2+k}(\text{supC}_{2+k})$. Thus, xa is in $P_k(M)$. The rest of the proof is the same as in [12, Theorem 11]. \square

In this Theorem, a relatively large number of properties that the coordinator, the local plants and the specification have to satisfy is assumed. However, a polynomial algorithm extending the coordinator event set so that the language K becomes conditionally

decomposable has already been discussed, see [11]. In addition, to ensure that the projection P_k^{i+k} is an $(P_i^{i+k})^{-1}(L(G_i))$ -observer and LCC for the language $(P_i^{i+k})^{-1}(L(G_i))$, the coordinator event set can again be extended so that the conditions are fulfilled [24,5].

Conditions of Theorem 10 imply that the projection P_k is LCC for the language L .

Lemma 6. *Let G_i over Σ_i be generators, for $i = 1, 2$. Let $\Sigma = \Sigma_1 \cup \Sigma_2$, and let $P_i : \Sigma^* \rightarrow \Sigma_i^*$, for $i = 1, 2, k$ and $\Sigma_k \subseteq \Sigma$, be projections. If $\Sigma_1 \cap \Sigma_2$ is a subset of Σ_k and the projection P_k^{i+k} is LCC for the language $(P_i^{i+k})^{-1}(L(G_i))$, for $i = 1, 2$, then the projection P_k is LCC for the language $L = L(G_1 \parallel G_2 \parallel G_k)$.*

Proof. For a word s in L and an event σ_u in $\Sigma_{k,u}$, assume that there exists a word u in $(\Sigma \setminus \Sigma_k)^*$ such that $su\sigma_u$ is in L . Then $P_{i+k}(su\sigma_u) = P_{i+k}(s)P_{i+k}(u)\sigma_u$ is in $(P_i^{i+k})^{-1}(L(G_i))$ implies that there exists a word v_i in $(\Sigma_{i+k,u} \setminus \Sigma_k)^*$, for $i = 1, 2$, such that $P_{i+k}(s)v_i\sigma_u$ is in $(P_i^{i+k})^{-1}(L(G_i))$. As $P_k(v_i) = \varepsilon$, $P_i(v_i) = v_i$ and we get that $P_i(s)P_i(v_i)P_i(\sigma_u)$ is in $L(G_i)$, for $i = 1, 2, k$. Consider a word u' in $\{v_1\} \parallel \{v_2\}$. Then $P_i(u') = v_i$ and, thus, $su'\sigma_u$ is in L . Moreover, u' is in $(\Sigma_u \setminus \Sigma_k)^*$. \square

It is an open problem how to verify that the projection P_{i+k} is LCC for the language L without computing the whole plant. In such a case and with the coordinator language included in the corresponding projection of the plant language, the solution computed using our coordination control architecture coincides with the global optimal solution given by the supremal controllable sublanguage of the specification.

Theorem 11. *Consider the setting of Theorem 10. If, in addition, $L(G_k)$ is a subset of $P_k(L)$ and the projection P_{i+k} is LCC for the language L , for $i = 1, 2$, then*

$$\sup C(K, L, \Sigma_u) = \sup cC(K, L, (\Sigma_{1,u}, \Sigma_{2,u}, \Sigma_{k,u})).$$

Proof. It was shown in [12, Theorem 15] that the projection P_k is an L -observer. Moreover, by Lemma 6, the projection P_k is LCC for the language L . Let $\sup C$ denote $\sup C(K, L, \Sigma_u)$. We prove that the language $P_k(\sup C)$ is controllable with respect to $L(G_k)$. Consider a word t in $P_k(\sup C)$ and an event a in $\Sigma_{k,u}$ such that the word ta is in $L(G_k)$, which is a subset of $P_k(L)$. We proved in [12, Theorem 15] that there exist words s in $\sup C$ and u in $(\Sigma \setminus \Sigma_k)^*$ such that sua is in L and $P_k(sua) = ta$. By the LCC property of the projection P_k , there exists a word u' in $(\Sigma_u \setminus \Sigma_k)^*$ such that $su'a$ is in L . By controllability of the language $\sup C$ with respect to L , the word $su'a$ is in $\sup C$, that is, $P_k(su'a) = ta$ is in $P_k(\sup C)$. Thus, (1) of Definition 3 holds. By [12, Theorem 15], the projection P_{i+k} is an L -observer, for $i = 1, 2$. To prove (2) of Definition 3, consider a word t in $P_{i+k}(\sup C)$, for $1 \leq i \leq 2$, and an event a in $\Sigma_{i+k,u}$ such that the word ta is in $L(G_i) \parallel P_k(\sup C)$. We proved in [12, Theorem 15] that there exist words s in $\sup C$ and u in $(\Sigma \setminus \Sigma_k)^*$ such that sua is in L and $P_{i+k}(sua) = ta$. As the projection P_{i+k} is LCC for the language L , there exists a word u' in $(\Sigma_u \setminus \Sigma_{1+k})^*$ such that $su'a$ is in L . Then controllability of $\sup C$ with respect to L implies that $su'a$ is in $\sup C$, that is, $P_{i+k}(su'a) = ta$ is in $P_{i+k}(\sup C)$. The other inclusion is the same as in [12, Theorem 15]. \square

Finally, a natural and simple extension to more than two local subsystems with one central coordinator is sketched. All concepts and results carry over to this general case of n subsystems, where the coordinator event set Σ_k should contain all shared

events (events common to two or more subsystems). Conditional decomposability is then simply decomposability with respect to event sets $(\Sigma_i)_{i=1}^n$ and Σ_k , cf. Section 3. It is a very good news for large systems that conditional decomposability can be checked in polynomial time with respect to the number of components as has been noticed in [11]. Note that unlike the previous form of conditional controllability, Definition 3 can be extended to the general case of n subsystems in an obvious way. Namely, conditions (2) and (3) are replaced by n conditions of the form $P_{i+k}(K)$ is controllable with respect to $L(G_i) \parallel P_k(K)$ and $\Sigma_{i+k,u}$.

Note, however, that for many large-scale systems a single central coordinator might be of little (if any) help due to too many events to be included in the coordinator event sets so that the conditions presented in this paper are satisfied (in particular, conditional decomposability, LCC, and observer conditions). It is always possible to relax some of the assumptions with the price of losing optimality, but in future publications we will rather propose multi-level coordination architectures with several layers of coordinators together with different optimality conditions corresponding to a given multi-level coordination architecture.

8 Conclusion

We have revised, simplified, and extended the coordination control scheme for discrete-event systems. These results have been used, for the case of prefix-closed languages, in the implementation of the coordination control plug-in for libFAUDES. We have identified cases, where supremal conditionally-controllable sublanguages can be computed even in the case of non-prefix-closed specification languages, and proposed coordinators for nonblockingness in addition to coordinators for safety developed in our earlier publications. Note that a general procedure for the computation of supremal conditionally-controllable sublanguages in the case of non-prefix-closed specification languages is still missing.

Another aspect that requires further investigation is the generalization of coordination control from the current case of one central coordinator to multilevel coordination control with several coordinators on different levels. In fact, one central coordinator is typically not enough in the case of large number of local subsystems, because too many events must be communicated (added into the coordinator event set) between the coordinator and local subsystems. This general architecture will be computationally more efficient, because less events need to be communicated. In the multi-level coordination control the subsystems will be organized into different groups and each group will have a coordinator meaning that only events from a given group will be communicated among subsystems of the same group via the coordinator.

A Auxiliary results

In this section, we list auxiliary results required in the paper.

Lemma 7 (Proposition 4.6, [4]). *Let L_i over Σ_i , for $i = 1, 2$, be prefix-closed languages, and let K_i be a controllable sublanguage of L_i with respect to L_i and $\Sigma_{i,u}$. Let $\Sigma =$*

$\Sigma_1 \cup \Sigma_2$. If K_1 and K_2 are synchronously nonconflicting, then $K_1 \parallel K_2$ is controllable with respect to $L_1 \parallel L_2$ and Σ_u .

Lemma 8 ([12]). Let K be a subset of a language L , and L be a subset of a language M over Σ such that K is controllable with respect to \bar{L} and Σ_u , and L is controllable with respect to \bar{M} and Σ_u . Then K is controllable with respect to \bar{M} and Σ_u .

Lemma 9 ([30]). Let $P_k : \Sigma^* \rightarrow \Sigma_k^*$ be a projection, and let L_i be a language over Σ_i , where Σ_i is a subset of Σ , for $i = 1, 2$, and $\Sigma_1 \cap \Sigma_2$ is a subset of Σ_k . Then $P_k(L_1 \parallel L_2) = P_k(L_1) \parallel P_k(L_2)$.

Lemma 10 ([12]). Let L_i be a language over Σ_i , for $i = 1, 2$, and let $P_i : (\Sigma_1 \cup \Sigma_2)^* \rightarrow \Sigma_i^*$ be a projection. Let A be a language over $\Sigma_1 \cup \Sigma_2$ such that $P_1(A)$ is a subset of L_1 and $P_2(A)$ is a subset of L_2 . Then A is a subset of $L_1 \parallel L_2$.

Lemma 11 ([19]). Let L_i be a language over Σ_i , for $i \in J$, and let $\bigcup_{k \neq \ell \in J} (\Sigma_k \cap \Sigma_\ell) \subseteq \Sigma_0$. If $P_{i,0} : \Sigma_i^* \rightarrow (\Sigma_i \cap \Sigma_0)^*$ is an L_i -observer, for $i \in J$, then $\overline{\|_{i \in J} L_i} = \overline{\|_{i \in J} \bar{L}_i}$ if and only if $\overline{\|_{i \in J} P_{i,0}(L_i)} = \overline{\|_{i \in J} P_{i,0}(\bar{L}_i)}$.

Lemma 12 ([9]). A language $K \subseteq (\Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_n)^*$ is conditionally decomposable with respect to event sets $\Sigma_1, \Sigma_2, \dots, \Sigma_n, \Sigma_k$ if and only if there exist languages $M_{i+k} \subseteq \Sigma_{i+k}^*$, $i = 1, 2, \dots, n$, such that $K = \overline{\|_{i=1}^n M_{i+k}}$.

Acknowledgements. Research of two first authors was supported by RVO: 67985840. In addition, the first author was supported by the Grant Agency of the Czech Republic under grant P103/11/0517 and the second author by the Grant Agency of the Czech Republic under grant P202/11/P028.

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