Succinctness and Tractability of Closure Operator Representations

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Abstract

It is widely known that closure operators on finite sets can be represented by sets of implications (also known as inclusion dependencies) as well as by formal contexts. In this article, we consider these two representation types, as well as generalizations of them: extended implication sets and context families. We discuss the mutual succinctness of these four representations and the tractability of certain operations used to compare and modify closure operators.

1. Introduction

Closure operators and closure systems are a basic notion in algebra and occur in various computer science scenarios such as logic programming or databases. One central task when dealing with closure operators algorithmically is to represent them in a succinct way while still allowing for their efficient computational usage. Formal concept analysis (FCA) naturally provides two complementary ways of representing closure operators: by means of *formal contexts* on one side and *implication sets* on the other. Although being complementary, these two representations share the property that they allow for tractable closure computation. In fact, this property is also exhibited by further representation types, which properly generalize the ones mentioned above: *context families* consist of several contexts and the closure is specified as the "simultaneous fixpoint" of all the separate contexts' closures; *extended implications* are implications where auxiliary elements are allowed.

For a given closure operator, the space needed to represent it in one or the other way may differ significantly: it is well known that there are closure operators whose minimal implicational representation is exponentially larger than their minimal contextual one and vice versa (see Section 3).

Thus, when designing algorithms which use and manipulate closure operators (as many FCA algorithms do) it is important to know which of the possible representation types allow for efficient storage and still guarantee fast (that is: PTime) execution of typical computations.

This paper investigates the four representation types in this respect. To this end, we will consolidate known results from diverse areas into one framework and provide some findings which are – to the best of our knowledge – novel and original to fill the remaining gaps. Our main results can be generalized as follows:

- We show that context families allow for succinct representation of both contexts and implications, and that extended implication sets can succinctly represent all the three other representation types. We also show that a succinct translation (i.e., one where the size of the result is polynomially bounded by the input) in all other directions is not possible.

- We clarify the complexities for comparing closure operators in different representations in terms of whether one is a refinement of the other. Interestingly, some of the investigated comparison tasks are tractable (i.e., time-polynomial), others are not (assuming P ≠ NP). We provide algorithms for the tractable cases and coNP-hardness arguments for the others.
- We go through standard manipulation tasks for closure operators (refinement by adding a closed set, coarsening through an implication, projection, meet and join in the lattice of closure operators) and clarify which are tractable and which are not.

This paper is a significantly refined and extended version of two precursor publications (Rudolph, 2012, 2014). All statements for which proofs are given are original to the best of our knowledge, unless explicitly stated otherwise.

2. Preliminaries

We start providing a condensed overview of the notions used in this paper. After recalling some complexity notations, we introduce closure operators as well as the four representation types we want to discuss in this article: (formal) contexts, context families, implication sets and extended implication sets.

2.1. Complexity Notations

In order to asymptotically compare sizes of data structures, we will make use of the Bachmann-Landau notation. In particular, we remind the reader that for two infinite sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, we write

- $a_n \in \Omega(b_n)$ if b_n is an asymptotic lower bound of a_n, i.e., there exists some k with $a_n \geq k \cdot b_n$ for sufficiently large n, and
- $a_n \in \Theta(b_n)$ if b_n is an asymptotic lower and upper bound of a_n, i.e., there exist some k_1 and k_2 with $k_1 \cdot b_n > a_n > k_2 \cdot b_n$ for sufficiently large n.

2.2. Closure Operators

We now introduce and formally define the central notion of this paper: closure operators.

Definition 1. *Let M be an arbitrary set. A function $\varphi : 2^M \to 2^M$ is called a* closure operator *on M if it is*

1. extensive*, i.e., $A \subseteq \varphi(A)$ for all $A \subseteq M$,*
2. monotone*, i.e., $A \subseteq B$ implies $\varphi(A) \subseteq \varphi(B)$ for all $A, B \subseteq M$, and*
3. idempotent*, i.e., $\varphi(\varphi(A)) = \varphi(A)$ for all $A \subseteq M$.*

A set $A \subseteq M$ is called closed *(or φ-*closed *in case of ambiguity), if $\varphi(A) = A$. The set of all closed sets $\{A \mid A = \varphi(A) \subseteq M\}$ is called* closure system *of φ.*

It is easy to show that for an arbitrary closure system $\mathcal{S}$, the corresponding closure operator φ can be reconstructed by

$$\varphi(A) = \bigcap_{B\in\mathcal{S},\ A\subseteq B} B.$$

Hence, there is a one-to-one correspondence between a closure operator and the according closure system.
In the following, we provide some closure operators which will serve as running examples in the course of the paper.

Example 2. *Considering $M = \{a, b, c, d, e\}$, the functions α, β, γ, and δ defined in the below table are all closure operators (due to extensivity, every closure operator φ satisfies $A \subseteq \varphi(A)$, thus for better readability, we underline elements of $\varphi(A) \setminus A$).*

A	$\alpha(A)$	$\beta(A)$	$\gamma(A)$	$\delta(A)$
$\{\}$	$\{\underline{a}\}$	$\{\}$	$\{\underline{a}\}$	$\{\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$\{b\}$	$\{\underline{a}, b, \underline{c}\}$	$\{b\}$	$\{\underline{a}, b, \underline{c}, \underline{e}\}$	$\{b\}$
$\{a, b\}$	$\{a, b, \underline{c}\}$	$\{a, b\}$	$\{a, b, \underline{c}, \underline{e}\}$	$\{a, b\}$
$\{c\}$	$\{\underline{a}, c\}$	$\{c\}$	$\{\underline{a}, c, \underline{e}\}$	$\{c\}$
$\{a, c\}$	$\{a, c\}$	$\{a, c, \underline{e}\}$	$\{a, c, \underline{e}\}$	$\{a, c\}$
$\{b, c\}$	$\{\underline{a}, b, c\}$	$\{b, c, \underline{e}\}$	$\{\underline{a}, b, c, \underline{e}\}$	$\{b, c\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c, \underline{e}\}$	$\{a, b, c, \underline{e}\}$	$\{a, b, c\}$
$\{d\}$	$\{\underline{a}, d\}$	$\{d\}$	$\{\underline{a}, \underline{c}, d, \underline{e}\}$	$\{d\}$
$\{a, d\}$	$\{a, d\}$	$\{a, d, \underline{e}\}$	$\{a, \underline{c}, d, \underline{e}\}$	$\{a, d\}$
$\{b, d\}$	$\{\underline{a}, b, \underline{c}, d, \underline{e}\}$	$\{b, d, \underline{e}\}$	$\{\underline{a}, b, \underline{c}, d, \underline{e}\}$	$\{b, d, \underline{e}\}$
$\{a, b, d\}$	$\{a, b, \underline{c}, d, \underline{e}\}$	$\{a, b, d, \underline{e}\}$	$\{a, b, \underline{c}, d, \underline{e}\}$	$\{a, b, d, \underline{e}\}$
$\{c, d\}$	$\{\underline{a}, c, d, \underline{e}\}$	$\{c, d\}$	$\{\underline{a}, c, d, \underline{e}\}$	$\{c, d\}$
$\{a, c, d\}$	$\{a, c, d, \underline{e}\}$	$\{a, c, d, \underline{e}\}$	$\{a, c, d, \underline{e}\}$	$\{a, c, d, \underline{e}\}$
$\{b, c, d\}$	$\{\underline{a}, b, c, d, \underline{e}\}$	$\{b, c, d, \underline{e}\}$	$\{\underline{a}, b, c, d, \underline{e}\}$	$\{b, c, d, \underline{e}\}$
$\{a, b, c, d\}$	$\{a, b, c, d, \underline{e}\}$	$\{a, b, c, d, \underline{e}\}$	$\{a, b, c, d, \underline{e}\}$	$\{a, b, c, d, \underline{e}\}$
$\{e\}$	$\{\underline{a}, e\}$	$\{e\}$	$\{\underline{a}, e\}$	$\{e\}$
$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$
$\{b, e\}$	$\{\underline{a}, b, \underline{c}, e\}$	$\{b, e\}$	$\{\underline{a}, b, \underline{c}, e\}$	$\{b, e\}$
$\{a, b, e\}$	$\{a, b, \underline{c}, e\}$	$\{a, b, e\}$	$\{a, b, \underline{c}, e\}$	$\{a, b, e\}$
$\{c, e\}$	$\{\underline{a}, c, e\}$	$\{c, e\}$	$\{\underline{a}, c, e\}$	$\{c, e\}$
$\{a, c, e\}$	$\{a, c, e\}$	$\{a, c, e\}$	$\{a, c, e\}$	$\{a, c, e\}$
$\{b, c, e\}$	$\{\underline{a}, b, c, e\}$	$\{b, c, e\}$	$\{\underline{a}, b, c, e\}$	$\{b, c, e\}$
$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$
$\{d, e\}$	$\{\underline{a}, \underline{c}, d, e\}$	$\{d, e\}$	$\{\underline{a}, \underline{c}, d, e\}$	$\{d, e\}$
$\{a, d, e\}$	$\{a, \underline{c}, d, e\}$	$\{a, d, e\}$	$\{a, \underline{c}, d, e\}$	$\{a, d, e\}$
$\{b, d, e\}$	$\{\underline{a}, b, \underline{c}, d, e\}$	$\{b, d, e\}$	$\{\underline{a}, b, \underline{c}, d, e\}$	$\{b, d, e\}$
$\{a, b, d, e\}$	$\{a, b, \underline{c}, d, e\}$	$\{a, b, d, e\}$	$\{a, b, \underline{c}, d, e\}$	$\{a, b, d, e\}$
$\{c, d, e\}$	$\{\underline{a}, c, d, e\}$	$\{c, d, e\}$	$\{\underline{a}, c, d, e\}$	$\{c, d, e\}$
$\{a, c, d, e\}$	$\{a, c, d, e\}$	$\{a, c, d, e\}$	$\{a, c, d, e\}$	$\{a, c, d, e\}$
$\{b, c, d, e\}$	$\{\underline{a}, b, c, d, e\}$	$\{b, c, d, e\}$	$\{\underline{a}, b, c, d, e\}$	$\{b, c, d, e\}$
$\{a, b, c, d, e\}$	$\{a, b, c, d, e\}$	$\{a, b, c, d, e\}$	$\{a, b, c, d, e\}$	$\{a, b, c, d, e\}$

Closure operators can be compared with respect to the inclusion of their respective closed sets.

Definition 3. *Given two closure operators φ and ψ on M, φ is called* finer than *ψ (written $\varphi \preceq \psi$, alternatively we also say ψ is* coarser than *φ) if every ψ-closed set is also φ-closed. We call φ and ψ* equivalent *(written $\varphi \equiv \psi$), if both $\varphi \preceq \psi$ and $\psi \preceq \varphi$.*

Note that, as a straightforward consequence from this definition, $\varphi \preceq \psi$ holds exactly if $\varphi(A) \subseteq \psi(A)$ for all $A \subseteq M$. Moreover, $\varphi \equiv \psi$ holds if and only if $\varphi(A) = \psi(A)$ for all $A \subseteq M$.

Example 4. *For the closure operators defined in Example 2, we observe $\delta \preceq \alpha$ and $\beta \preceq \gamma$.*

It is well-known that the set of all closure operators together with the "finer than" relation constitutes a complete lattice (Caspard and Monjardet (2003) give an excellent and comprehensive treatise about this subject). The lattice operations can be defined as follows: $\varphi \wedge \psi$ is the closure operator mapping any $A \subseteq M$ to $\varphi(A) \cap \psi(A)$ whereas $\varphi \vee \psi$ is the closure operator that maps $A \subseteq M$ to the smallest superset of A that is closed under φ and ψ (which, for finite sets, can be obtained by alternatingly applying φ and ψ to A until a fixpoint is reached). The finest closure operator is the identity function mapping every set to itself. The coarsest closure operator maps every input set to M.

Example 5. *For the closure operators defined in Example 2 we obtain $\gamma \equiv \alpha \vee \beta$ and $\delta \equiv \alpha \wedge \beta$.*

To this date, the precise numbers of closure operators on finite sets are known up to $|M| = 7$:

$\lvert M\rvert$	number of closure operators on M	reference
1	2	
2	7	
3	61	
4	2,480	
5	1,385,552	Higuchi, 1998
6	75,973,751,474	Habib and Nourine, 2005
7	14,087,648,235,707,352,472	Colomb et al., 2010

Moreover, general lower and upper bounds have been determined (Burosch et al., 1991), according to which the number of closure operators an an n-element set is between $2^{\binom{n}{\lfloor n/2 \rfloor}}$ and $2^{2\sqrt{2}\binom{n}{\lfloor n/2 \rfloor}(1+o(1))}$. Thereby, the lower bound can be exploited to obtain a first negative result regarding succinct representability of closure operators in general.

Proposition 6. *There is no uniform representation of closure operators that requires at most polynomial space w.r.t. $|M|$.*

Proof. Suppose the contrary, i.e., that there exists some fixed k such that every closure operator on M can be expressed by a word of length $|M|^k$ over some alphabet Σ of bounded size, say $|\Sigma| = \ell$. Obviously, there are $\ell^{(|M|^k)}$ such words in total. For sufficiently large M, we then obtain

$$\ell^{(|M|^k)} = 2^{\log_2(\ell)(|M|^k)} < 2^{(2^{\lfloor |M|/2 \rfloor})} < 2^{\binom{|M|}{\lfloor |M|/2 \rfloor}}.$$

Thus, there are fewer words of the required length than there are distinct closure operators. $\square$

Finally, we introduce the notion of a projection of a closure operator.

Definition 7. *Given a closure operator φ on a set M and some set $N \subseteq M$, the* projection of φ to *N, written $\varphi|_N$ is a closure operator on N with $\varphi|_N(A) = \varphi(A) \cap N$ for all $A \subseteq N$.*

Next we introduce four ways of representing closure operators. Thereby and in what follows, we will restrict our considerations to closure operators over finite sets, which is a reasonable assumption when investigating succinctness and complexity properties.

2.3. Contexts

Following the normal line of argumentation of FCA (Ganter and Wille, 1997), we use formal contexts as data structure to encode closure operators.

Definition 8. *A* formal context $\mathbb{K}$ *is a triple* (G, M, I) *with some set* G *called* objects*, some set* M *called* attributes*, and a relation* $I \subseteq G \times M$ *called* incidence relation*. The* size *of* $\mathbb{K}$ *(written:* $\#\mathbb{K}$*) is defined as* $|G| \cdot |M|$*, i.e., as the number of bits required to store* I*.*

For the sake of brevity, we will sometimes just write context instead of formal context. Contexts are often conveniently represented by means of crosstables.

Example 9. *The following crosstable represents a formal context* $\mathbb{K} = (G, M, I)$ *with object set* $G = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ *and attribute set* $M = \{a, b, c, d, e\}$*.*

	a	b	c	d	e
g_1	×		×		
g_2	×	×	×		
g_3	×			×	
g_4	×				×
g_5	×		×		×
g_6	×	×	×		×
g_7	×		×	×	×

This basic data structure can then be used to define operations on sets of objects or attributes, respectively.

Definition 10. *Let* $\mathbb{K} = (G, M, I)$ *be a formal context. We define a function* $(\cdot)^I : 2^G \to 2^M$ *with* $A^I := \{m \mid gIm \text{ for all } g \in A\}$ *for* $A \subseteq G$*. Furthermore, we use the same notation to define the function* $(\cdot)^I : 2^M \to 2^G$ *where* $B^I := \{g \mid gIm \text{ for all } m \in B\}$ *for* $B \subseteq M$*. For convenience, we sometimes write* g^I *instead of* $\{g\}^I$ *and* m^I *instead of* $\{m\}^I$*.*

Applied to an object set, this function yields all attributes common to these objects; by applying it to an attribute set we get the set of all objects having those attributes. The following facts are consequences of the above definitions:

- $(\cdot)^{II}$ is a closure operator on G as well as on M.
- For $A \subseteq G$, A^I is a $(\cdot)^{II}$-closed set and
- for $B \subseteq M$, B^I is a $(\cdot)^{II}$-closed set.

Example 11. *Looking at the formal context from Example 9, we find for instance* $\{b\}^{II} = \{g_2, g_6\}^I = \{a, b, c\}$ *and* $\{b, d\}^{II} = \emptyset^I = \{a, b, c, d, e\}$*. Indeed,* $(\cdot)^{II}$ *coincides with* α *from Example 2.*

In the following, we will focus only on the closure operator on attribute sets and exploit the fact that this closure operator is independent from the concrete object set G; it suffices to know the set of the context's so-called "object intents", i.e., the sets of attributes associated to each object. Thus, we will directly use intent sets, that is: families $\mathcal{F}$ of subsets of M, to represent formal contexts.

Example 12. *Obviously, the context from Example 9 can be represented by the family* $\mathcal{F} = \{F_1, \ldots, F_7\}$ *with* $F_1 = \{a, c\}$*,* $F_2 = \{a, b, c\}$*,* $F_3 = \{a, d\}$*,* $F_4 = \{a, e\}$*,* $F_5 = \{a, c, e\}$*,* $F_6 = \{a, b, c, e\}$ *and* $F_7 = \{a, c, d, e\}$*.*

Definition 13. *Given a family $\mathcal{F} \subseteq 2^M$, we let $\mathbb{K}(\mathcal{F})$ denote the formal context (G, M, I) with $G = \mathcal{F}$ and, for an $A \in \mathcal{F}$, we let AIm exactly if $m \in A$. Given $B \subseteq M$, we use the notation $B^{\mathcal{F}}$ to denote the attribute closure B^{II} in $\mathbb{K}(\mathcal{F})$ and let $\#\mathcal{F} = \#\mathbb{K}(\mathcal{F}) = |\mathcal{F}| \cdot |M|$.*

For the sake of simplicity we will from now on refer to $\mathcal{F}$ as contexts (on M). It is easy to see that $(\cdot)^{\mathcal{F}}$ can be alternatively defined via $A^{\mathcal{F}} = \bigcap_{A \subseteq B \in \mathcal{F}} B$ for every $A \subseteq M$. This gives rise to the following algorithm and complexity result.

Algorithm 1 `closureContext`

Input: context $\mathcal{F}$ on M, set $A \subseteq M$
Output: $C = A^{\mathcal{F}}$

```
C := M
for each B ∈ F do
    if A ⊆ B then
        C := C ∩ B
    end if
end for
output C
```

Proposition 14. *For any context $\mathcal{F}$ on a set M and any set $A \subseteq M$, the closure $A^{\mathcal{F}}$ can be computed in $O(\#\mathcal{F}) = O(|\mathcal{F}| \cdot |M|)$ time.*

Given an arbitrary context $\mathcal{F}$ representing some closure operator φ on some set M, the question whether there exists another $\mathcal{F}'$ representing φ and satisfying $\#\mathcal{F}' < \#\mathcal{F}$ – and if so, how to compute it – is straightforwardly solved by noting that this coincides with the question if $\mathbb{K}(\mathcal{F})$ is row-reduced (Ganter and Wille, 1997) and how to row-reduce it. Algorithm 2 displays the according method cast in our representation via set families which directly allows to establish the subsequent complexity result.

Algorithm 2 `minimizeContext`

Input: context $\mathcal{F}$ on M
Output: size-minimal context $\mathcal{F}'$ such that $(\cdot)^{\mathcal{F}} \equiv (\cdot)^{\mathcal{F}'}$

```
F' := F
for each A ∈ F' do
    if A = A^(F' \ {A}) then
        F' := F' \ {A}
    end if
end for
output F'
```

Proposition 15. *Given a context $\mathcal{F}$ on M, a size-minimal context $\mathcal{F}'$ with $(\cdot)^{\mathcal{F}} \equiv (\cdot)^{\mathcal{F}'}$ can be computed in $O(|\mathcal{F}|^2 \cdot |M|) = O(|\mathcal{F}| \cdot \#\mathcal{F}) \leq O(\#\mathcal{F}^2)$ time.*

We note that for a given closure operator φ, the minimal $\mathcal{F}$ with $\varphi \equiv (\cdot)^{\mathcal{F}}$ is uniquely determined. It consists of all those φ-closed sets that cannot be obtained by intersecting other φ-closed sets. We will denote this minimal context by $\mathcal{F}(\varphi)$.

Example 16. *Considering the context from Example 9, we see that the set $\{a, c, e\}$ (associated with g_5) can be represented as the intersection of the sets $\{a, b, c, e\}$ (associated with g_6) and $\{a, c, d, e\}$ (associated with g_7) and therefore $\{a, c, e\}^{\mathcal{F}\setminus\{a,c,e\}} = \bigcap_{\{a,c,e\}\subseteq B\in\mathcal{F}} B = \{a, b, c, e\} \cap \{a, c, d, e\} = \{a, c, e\}$. Thus $\{a, c, e\}$ will be removed from $\mathcal{F}$ by the minimization algorithm.*

As a final observation regarding contexts, we note that the closure operator associated to a context coincides with the infimum of all the "one-line contexts" it is composed of, in the lattice of closure operators.

Proposition 17. *Let $\mathcal{F}$ be a context on a set M. Then $(\cdot)^{\mathcal{F}} \equiv \bigwedge_{A\in\mathcal{F}}(\cdot)^{\{A\}}$.*

Proof. Let $\varphi = \bigwedge_{A\in\mathcal{F}}(\cdot)^{\{A\}}$. Then for any $B \subseteq M$, the following holds: $\varphi(B) = \bigcap_{A\in\mathcal{F}} B^{\{A\}} = \bigcap_{B\subseteq A\in\mathcal{F}} A = B^{\mathcal{F}}$. □

2.4. Context Families

The notion of contexts can be extended to that of context families.

Definition 18. *A* context family *on a set M is a finite set $\mathfrak{F} = \{\mathcal{F}_1, \dots \mathcal{F}_n\}$ of formal contexts on M. The size of $\mathfrak{F}$ (written: $\#\mathfrak{F}$) is defined as $\sum_{i=1}^{n} \#\mathcal{F}_i$.*
The closure operator $(\cdot)^{\mathfrak{F}}$ associated with $\mathfrak{F}$ is defined via its closed sets: A is $(\cdot)^{\mathfrak{F}}$-closed if it is $(\cdot)^{\mathcal{F}}$-closed for every $\mathcal{F} \in \{\mathcal{F}_1, \dots \mathcal{F}_n\}$.

Note that the provided definition of $(\cdot)^{\mathfrak{F}}$ can be equivalently expressed by $(\cdot)^{\mathfrak{F}} := \bigvee_{\mathcal{F}\in\mathfrak{F}}(\cdot)^{\mathcal{F}}$ using the join operation $\bigvee$ in the lattice of closure operators.
This type of data structure has been investigated in an area of computer science called model-based reasoning (Eiter et al., 1998) and, as we will see, it is more succinct than plain contexts. On the other hand, this improvement seems to come at a prize; the obvious upper bound for closure computation is higher than for plain contexts, although still polynomial:

Proposition 19. *For any context family $\mathfrak{F}$ on a set M and any set $A \subseteq M$, the closure $A^{\mathfrak{F}}$ can be computed in $O(|\mathfrak{F}| \cdot |M|^2) = O(\#\mathfrak{F} \cdot |M|) \leq O(\#\mathfrak{F}^2)$ time.*

Proof. According to the definition, $A^{\mathfrak{F}}$ is the smallest simultaneous fixpoint of $(\cdot)^{\mathcal{F}_1}, \dots, (\cdot)^{\mathcal{F}_n}$ that contains A. Thanks to monotonicity and finiteness of M, such a fixpoint can be obtained by $|M|$-fold application of $(\cdot)^{\mathcal{F}_1\cdots\mathcal{F}_n}$ (that is, the composition of the closure operators $(\cdot)^{\mathcal{F}_1} \dots (\cdot)^{\mathcal{F}_n}$) to A. Exploiting Proposition 14, a one-fold application requires $\sum_{i=1}^{n} O(\#\mathcal{F}) = O(\sum_{i=1}^{n} \#\mathcal{F}) = O(\#\mathfrak{F})$ time, which leads to the above result for $|M|$-fold application. □

This finding can be seen as a special case of a more general result (Eiter et al., 1998), according to which checking if a propositional formula in CNF with m conjuncts is entailed by the Horn theory represented by a context family is feasible in $O(m \cdot \#\mathfrak{F})$ time.

2.5. Implication Sets

Given a set of attributes, *implications* on that set are logical expressions that can be used to describe certain attribute correspondences which are valid for all objects in a formal context.

Definition 20. *Let M be an arbitrary set. An* implication *on M is a pair (A, B) with $A, B \subseteq M$. To support intuition we write $A \to B$ instead of (A, B). We say an implication $A \to B$* holds *for an attribute set C (also: C* respects *$A \to B$), if $A \nsubseteq C$ or $B \subseteq C$. Moreover, an implication* ἰ holds *(or: is* valid*) in a formal context $\mathcal{F}$ if it holds for all sets $C \in \mathcal{F}$. We then write $\mathcal{F} \models$ ἰ.*

The size of an implication set $\Im$ (written: $\#\Im$) is defined as $|\Im| \cdot |M|$. Given a set $A \subseteq M$ and a set $\Im$ of implications on M, we write $A^{\Im}$ for the smallest set that contains A and respects all implications from $\Im$. (Since those two requirements are preserved under intersection, the existence of a smallest such set is assured).

Note that the above definition implies that in an implication $A \to B$, both A and B are interpreted as conjunctions. In particular, any implication with $B = \emptyset$ is vacuously true.
It can be easily shown that an implication $A \to B$ is valid in a context $\mathcal{F}$ exactly if $B \subseteq A^{\mathcal{F}}$. Furthermore it is obvious that for any set $\Im$ of implications on M, the operation $(\cdot)^{\Im}$ is a closure operator on M.

Example 21. *The closure operator α from Example 2 coincides with the closure operator corresponding to the implication set consisting of the implications $\emptyset \to \{a\}$, $\{b\} \to \{c\}$, $\{b, d\} \to \{e\}$, $\{c, d\} \to \{e\}$, and $\{d, e\} \to \{c\}$*

The naïve algorithm for computing $A^{\Im}$ for some set $A \subseteq M$ and implication set $\Im$ requires to pass through the implication set $|\Im|$ times in the worst case, giving rise to a runtime of $O(|\Im|^2 \cdot |M|)$. However, one can do better as shown by Maier (1983) as well as Dowling & Gallier (1984), who provided Algorithm 3, also known as LINCLOSURE, with a better time complexity.

Algorithm 3 `closureImpSet`

Input: implication set $\Im$ on M, set $A \subseteq M$
Output: $D = A^{\Im}$

for each $B \to C \in \Im$ **do**
  $count_{B \to C} := |B|$
  for each $m \in B$ **do**
    $L_m := L_m \cup \{B \to C\}$
  end for
end for
$D := A$
$E := A$
while $E \neq \emptyset$ **do**
  pick $m \in D$
  $E := E \setminus \{m\}$
  for each $B \to C \in L_m$ **do**
    $count_{B \to C} := count_{B \to C} - 1$
    if $count_{B \to C} = 0$ **then**
      $E := E \cup (C \setminus D)$
      $D := D \cup C$
    end if
  end for
end while
output D

Proposition 22 (Maier 1983, Dowling & Gallier 1984)**.** *For any attribute set $B \subseteq M$ and set $\Im$ of implications, $B^{\Im}$ can be computed in $O(\#\Im) = O(|\Im| \cdot |M|)$ time.*

Like in the case of the contextual encoding, also here it is natural to ask for a size-minimal set of implications that corresponds to a certain closure operator.
Although there is in general no unique minimal implication set for a given closure operator φ, the so-called Duquenne-Guigues base or stem base (Guigues and Duquenne, 1986) is often used as a (minimal) canonical representation.

Definition 23. *Given a closure operator φ on a finite set M, a set $A \subseteq M$ is called* pseudoclosed *if $\varphi(A) \neq A$ and for every pseudoclosed set B strictly contained in A (i.e., $B \subset A$) holds $\varphi(A) \subseteq B$. The* Duquenne-Guigues base *or* stem base *of φ, denoted by $\Im(\varphi)$ is the implication set*

$$\{P \to \varphi(P) \mid P \text{ pseudoclosed with respect to } \varphi\}.$$

It is worth mentioning that the definition of pseudoclosed sets, although recursive, is well since we assume finiteness of M.

Example 24. *The Duquenne-Guigues base of the closure operator α from Example 2 contains the implications $\emptyset \to \{a\}$, $\{a,b\} \to \{a,b,c\}$, $\{a,c,d\} \to \{a,c,d,e\}$, and $\{a,d,e\} \to \{a,c,d,e\}$. Note that the cardinality (and hence the size) of this implication set is smaller than that of the one given in Example 21.*

It has been shown that $(\cdot)^{\Im(\varphi)} \equiv \varphi$ and that every implication set $\Im$ satisfying $(\cdot)^{\Im} \equiv \varphi$ has equal or greater cardinality than $\Im(\varphi)$.
Algorithm 4 provides a well-known way to turn an arbitrary implication set into an equivalent Duquenne-Guigues base (Day, 1992; Wild, 1991; Rudolph, 2007), giving rise to the subsequent complexity result.

Algorithm 4 `minimizeImpSet`

Input: implication set $\Im$ on M
Output: size-minimal implication set $\Im'$
  such that $(\cdot)^{\Im} \equiv (\cdot)^{\Im'}$

```
1:  $\widetilde{\Im} := \emptyset$
2:  for each $A \to B \in \Im$ do
3:      $\widetilde{\Im} := \widetilde{\Im} \cup \{A \to (A \cup B)^{\Im}\}$
4:  end for
5:  $\Im' := \emptyset$
6:  for each $A \to B \in \widetilde{\Im}$ do
7:      delete $A \to B$ from $\widetilde{\Im}$
8:      $C := A^{\widetilde{\Im} \cup \Im'}$
9:      if $C \neq B$ then
10:         $\Im' := \Im' \cup \{C \to B\}$
11:     end if
12: end for
13: output $\Im'$
```

Proposition 25 (Day 1992)**.** *Given a set $\Im$ of implications on M, a size-minimal $\Im'$ with $(\cdot)^{\Im} \equiv (\cdot)^{\Im'}$ can be computed in $O(|\Im|^2 \cdot |M|) = O(|\Im| \cdot \#\Im) \leq O(\#\Im^2)$ time.*

The algorithm performs a 2-pass processing of the implication set. Note that both passes can be performed *in situ* (i.e., by overwriting the input with the output) which would require only $O(|M|)$ additional memory.
It is worth noting that the existence of a polynomial minimization algorithm presented here hinges on the way how we defined implication sets and their size. If we, for instance, allow only implications with one-element conclusion (which correspond to propositional Horn clauses), minimization cannot be carried out in polynomial time (a comprehensive treatise on the problem being given by Boros et al. (2013)), no matter if the size is defined depending on the number of implications (Ausiello et al., 1986) or on the number of accumulated element occurrences (Maier, 1980).
As a final observation regarding implications, we note that the closure operator associated to an implication set coincides with the supremum of all the closure operators corresponding to the "single-implication subsets" of that implication set, in the lattice of closure operators.

Proposition 26. *Let $\mathfrak{I}$ be an implication set on a set M. Then* $(\cdot)^{\mathfrak{I}} \equiv \bigvee_{\mathfrak{i}\in\mathfrak{I}}(\cdot)^{\{\mathfrak{i}\}}$.

Proof. Let $\varphi = \bigvee_{\mathfrak{i}\in\mathfrak{I}}(\cdot)^{\{\mathfrak{i}\}}$. Then some $B \subseteq M$ is φ-closed iff it is $(\cdot)^{\{\mathfrak{i}\}}$-closed for every $\mathfrak{i} \in \mathfrak{I}$. This is the case iff B respects all implications $\mathfrak{i} \in \mathfrak{I}$, i.e., B is $(\cdot)^{\mathfrak{I}}$-closed. □

2.6. Extended Implication Sets

We will now slightly generalize the notion of implications by allowing for "auxiliary elements" that do not belong to M.

Definition 27. *An* extended implication set *on M is an implication set over some set $N \supseteq M$ (referred to as* total attribute set*) where the elements of $N \setminus M$ are called* auxiliary elements. *The size of an extended implication set $\mathcal{K}$ (written: $\#\mathcal{K}$) is defined as $|\mathcal{K}| \cdot |N|$. Given an extended implication set $\mathcal{K}$ over M, we associate with it the closure operator $(\cdot)^{\mathcal{K}}|_M$.*

Example 28. *The closure operator β from Example 2 can be represented by the following extended implication using auxiliary attributes f and g:*

$$a \to f \qquad b \to f \qquad c \to g \qquad d \to g \qquad f, g \to e.$$

We will see later that allowing for auxiliary elements enables a more succinct representation of closure operators. The complexities for closure computation follow directly from those for plain implication sets.

3. Mutual Succinctness

Given the four encodings of closure operators introduced above, a question which arises naturally is whether one encoding is superior to the other in terms of memory required to store it. First of all, note that for a given M, we will find a representation of any of the four types whose size is bounded by $2^{|M|} \cdot |M|$, i.e., at most exponential in the size of M.
The following proposition shows that for some φ, $\#\mathcal{F}(\varphi)$ is exponentially larger than $\#\mathfrak{I}(\varphi)$.

Proposition 29. *There exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of closure operators such that $\#\mathcal{F}(\varphi_n) \in \Theta(n\cdot 2^n)$ whereas $\#\mathfrak{I}(\varphi_n) \in \Theta(n^2)$.*

Proof. We define φ_n as the closure operator on the set $M_n = \{1, \ldots, 2n\}$ that maps a set $A \subseteq M$ to M in case A contains both an odd number and its successor, and otherwise just maps A to A. Clearly, φ_n can be represented by the implication set $\mathfrak{I}_b$ containing the implication $\{2i-1, 2i\} \to M_n$ for every $i \in \{1, \ldots, n\}$. Then, we obtain $\#\mathfrak{I}(\varphi_n) = 2n^2$. On the other hand, $\mathcal{F}(\varphi_n) = \{\{2k - a_k | 1 \leq k \leq n\} \mid \langle a_1, \ldots, a_n\rangle \in \{0,1\}^n\}$, as schematically displayed in the following crosstable:

	1	2	...	$2n{-}3$	$2n{-}2$	$2n{-}1$	$2n$
g_1	×		...	×		×	
g_2	×		...	×			×
g_3	×		...		×	×	
g_4	×		...		×		×
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
g_{2^n-1}		×	...		×	×	
g_{2^n}		×	...		×		×

Therefore, we obtain $\#\mathcal{F}(\varphi_n) = 2^n \cdot 2n$. □

This shows that plain contexts cannot succinctly (that is: with only polynomial increase in size) represent closure operators defined via implication sets. However, we will next show that this can be achieved by context families. To this end, we first define the notion of a one-implication-context.

Definition 30. *For an implication* $\mathfrak{i} = A \to B$ *on some set* M*, the* one-implication-context $\mathcal{F}_\mathfrak{i}$ *is defined by* $\mathcal{F}_\mathfrak{i} = \{M \setminus \{m\} \mid m \in (M \setminus B) \cup A\} \cup \{M \setminus \{m, m'\} \mid m \in B \setminus A, m' \in A\}$.

It is not hard to verify that $\mathcal{F}_\mathfrak{i}$ is the unique context which is reduced, in which $\mathfrak{i}$ holds and that satisfies that every other implication holding therein is a logical consequence of $\mathfrak{i}$ (in other words, the closure operators $(\cdot)^{\mathcal{F}_\mathfrak{i}}$ and $(\cdot)^{\{\mathfrak{i}\}}$ coincide). In other words, whenever $B \setminus A$ is nonempty, the stem base of $\mathcal{F}_{A\to B}$ will contain exactly the implication $A \to A \cup B$. We omit a proof here as this is a special case of Proposition 58 presented later. Furthermore, we obtain $\#\mathcal{F}_\mathfrak{i} < |M|^3$. Now, given an implication set, we can obtain a context family by putting together all the corresponding one-implication-contexts:

Definition 31. *For an implication set* $\mathfrak{I} = \{\mathfrak{i}_1, \ldots, \mathfrak{i}_n\}$ *on some set* M*, the associated context family* $\mathfrak{F}(\mathfrak{I})$ *is defined by* $\mathfrak{F}(\mathfrak{I}) = \{\mathcal{F}_{\mathfrak{i}_1}, \ldots, \mathcal{F}_{\mathfrak{i}_n}\}$.

Proposition 32. *For any implication set* $\mathfrak{I}$ *on some set* M *holds* $(\cdot)^{\mathfrak{I}} \equiv (\cdot)^{\mathfrak{F}(\mathfrak{I})}$*. Moreover,* $\#\mathfrak{F}(\mathfrak{I}) < \#\mathfrak{I} \cdot |M|^2$.

Proof. We find $(\cdot)^{\mathfrak{I}} \equiv \bigvee_{\mathfrak{i}\in\mathfrak{I}} (\cdot)^{\{\mathfrak{i}\}} \equiv \bigvee_{\mathfrak{i}\in\mathfrak{I}} (\cdot)^{\mathcal{F}_\mathfrak{i}} \equiv (\cdot)^{\mathfrak{F}(\mathfrak{I})}$. The given bound is straightforward. □

This shows that for every implication set, there exists a context family that is only polynomially larger and represents the same closure operator.

We now turn our attention to the other direction, asking if implications allow for a succinct representation of contextually specified closure operators. It is known that this is not the case: as a consequence of a result on the number of pseudo-intents (Kuznetsov, 2004; Mannila and Räihä, 1992), we know that for some φ, $\#\mathfrak{I}(\varphi)$ is exponentially larger than $\#\mathcal{F}(\varphi)$.

Proposition 33 (Kuznetsov 2004, Mannila & Räihä 1992)**.** *There exists a sequence* $(\varphi_n)_{n\in\mathbb{N}}$ *of closure operators such that* $\#\mathcal{F}(\varphi_n) \in \Theta(n^2)$ *but* $\#\mathfrak{I}(\varphi_n) \in \Theta(2^n)$.

We provide here the construction given by (Kuznetsov, 2004) but refer the reader to the original source for the complete argument. Given an n, we define $M = \{0, \ldots, 2n\}$ and the context $\mathcal{F} = \{F_1, \ldots, F_{3n}\}$ with $F_i = \{1, \ldots, 2n\} \setminus \{i, n+i\}$ for $1 \leq i \leq n$ as well as $F_{n+i} = \{0, 1, \ldots, 2n\} \setminus \{i\}$ for $1 \leq i \leq 2n$. The size of the context is $6n^2 + 3n$ while the corresponding closure operator has 2^n pseudo-closed sets, and therefore every implication set giving rise to this closure operator must have at least 2^n implications.
This result implies that in general, one cannot avoid the exponential blow-up if a contextually represented closure operator is to be represented by means of implications on the set M.
However, as the following definition and theorem show, this does not hold for extended implication sets. In fact, we show that the exponential blow-up can then always be avoided.

Definition 34. *Given a context $\mathcal{F}$ on a set M, let M^+ denote the set M extended by one new attribute m_F for each $F \in \mathcal{F}$. Then we define $\mathfrak{I}_{\mathcal{F}}$ as the extended implication set containing for every $m \in M$ the two implications $\{m\} \to \{m_F \mid F \in \mathcal{F}, m \notin F\}$ and $\{m_F \mid F \in \mathcal{F}, m \notin F\} \to \{m\}$.*

Example 35. *Consider the context $\mathcal{F}$ from Example 12 (and displayed in Example 9). The corresponding extended implication set $\mathfrak{I}_{\mathcal{F}}$ is*

$$\begin{array}{ll}
\{a\} \to \emptyset & \emptyset \to \{a\} \\
\{b\} \to \{m_{F_1}, m_{F_3}, m_{F_4}, m_{F_5}, m_{F_7}\} & \{m_{F_1}, m_{F_3}, m_{F_4}, m_{F_5}, m_{F_7}\} \to \{b\} \\
\{c\} \to \{m_{F_3}, m_{F_4}\} & \{m_{F_3}, m_{F_4}\} \to \{c\} \\
\{d\} \to \{m_{F_1}, m_{F_2}, m_{F_4}, m_{F_5}, m_{F_6}\} & \{m_{F_1}, m_{F_2}, m_{F_4}, m_{F_5}, m_{F_6}\} \to \{d\} \\
\{e\} \to \{m_{F_1}, m_{F_2}, m_{F_3}\} & \{m_{F_1}, m_{F_2}, m_{F_3}\} \to \{e\}.
\end{array}$$

Theorem 36. *Let $\mathcal{F}$ be a context on some set M. Then $\#\mathfrak{I}_{\mathcal{F}} = 2 \cdot |M| \cdot |M^+| = 2 \cdot |M| \cdot (|M| + |\mathcal{F}|) \leq 4 \cdot (\#\mathcal{F})^2$. Moreover, $(\cdot)^{\mathcal{F}} \equiv (\cdot)^{\mathfrak{I}_{\mathcal{F}}}|_M$, that is, $A^{\mathcal{F}} = A^{\mathfrak{I}_{\mathcal{F}}} \cap M$ for all $A \subseteq M$.*

Proof. The first claim is obvious.
For the second claim, we first show that for an arbitrary set $A \subseteq M$, one obtains $A^{\mathfrak{I}_{\mathcal{F}}} = B \cup C$ with $B = \{m_F \mid F \in \mathcal{F}, A \nsubseteq F\}$ and $C = \{m \mid \{m_F \mid F \in \mathcal{F}, m \notin F\} \subseteq B\}$. To show $A^{\mathfrak{I}_{\mathcal{F}}} \subseteq B \cup C$ we note that $A \subseteq B \cup C$ and that $B \cup C$ is $\mathfrak{I}_{\mathcal{F}}$-closed: $B \cup C$ satisfies all implications of the type $\{m_F \mid F \in \mathcal{F}, m \notin F\} \to \{m\}$ by definition of C. To check implications of the second type, $\{m\} \to \{m_F \mid F \in \mathcal{F}, m \notin F\}$, we note that

$$\begin{aligned}
C = \; & \{m \mid \{m_F \mid F \in \mathcal{F}, m \notin F\} \subseteq B\} \\
= \; & \{m \mid \{m_F \mid F \in \mathcal{F}, m \notin F\} \subseteq \{m_F \mid F \in \mathcal{F}, A \nsubseteq F\}\} \\
= \; & \{m \mid \forall F \in \mathcal{F} : m \notin F \to A \nsubseteq F\}
\end{aligned}$$

Now, picking an $m \in C$, we find that every m_F for which $m \notin F$ must also satisfy $A \nsubseteq F$ and therefore $m_F \in B$ so we find all implications of the second type satisfied.
Further, we show $B \cup C \subseteq A^{\mathfrak{I}_{\mathcal{F}}}$, by proving $B \subseteq A^{\mathfrak{I}_{\mathcal{F}}}$ and $C \subseteq A^{\mathfrak{I}_{\mathcal{F}}}$ separately.
We obtain $B = \{m_F \mid F \in \mathcal{F}, A \nsubseteq F\} \subseteq A^{\mathfrak{I}_{\mathcal{F}}}$ due to the following: given an $F \in \mathcal{F}$ with $A \nsubseteq F$, we find an $m \in A$ with $m \notin F$ and thus an implication $m \to \{m_F, \ldots\}$ contained in $\mathfrak{I}_{\mathcal{F}}$, therefore $A^{\mathfrak{I}_{\mathcal{F}}}$ must contain m_F.
We then also obtain $C := \{m \mid \{m_F \mid F \in \mathcal{F}, m \notin F\} \subseteq B\} \subseteq A^{\mathfrak{I}_{\mathcal{F}}}$ by the following argument: picking an $m \in C$, we find the implication $\{m_F \mid F \in \mathcal{F}, m \notin F\} \to \{m\}$ contained in $\mathfrak{I}_{\mathcal{F}}$. On the other hand, we already know $B \subseteq A^{\mathfrak{I}_{\mathcal{F}}}$ and $B \supseteq \{m_F \mid F \in \mathcal{F}, m \notin F\}$, hence $m \in A^{\mathfrak{I}_{\mathcal{F}}}$.
Finally, we obtain $A^{\mathfrak{I}_{\mathcal{F}}}|_M = A^{\mathfrak{I}_{\mathcal{F}}} \cap M = C = \{m \mid \forall F \in \mathcal{F} : m \notin F \to A \nsubseteq F\} = \{m \mid \forall F \in \mathcal{F} : A \subseteq F \to m \in F\} = \bigcap_{F \in \mathcal{F}, A \subseteq F} F = A^{\mathcal{F}}$ for any $A \subseteq M$. $\square$

Thus, we obtain a polynomially size-bounded implicational representation of a context. In our view this is a remarkable – although not too intricate – insight as it seems to challenge the practical relevance of computationally hard problems w.r.t. pseudo-intents (recognizing, enumerating, counting), on which theoretical FCA research has been focusing lately (Kuznetsov and Obiedkov, 2006; Rudolph, 2007; Kuznetsov and Obiedkov, 2008; Sertkaya, 2009b,a; Distel, 2010).
What remains to be clarified is the mutual succinctness of context families vs. extended implication sets. Can they be polynomially transformed into each other, is one strictly more succinct than the other or are they incomparable in that respect?
We will first show that extended implication sets can indeed polynomially express closure operators which are defined via context families. The idea is to translate the contexts separately into extended implication sets as defined before and then take the union over those implication sets. However, one has to make sure that the auxiliary attribute sets of the created extended implication sets are mutually disjoint.

Definition 37. *Given a context family $\mathfrak{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$, we obtain the corresponding extended implication set $\mathfrak{I}_{\mathfrak{F}}$ as the union $\bigcup_{i=1}^{n} \mathrm{rename}(\mathfrak{I}_{\mathcal{F}_i}, i)$ where* rename *is a function replacing every auxiliary attribute $m \notin M$ occurring in $\mathfrak{I}_{\mathcal{F}_i}$ by a fresh attribute denoted by (m, i).*

Proposition 38. *Given a context family $\mathfrak{F}$ on a set M, we obtain $(\cdot)^{\mathfrak{I}_{\mathfrak{F}}}|_M \equiv (\cdot)^{\mathfrak{F}}$. Moreover, $\#\mathfrak{I}_{\mathfrak{F}} = 2(|\mathfrak{F}| \cdot |M|^2 + \#\mathfrak{F}) < 4\#\mathfrak{F}^2$.*

Proof. The second claim is obvious. For the first claim, we show that every $(\cdot)^{\mathfrak{I}_{\mathfrak{F}}}|_M$-closed set is $(\cdot)^{\mathfrak{F}}$-closed and vice versa. Let A be $(\cdot)^{\mathfrak{I}_{\mathfrak{F}}}|_M$-closed. Then, by construction of $\mathfrak{I}_{\mathfrak{F}}$, A it must be $(\cdot)^{\mathfrak{I}_{\mathcal{F}}}|_M$-closed for every $\mathcal{F} \in \mathfrak{F}$, hence it must be $(\cdot)^{\mathfrak{F}}$-closed. Next, let B be $(\cdot)^{\mathfrak{F}}$-closed. By definition, this means B is $(\cdot)^{\mathcal{F}}$-closed for every $\mathcal{F} \in \mathfrak{F}$ and hence also $(\cdot)^{\mathfrak{I}_{\mathcal{F}}}|_M$-closed for every $\mathcal{F} \in \mathfrak{F}$. Then, by construction of $\mathfrak{I}_{\mathfrak{F}}$, it must be also $(\cdot)^{\mathfrak{I}_{\mathfrak{F}}}|_M$-closed. ☐

The final question, if every extended implication set has a polynomial-sized context family counterpart, is the last missing piece to the big picture about succinctness of representation types. The question must be answered negatively and we do so by providing a sequence of closure operators having a size-polynomial representation as extended implication set but not as context family.

Proposition 39. *There exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of closure operators that can be represented by a sequence $(\mathfrak{K}_n)_{n\in\mathbb{N}}$ of extended implication sets with $\#\mathfrak{K}_n \in \Theta(n^4)$ but any sequence $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ of context families representing $(\varphi_n)_{n\in\mathbb{N}}$ satisfies $\#\mathfrak{F}_n \in \Omega(2^n)$.*

Proof. For some natural number ℓ, we let $M_\ell = \{even\} \cup \{zero_i, one_i \mid 1 \leq i \leq \ell\}$. Next, for any $S \subseteq \{1, \ldots, \ell\}$ we define $Y_S := \{one_i \mid i \in S\} \cup \{zero_i \mid i \notin S\}$. Now, let $\psi_\ell(A) = A \cup \{even\}$ whenever there is some $S \subseteq \{1, \ldots, \ell\}$ of even cardinality for which $Y_S \subseteq A$. Otherwise, let $\psi_\ell(A) = A$. It can be easily verified that ψ_ℓ is indeed a closure operator.
We next note that ψ_ℓ can be represented by the extended implication set $\mathfrak{K}'_\ell$ with auxiliary attributes $\{even_i, odd_i \mid 1 \leq i \leq \ell\}$ containing the implications

$$
\begin{array}{rcll}
one_1 & \rightarrow & odd_1 & \\
zero_1 & \rightarrow & even_1 & \\
odd_i, one_{i+1} & \rightarrow & even_{i+1} & \text{for all } 1 \leq i < \ell \\
even_i, one_{i+1} & \rightarrow & odd_{i+1} & \text{for all } 1 \leq i < \ell \\
odd_i, zero_{i+1} & \rightarrow & odd_{i+1} & \text{for all } 1 \leq i < \ell \\
even_i, zero_{i+1} & \rightarrow & even_{i+1} & \text{for all } 1 \leq i < \ell \\
even_n & \rightarrow & even &
\end{array}
$$

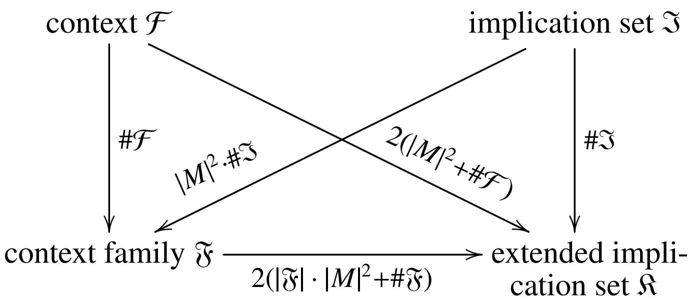

Figure 1: Overview about polynomial translatability between the four representation types. Arrows indicate the existence of a polynomial translation and the arrow labels indicate the upper bounds for the size of the resulting data structure. If no arrow exists between two representation types, no polynomial translation exists.

It is rather easy to see that $\#\mathfrak{K}'_\ell \in \Theta(\ell^2)$ and $\mathfrak{K}'_\ell$ implements the wanted behavior.
For the second part, let $\mathfrak{F}'_\ell$ be a context family with the desired behavior. Then, by definition, for any set $S \subseteq \{1, \ldots, \ell\}$ holds $even \in Y_S^{\mathfrak{F}'_\ell}$ iff[1]

$$\bigvee_{\mathcal{F} \in \mathfrak{F}'_\ell} \bigwedge_{\substack{A \in \mathcal{F} \\ even \notin A}} \bigvee_{m' \in M_\ell \setminus A} m' \in Y_S .$$

Consequently, S contains an even number of elements, iff $\{p_i \mapsto true \mid i \in S\} \cup \{p_i \mapsto false \mid i \notin S\}$ is a truth assignment for the propositional formula

$$\bigvee_{\mathcal{F} \in \mathfrak{F}'_\ell} \bigwedge_{\substack{A \in \mathcal{F} \\ even \notin A}} \bigvee_{m_k \in M_\ell \setminus A} \begin{cases} p_k & \text{if } m_k = one_k \\ \neg p_k & \text{otherwise} \end{cases} .$$

Note that this propositional formula has linear size compared to $\mathfrak{F}'_\ell$ and, by definition, it encodes a parity function over $p_1, \ldots, p_\ell$. Note that, due to its structure, this formula gives rise to a Boolean circuit of depth 3 whose size is linearly bounded by the size of $\mathfrak{F}'_\ell$. However, as was shown by Håstad (1987, Theorem 5.1), there are no Boolean circuits of depth k and size $2^{1/10^{k/(k-1)} \ell^{1/(k-1)}}$ that compute parity on ℓ input bits, for sufficiently large ℓ. Therefore $\#\mathfrak{F}'_\ell \in \Omega((2^{1/\sqrt{1000}})^{\sqrt{\ell}})$.
Finally, we define a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of closure operators by letting $\varphi_n = \psi_{1000n^2}$. We obtain corresponding sequences $(\mathfrak{K}_n)_{n \in \mathbb{N}}$ and $(\mathfrak{F}_n)_{n \in \mathbb{N}}$ of representations by letting $\mathfrak{K}_n = \mathfrak{K}'_{1000n^2}$ and $\mathfrak{F}_n = \mathfrak{F}'_{1000n^2}$. Then we obtain $\#\mathfrak{K}_n \in \Theta(n^4)$ and $\#\mathfrak{F}_n \in \Omega(2^n)$, concluding our proof. □

Figure 1 provides a visual summary of this section. Note that the non-existence of polynomial translations from context families to contexts and from extended implication sets to implication sets follows from the existing translations and the known non-existence of polynomial translations between contexts and implication sets.

[1]Note that in this proof, the symbols $\bigvee$ and $\bigwedge$ stand for logical connectives, whereas in the rest of the paper, they denote lattice operations.

Algorithm 5 `finerThanContext`

Input: closure operator φ on set M,
context $\mathcal{F}$

Output: YES if $\varphi \leq (\cdot)^{\mathcal{F}}$, NO otherwise

```
for each A ∈ F do
    if A ≠ φ(A) then
        output NO
        exit
    end if
end for
output YES
```

4. Algorithms for Managing Closure Operators

4.1. Finer or Coarser?

Depending on how closure operators are represented, there are several ways of checking if one is finer than the other.
We will start with the two basic representation types, contexts and implication sets, and establish results for the cases where this check is tractable, i.e., can be done in polynomial time.

Theorem 40. *Let φ be a closure operator on a set M for which computing of closures can be performed in t_φ time. Then, the following hold:*

- *For a context $\mathcal{F}$ on M, the problem $\varphi \leq (\cdot)^{\mathcal{F}}$ can be decided in $|\mathcal{F}| \cdot t_\varphi$ time.*
- *For an implication set $\mathfrak{I}$ on M, the problem $(\cdot)^{\mathfrak{I}} \leq \varphi$ can be decided in $|\mathfrak{I}| \cdot t_\varphi$ time.*

Proof. Algorithm 5 provides a solution for the first case. It verifies that every element (in FCA terms: every object intent) of $\mathcal{F}$ is φ-closed, this suffices to guarantee that all $\mathcal{F}$-closed sets are φ-closed since every $\mathcal{F}$-closed set is an intersection of elements of $\mathcal{F}$ and φ-closed sets are closed under intersections (since this holds for every closure operator).
Algorithm 6 provides a solution for the second case. To ensure that every φ-closed set is also $(\cdot)^{\mathfrak{I}}$-closed, it suffices to show that every φ-closed set respects all implications from $\mathfrak{I}$. Whether every φ-closed set respects an implication $A \to B \in \mathfrak{I}$ can in turn be verified by checking if $B \subseteq \varphi(A)$. □

The results established in the above theorem give rise to precise polynomial complexity bounds for seven of the 16 possible comparisons between the different representation types of closure operators.

Corollary 41. *Given contexts $\mathcal{F}, \mathcal{F}'$, a context family $\mathfrak{F}$, implication sets $\mathfrak{I}, \mathfrak{I}'$ and an extended implication set $\mathfrak{R}$ on some set M, it is possible to check*

- $(\cdot)^{\mathcal{F}} \leq (\cdot)^{\mathcal{F}'}$ *in time* $O(|\mathcal{F}| \cdot |\mathcal{F}'| \cdot |M|) = O(\#\mathcal{F} \cdot \#\mathcal{F}'/|M|)$,
- $(\cdot)^{\mathfrak{I}} \leq (\cdot)^{\mathfrak{I}'}$ *in time* $O(|\mathfrak{I}| \cdot |\mathfrak{I}'| \cdot |M|) = O(\#\mathfrak{I} \cdot \#\mathfrak{I}'/|M|)$,

Algorithm 6 `coarserThanImpSet`

Input: closure operator φ on set M,
implication set $\mathfrak{I}$

Output: YES if $(\cdot)^{\mathfrak{I}} \leq \varphi$, NO otherwise

for each $A \to B \in \mathfrak{I}$ **do**
  if $B \nsubseteq \varphi(A)$ **then**
    output NO
    exit
  end if
end for
output YES

- $(\cdot)^{\mathfrak{I}} \leq (\cdot)^{\mathcal{F}}$ *in time* $O(|\mathcal{F}| \cdot |\mathfrak{I}| \cdot |M|) = O(\#\mathcal{F} \cdot \#\mathfrak{I}/|M|)$,
- $(\cdot)^{\mathfrak{F}} \leq (\cdot)^{\mathcal{F}}$ *in time* $O(\sum_{\mathcal{F}' \in \mathfrak{F}} |\mathcal{F}'| \cdot |\mathcal{F}| \cdot |M|^2) = O(\#\mathfrak{F} \cdot \#\mathcal{F})$,
- $(\cdot)^{\mathfrak{I}} \leq (\cdot)^{\mathfrak{R}}$ *in time* $O(|\mathfrak{I}| \cdot |\mathfrak{R}| \cdot |N|) = O(\#\mathfrak{I} \cdot \#\mathfrak{R}/|M|)$,
- $(\cdot)^{\mathfrak{I}} \leq (\cdot)^{\mathfrak{F}}$ *in time* $O(\sum_{\mathcal{F}' \in \mathfrak{F}} |\mathcal{F}'| \cdot |\mathfrak{I}| \cdot |M|^2) = O(\#\mathfrak{F} \cdot \#\mathfrak{I})$, *and*
- $(\cdot)^{\mathfrak{R}} \leq (\cdot)^{\mathcal{F}}$ *in time* $O(|\mathcal{F}| \cdot |\mathfrak{R}| \cdot |N|) = O(\#\mathcal{F} \cdot \#\mathfrak{R}/|M|)$.

Surprisingly, the ensuing question – whether it is possible to establish a polynomial time complexity bound for the missing comparison cases – has to be denied assuming P $\neq$ NP. The corresponding findings are based on the following theorem. This result in a slightly different formulation is already known in other communities (Gottlob and Libkin, 1990), but we give a direct proof for the sake of self-containedness.

Theorem 42. *The problem of deciding if* $(\cdot)^{\mathcal{F}} \leq (\cdot)^{\mathfrak{I}}$ *for some context* $\mathcal{F}$ *and an implication set* $\mathfrak{I}$ *on some set* M *is* coNP*-complete.*

Proof. To show coNP membership, we note that $(\cdot)^{\mathcal{F}} \nleq (\cdot)^{\mathfrak{I}}$ if and only if there is a set A which is $(\cdot)^{\mathfrak{I}}$-closed but not $(\cdot)^{\mathcal{F}}$-closed. Clearly, we can guess such a set and check the above properties in polynomial time.
We show coNP hardness by a polynomial reduction of the problem to 3SAT (Karp, 1972). Given a set $C = \{C_1, \ldots, C_k\}$ of 3-clauses (i.e. $|C_i| = 3$) over a set of literals $L = \{p_1, \neg p_1, \ldots p_\ell, \neg p_\ell\}$, we let $M = L$ and define

$$\mathfrak{I} := \{\{p_i, \neg p_i\} \to M \mid p_i \in L\}$$

as well as

$$\mathcal{F} := \{M \setminus (C_i \cup \{m\}) \mid C_i \in C, m \in M\}.$$

We now show that there is a set A with $A^{\mathfrak{I}} = A$ but $A^{\mathcal{F}} \neq A$ if and only if there is a valuation on $\{p_1, \ldots, p_\ell\}$ for which C is satisfied.
For the "if" direction assume $val : \{p_1, \ldots, p_\ell\} \to \{true, false\}$ to be that valuation and define $A := \{p_i \mid val(p_i) = true\} \cup \{\neg p_i \mid val(p_i) = false\}$. Obviously, A is $(\cdot)^{\mathfrak{I}}$-closed. On the other hand, since by definition A must contain one element from each $C_i \in C$, we have that $F \nsubseteq A$ for all $F \in \mathcal{F}$ and hence $A^{\mathcal{F}} = M \neq A$.

$\leq$	context	implications	context family	extended implications
context	$\#\mathcal{F} \cdot \#\mathcal{F}'/\lvert M\rvert$	coNP-hard	coNP-hard	coNP-hard
implications	$\#\mathcal{F} \cdot \#\mathfrak{I}/\lvert M\rvert$	$\#\mathfrak{I} \cdot \#\mathfrak{I}'/\lvert M\rvert$	$\#\mathfrak{F} \cdot \#\mathfrak{I}$	$\#\mathfrak{I} \cdot \#\mathfrak{K}/\lvert M\rvert$
context family	$\#\mathfrak{F} \cdot \#\mathcal{F}$	coNP-hard	coNP-hard	coNP-hard
extended implications	$\#\mathcal{F} \cdot \#\mathfrak{K}/\lvert M\rvert$	coNP-hard	coNP-hard	coNP-hard

Table 1: Upper bounds for time complexities for checking the $\leq$ relation depending on the representation types.

For the "only if" direction, assume $A^{\mathfrak{I}} = A$ but $A^{\mathcal{F}} \neq A$. By construction of $\mathcal{F}$, the latter can only be the case if A contains one element of each $C_i \in C$. Thus, the valuation $val : \{p_1, \ldots, p_\ell\} \to \{true, false\}$ with

$$val(p_i) = \begin{cases} true & \text{if } p_i \in A \\ false & \text{otherwise} \end{cases}$$

witnesses the satisfiability of C. □

In fact, this negative result allows us to infer equally negative results for all remaining eight open cases.

Proposition 43. *For arbitrary context $\mathcal{F}$, context families $\mathfrak{F}, \mathfrak{F}'$, implication set $\mathfrak{I}$, and extended implication sets $\mathfrak{K}, \mathfrak{K}'$ on some set M, each of the following checks is* coNP*-hard: $\mathcal{F} \leq \mathfrak{F}$, $\mathcal{F} \leq \mathfrak{K}$, $\mathfrak{F} \leq \mathfrak{I}$, $\mathfrak{F} \leq \mathfrak{I}$, $\mathfrak{F} \leq \mathfrak{F}'$, $\mathfrak{F} \leq \mathfrak{K}$, $\mathfrak{K} \leq \mathfrak{I}$, $\mathfrak{K} \leq \mathfrak{F}$, and $\mathfrak{K} \leq \mathfrak{K}'$.*

Proof. The coNP-hard problem $\mathcal{F} \leq \mathfrak{I}$ can be polynomially translated in any of the above problems employing the translations given in Section 3. □

Table 1 summarizes the situation providing the time complexities for the tractable cases.

4.2. Projection

Next we investigate for all four representation types, whether a succinct representation of the projection of a closure operator to a subset $N \subseteq M$ of the attributes exists. The findings are mostly trivial or simple consequences of earlier results. We start with contexts, where it is straightforward that the result can be obtained by element-wise projection.

Proposition 44. *Given a formal context $\mathcal{F}$ on a set M, its projection to some set N can be expressed by $\mathcal{F}_N = \{A \cap N \mid A \in \mathcal{F}\}$. Moreover, $\#\mathcal{F}_N = |\mathcal{F}| \cdot |N| = \#\mathcal{F} \cdot |N|/|M| \leq \#\mathcal{F}$.*

Proof. Let $B \subseteq N$. Then

$$B^{\mathcal{F}_N} = \bigcap_{B \subseteq A \cap N, A \in \mathcal{F}} A \cap N = N \cap \bigcap_{B \subseteq A \in \mathcal{F}} A = B^{\mathcal{F}} \cap N.$$

□

Turning to implications, we note that if a polynomial-size representation of the projection existed, this would imply the existence of a polynomial translation from extended implication sets into implication sets (by projecting away all the auxiliary attributes), contradicting our finding in Section 3. We can even use our considerations from that section to provide an example where exponential blow-up is unavoidable.

Proposition 45. *There exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of closure operators on sets $M_1^*, M_2^*, \dots$ as well as a sequence of sets $M_1, M_2, \dots$ with $M_i \subseteq M_i^*$ such that $\#\mathfrak{I}(\varphi_n) \in \Theta(n^2)$ but $\#\mathfrak{I}(\varphi_n|_{M_n}) \in \Theta(2^n)$.*

Proof. Consider the closure operators φ_n defined via contexts $\mathcal{F}_n$ given by Kuznetsov (given in this paper right after Proposition 33). As shown by Kuznetsov, the corresponding minimal implication sets $\mathfrak{I}(\varphi_n)$ are of exponential size. On the other hand, each context $\mathcal{F}_n$ can be translated into an extended implication set $\mathfrak{I}_{\mathcal{F}_n}$ with only linear blow-up, using the translation given in Definition 34. Clearly, the projection of $(\cdot)^{\mathfrak{I}_{\mathcal{F}_n}}$ to the original attribute sets M_n must yield $(\cdot)^{\mathfrak{I}(\varphi_n)}$. Hence we have found a sequence of quadratically growing implication sets whose projections grow exponentially. $\square$

For extended implications, the case is trivial: the attributes which are to be projected away are simply redefined to be auxiliary attributes.

Proposition 46. *Given an extended implication set $\mathfrak{K}$ on a set M, its projection to some set N can be expressed by itself, i.e., $\mathfrak{K}_N = \mathfrak{K}$. We obtain $\#\mathfrak{K}_N = \#\mathfrak{K}$.*

Last, we consider the context family representation type. Again we can show indirectly that no polynomial representation of projections can exist: assuming its existence, we could polynomially translate extended implications on M with total attribute set M' into context families by first using the polynomial implication-to-context family translation detailed in Section 3 to arrive at a context family on M' and then polynomially project away the auxiliary attributes in M'. However we know that such a translation cannot exist. Again, we can use arguments established in Section 3 to even show exponential blow-up.

Proposition 47. *There exists a sequence $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ of context families on sets $M_1^*, M_2^*, \dots$ as well as a sequence of sets $M_1, M_2, \dots$ with $M_i \subseteq M_i^*$ such that $\#\mathfrak{F}_n \in \Theta(n^6)$ but for any sequence $(\mathfrak{F}'_n)_{n\in\mathbb{N}}$ of context families with $(\cdot)^{\mathfrak{F}'_n} = (\cdot)^{\mathfrak{F}_n}|_{M_n}$ holds $\#\mathfrak{F}'_n \in \Omega(2^n)$.*

Proof. Consider the closure operators $(\cdot)^{\mathfrak{K}_n}$ with $\mathfrak{K}_n$ taken from the proof of Proposition 39. Note that $(\cdot)^{\mathfrak{K}_n}$ does not coincide with φ_n from that proof, since we interpret the extended implication sets $\mathfrak{K}_n$ on M_n given there as implication sets on their total attribute sets (let us denote them by M_n^*). Then we can express the closure operators $(\cdot)^{\mathfrak{K}_n}$ by context families $\mathfrak{F}_n = \mathfrak{F}(\mathfrak{K}_n)$ as defined in Definition 31. These context families are of size $\Theta(n^6)$. On the other hand, according to the proof of Proposition 39, we know that any sequence $(\mathfrak{F}'_n)_{n\in\mathbb{N}}$ of context families satisfying $(\cdot)^{\mathfrak{K}_n}|M_n \equiv (\cdot)^{\mathfrak{F}_n}$ must be of size $\Omega(2^n)$. $\square$

4.3. Adding a Closed Set

We now consider the task of making a closure operator φ minimally "finer" by requiring that a given set A be a closed set.

Definition 48. *Given a closure operator φ on M and some $A \subseteq M$, the* A-refinement *of φ (written $\varphi{\downarrow}A$) is defined as the coarsest closure operator ψ with $\psi \leq \varphi$ and $\psi(A) = A$.*

Alternatively, the A-refinement can also be defined by $\varphi{\downarrow}A = \varphi \wedge (\cdot)^{\{A\}}$ using the infimum operation on the lattice of closure operators.

It is straightforward to show, using McKinsey's theorem (McKinsey, 1943), that B is a $\varphi{\downarrow}A$-closed set exactly if it is φ-closed or the intersection of A and a φ-closed set. Clearly, if a closure operator is represented as formal context, refinements can be computed by simply adding a row to the context. The obtained context will in general not be size-minimal even if the original context is.

Proposition 49. *Given a context $\mathcal{F}$ on M and some $A \in M$, let $\mathcal{F}' = \mathcal{F} \cup \{A\}$. Then $(\cdot)^{\mathcal{F}'} \equiv (\cdot)^{\mathcal{F}} \downarrow A$ and $\#\mathcal{F}' \leq \#\mathcal{F} + |M|$.*

Proof. Direct consequence of Proposition 17. $\square$

If, however, the closure operator is represented as a context family, exponential blow-up may occur.[2]

Proposition 50. *There exist sequences $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ with $\mathfrak{F}_i$ being a context family over some set M_i^* and $A_i \subseteq M_i^*$ of context families and such that $\#\mathfrak{F}_n \in \Theta(n^3)$ and $|A_n| = \Theta(n)$ but for any sequence $(\mathfrak{F}'_n)_{n\in\mathbb{N}}$ of context families with $(\cdot)^{\mathfrak{F}'_n} = (\cdot)^{\mathfrak{F}_n \downarrow A_n}$ holds $\#\mathfrak{F}'_n \in \Theta(2^n)$.*

Proof. The proof proceeds as follows: we will show that, for any context family $\mathfrak{F}$ on some set M^* and some subset $M \subseteq M^*$, we can construct a context family $\mathfrak{G}$ on M with $(\cdot)^{\mathfrak{F}}|_M \equiv (\cdot)^{\mathfrak{G}}$. The construction of $\mathfrak{G}$ is linear up to one computation of a refinement w.r.t. M, but due to Proposition 47, the size of $\mathfrak{G}$ must be exponential wrt. the size of $\mathfrak{F}$, which provides us with the claimed lower bound.
Given $\mathfrak{F}$ and M, let now $\mathfrak{F}'$ be a size-minimal context family with $(\cdot)^{\mathfrak{F}'} \equiv (\cdot)^{\mathfrak{F}} \downarrow M$. Note that we then obtain $B^{\mathfrak{F}'} \subseteq M$ whenever $B \subseteq M$. Consequently $(\cdot)^{\mathfrak{F}'}$ coincides with $(\cdot)^{\mathfrak{F}}|_M$ for all arguments $B \subseteq M$. Also, as a straightforward consequence, we obtain $B^{\mathcal{F}} \subseteq M$ for every $B \subseteq M$ and $\mathcal{F} \in \mathfrak{F}'$.
Now let $\mathfrak{G}$ be the context family on M defined by $\mathfrak{G} = \{\{A \cap M \mid A \in \mathcal{F}\} \mid \mathcal{F} \in \mathfrak{F}'\}$, that is, $\mathfrak{G}$ is obtained from $\mathfrak{F}'$ by eliminating from its contexts all attributes not in M. For some $B \subseteq M$ we then have that $B^{\mathfrak{G}}$ is the smallest superset of B closed under every $(\cdot)^{\mathcal{G}}$ with $\mathcal{G} \in \mathfrak{G}$, which by construction is the smallest superset of B closed under every $(\cdot)^{\mathcal{F}}$ with $\mathcal{F} \in \mathfrak{F}'$. Thereby, we obtain that $\mathfrak{G}$ coincides with $(\cdot)^{\mathfrak{F}}|_M$.
Now, employing Proposition 47 as well as the fact that $\#\mathfrak{G} \leq \#\mathfrak{F}'$ we obtain the claimed bounds. $\square$

Likewise, if the closure operator is represented in terms of implications, adding a closed set may incur exponential blow-up as shown in some recent work on belief revision in propositional Horn logic (Adaricheva et al., 2012).[3]

Proposition 51 (Adaricheva et al. 2012)**.** *There exists sequences $(\varphi_n)_{n\in\mathbb{N}}$ of closure operators and sets $(A_n)_{n\in\mathbb{N}}$ such that $\#\mathfrak{I}(\varphi_n \downarrow A_n) \in \Theta(2^n)$ whereas $\#\mathfrak{I}(\varphi_n) \in \Theta(n^2)$.*

Again, we provide the construction here but refer to the literature for the comprehensive argument: given n, let $M_n = \{w\} \cup \{t_i, u_i, v_i \mid 1 \leq i \leq n\}$, let $\mathfrak{I}_n = \{t_i \to v_i;\ u_i \to v_i \mid 1 \leq i \leq n\} \cup \{v_1, \ldots, v_n \to w\}$ and let $A_n = \{w\} \cup \{t_i, u_i \mid 1 \leq i \leq n\}$.
As it turns out, the situation again changes when auxiliary attributes can be used. In this case a polynomial size implicational representation can be found.

[2] This finding and the closely related one of Proposition 69 correct wrong results presented in an earlier publication (Rudolph, 2014) where a construction was presented relying on the erroneous assumption that the lattice of closure operators were distributive. The author is grateful to Daniel Borchmann for catching the error.

[3] The author is indebted to Kira V. Adaricheva pointing him to a severe flaw in his earlier publication on the subject (Rudolph, 2012), where he erroneously claimed that a polynomial solution exists.

Definition 52. *Let $\mathfrak{K}$ be an extended implication set on M with the total attribute set N. Let $A \subseteq M$. Then we define an extended implication set $\mathfrak{K}{\downarrow}A$ on M with a total attribute set $N' := N \cup \{m' \mid m \in N \setminus A\} \cup \{tr\}$ (the new elements m' being copies of the respective m) as follows:*

$$\begin{aligned}\mathfrak{K}{\downarrow}A \quad = \quad & \{\{m\} \to \{m', tr\},\ \{m', tr\} \to \{m\} \mid m \in N \setminus A\} \cup \\ & \{(B \cap A) \cup \{m' \mid m \in B \setminus A\} \to (C \cap A) \cup \{m' \mid m \in C \setminus A\} \mid B \to C \in \mathfrak{K}\}\end{aligned}$$

Example 53. *Consider the implication set $\mathfrak{I}$ from Example 21, and let $A = \{b, d\}$. Then $\mathfrak{K}{\downarrow}A$ is composed of the implications:*

$$\begin{array}{rcl@{\qquad\qquad}rcl@{\qquad\qquad}rcl}
\{a\} & \to & \{a', tr\} & \{a', tr\} & \to & \{a\} & \{\} & \to & \{a'\} \\
\{c\} & \to & \{c', tr\} & \{c', tr\} & \to & \{c\} & \{b\} & \to & \{c'\} \\
\{e\} & \to & \{e', tr\} & \{e', tr\} & \to & \{e\} & \{b, d\} & \to & \{e'\} \\
 & & & & & & \{c', d\} & \to & \{e'\} \\
 & & & & & & \{d, e'\} & \to & \{c'\}
\end{array}$$

The intuition behind this encoding is to introduce "copies" of all attributes outside A and to use an implication set in which all those attributes are renamed into their copies. Moreover, a specific "trigger attribute" tr is implied by any of the original attributes from $M \setminus A$. Whenever tr is activated, all the introduced copies imply their original counterparts. The following proposition justifies this construction.

Proposition 54. *Given an extended implication set $\mathfrak{K}$ on M with N the total attribute set and some $A \in M$, we have $(\cdot)^{\mathfrak{K}\downarrow A}|_M \equiv (\cdot)^{\mathfrak{K}}|_M{\downarrow}A$ and $\#(\mathfrak{K}{\downarrow}A) \leq 2 \cdot \#\mathfrak{K} + |\mathfrak{K}| + 4|N|^2 + 2|N| < 9 \cdot \#\mathfrak{K}^2$ and for the total attribute set N' of $\mathfrak{K}{\downarrow}A$ holds $|N'| \leq 2|N| + 1$.*

Proof. Checking the provided size bounds is straightforward.
We now show the first claim by proving that a subset $S \subseteq M$ is $(\cdot)^{\mathfrak{K}\downarrow A}|_M$-closed iff it is $(\cdot)^{\mathfrak{K}}|_M$-closed or the intersection of A and some $(\cdot)^{\mathfrak{K}}|_M$-closed set.
We start with the "only if" direction, distinguishing two cases. Given some set $S \in M$ with $S \subseteq A$, we obtain $S^{\mathfrak{K}\downarrow A} = (S^{\mathfrak{K}} \cap A) \cup \{m' \mid m \in S^{\mathfrak{K}} \cap (N \setminus A)\}$ and, in particular, $S^{\mathfrak{K}\downarrow A}$ does not contain tr and hence also no $m \in N \setminus A$. Therefore $S^{\mathfrak{K}\downarrow A}|_M = S^{\mathfrak{K}}|_M \cap A$. Next assume $S \nsubseteq A$, i.e., there is some $m \in M$ with $m \in S \setminus A$. Then $tr \in S^{\mathfrak{K}\downarrow A}$ and therefore $S^{\mathfrak{K}\downarrow A} = S^{\mathfrak{K}} \cap A \cup \{m, m' \mid m \in S^{\mathfrak{K}} \cap (N \setminus A)\} \cup \{tr\} = S^{\mathfrak{K}} \cup \{m' \mid m \in S^{\mathfrak{K}} \cap (N \setminus A)\} \cup \{tr\}$. Hence we get $S^{\mathfrak{K}\downarrow A}|_M = S^{\mathfrak{K}}|_M$.
For the "if" direction, we distinguish the two cases. First, assume S is $(\cdot)^{\mathfrak{K}}|_M$-closed, i.e., $S^{\mathfrak{K}} \cap M = S$. Toward a contradiction, suppose that S is not $(\cdot)^{\mathfrak{K}\downarrow A}|_M$-closed, hence there is some $m \in M$ with $m \in S^{\mathfrak{K}\downarrow A} \setminus S$. If m is brought about by an implication of type $\{tr, m'\} \to \{m\}$, we also have $tr \in S^{\mathfrak{K}\downarrow A}$ and therefore $S^{\mathfrak{K}\downarrow A} \cap M = S^{\mathfrak{K}} \cap M = S$, a contradiction. Otherwise $m \in A$ but then we obtain $m \in S^{\mathfrak{K}} = S$ another contradiction. Second, assume S is the intersection of A and some $(\cdot)^{\mathfrak{K}}|_M$-closed set S'. Then we obtain $S^{\mathfrak{K}\downarrow A} = (A \cap S')^{\mathfrak{K}\downarrow A} \subseteq A^{\mathfrak{K}\downarrow A} \cap S'^{\mathfrak{K}\downarrow A} = A \cap S'^{\mathfrak{K}\downarrow A} = A \cap S'^{\mathfrak{K}} = A \cap S' = S$, which, together with the trivial $S \subseteq S^{\mathfrak{K}\downarrow A}$, shows $S^{\mathfrak{K}\downarrow A} = S$.

□

4.4. Adding an Implication

The task dual to the one from the preceding section is to make a given closure operator coarser by requiring that all closed sets of the coarsened version respect a given implication. In other words, all closed sets not respecting the implication are removed.

Definition 55. *Given a closure operator φ on M and some implication $\mathfrak{i} = A \to B$ with $A, B \subseteq M$, the* $\mathfrak{i}$-coarsening *of φ (written $\varphi\uparrow\mathfrak{i}$) is defined as the finest closure operator ψ with $\varphi \leq \psi$ and $B \subseteq \psi(A)$.*

Alternatively, the $\mathfrak{i}$-coarsening can also be defined by $\varphi\uparrow\mathfrak{i} = \varphi \vee (\cdot)^{\{\mathfrak{i}\}}$ using the supremum operation on the lattice of closure operators.
Clearly, if a closure operator is represented as implication set (extended or not), coarsenings can be computed by simply adding the implication to the set. Note that $\mathfrak{I}' := \mathfrak{I} \cup \{\mathfrak{i}\}$ will in general not be size-minimal.

Proposition 56. *Given a (possibly extended) implication set $\mathfrak{I}$ on M and some implication $\mathfrak{i}$ on M, we obtain $(\cdot)^{\mathfrak{I}'} \equiv (\cdot)^{\mathfrak{I}}\uparrow\mathfrak{i}$ with $\mathfrak{I}' = \mathfrak{I} \cup \{\mathfrak{i}\}$. Moreover, $|\mathfrak{I}'| = |\mathfrak{I}| + 1$.*

Proof. Direct consequence of Proposition 26. □

If the closure operator is represented by a context, a little more work is needed for this task. The idea behind the following definition is as follows: a set is closed w.r.t. the updated context $\mathcal{F}'$ iff it is closed w.r.t. the original context $\mathcal{F}$ and respects the new implication $\mathfrak{i}$. Thus, all $C \in \mathcal{F}$ respecting $\mathfrak{i}$ will be in $\mathcal{F}'$. For the other C, we have to add their $\mathfrak{i}$-respecting intersections with other sets, which essentially can only be intersections with sets D that do not contain the premise of $\mathfrak{i}$.

Definition 57. *Given a context $\mathcal{F}$ on M and some implication $\mathfrak{i} = A \to B$ on M, we define a new context $\mathcal{F}\uparrow\mathfrak{i}$ as follows*

$$\begin{aligned} \mathcal{F}\uparrow\mathfrak{i} \quad := \quad & \{C \mid C \in \mathcal{F} \text{ and } C \text{ respects } A \to B\} \cup \\ & \{C \cap D \mid C, D \in \mathcal{F},\ A \nsubseteq D \text{ and } C \text{ does not respect } A \to B\} \end{aligned}$$

Proposition 58. *Given a context $\mathcal{F}$ on M and some implication $\mathfrak{i}$ on M, we have $(\cdot)^{\mathcal{F}\uparrow\mathfrak{i}} \equiv (\cdot)^{\mathcal{F}}\uparrow\mathfrak{i}$. Moreover, we have $|\mathcal{F}\uparrow\mathfrak{i}| \leq |\mathcal{F}|^2$ and hence $\#\mathcal{F}\uparrow\mathfrak{i} \leq (\#\mathcal{F})^2/|M|$.*

Proof. It is easy to check that $\mathcal{F}\uparrow\mathfrak{i}$ satisfies the size bounds given in the second claim. We show the first claim by verifying that a set is $(\cdot)^{\mathcal{F}\uparrow\mathfrak{i}}$-closed if and only if it is $(\cdot)^{\mathcal{F}}$-closed and respects $A \to B$.
For the "if" direction, let S be an $(\cdot)^{\mathcal{F}}$-closed set that respects $A \to B$. This means that either $B \subseteq S$ or $A \nsubseteq S$. In the first case, note that every $C \in \mathcal{F}$ with $S \subseteq C$ respects $A \to B$ and thus each such C is contained in $\mathcal{F}\uparrow\mathfrak{i}$ as well. Since S is the intersection of all these C, it must itself be $(\cdot)^{\mathcal{F}\uparrow\mathfrak{i}}$-closed. In the second case, there must be some $C \in \mathcal{F}$ with $S \subseteq C$ and $A \nsubseteq C$. Thus we obtain

$$\begin{aligned} S &= \textstyle\bigcap_{S \subseteq D \in \mathcal{F}} D \\ &= (\textstyle\bigcap_{S \subseteq D \in \mathcal{F},\ D \text{ respects } A \to B} D) \cap (\bigcap_{S \subseteq D \in \mathcal{F},\ D \text{ violates } A \to B} D) \cap C \\ &= (\textstyle\bigcap_{S \subseteq D \in \mathcal{F},\ D \text{ respects } A \to B} D) \cap (\bigcap_{S \subseteq D \in \mathcal{F},\ D \text{ violates } A \to B} D \cap C) \end{aligned}$$

and see that S is an intersection of $(\cdot)^{\mathcal{F}\uparrow\mathfrak{i}}$-closed sets and hence itself $(\cdot)^{\mathcal{F}\uparrow\mathfrak{i}}$-closed.
For the "only if" direction, consider an arbitrary $(\cdot)^{\mathcal{F}\uparrow\mathfrak{i}}$-closed set S. It can be easily checked that all $C \in \mathcal{F}\uparrow\mathfrak{i}$ respect $A \to B$, hence also S does. Moreover, by definition, every $C \in \mathcal{F}\uparrow\mathfrak{i}$ is an intersection of elements of $\mathcal{F}$ and thus $(\cdot)^{\mathcal{F}}$-closed. □

For a context family $\mathfrak{F}$, there are two options of computing an implication-coarsening. One option is to exchange one context $\mathcal{F} \in \mathfrak{F}$ by $\mathcal{F}\uparrow\mathfrak{i}$. We will present the second option which will lead to a smaller blowup under reasonable assumptions.

Definition 59. *Given a context family* $\mathfrak{F}$ *on* M *and some implication* $\mathfrak{i}$ *on* M*, we define a new context family* $\mathfrak{F}{\uparrow}\mathfrak{i}$ *as* $\mathfrak{F} \cup \{\mathcal{F}_\mathfrak{i}\}$*, where* $\mathcal{F}_\mathfrak{i}$ *is the one-implication context for* $\mathfrak{i}$ *as defined in Definition 30.*

Proposition 60. *Given a context family* $\mathfrak{F}$ *on* M *and some implication* $\mathfrak{i}$ *on* M*, we have* $(\cdot)^{\mathfrak{F}\uparrow\mathfrak{i}} \equiv (\cdot)^{\mathfrak{F}}{\uparrow}\mathfrak{i}$*. Moreover, we have* $\#\mathfrak{F}{\uparrow}\mathfrak{i} < \#\mathfrak{F} + |M|^3$*.*

Proof. The upper size bound in the second claim is obvious. We prove the first claim by showing that the $(\cdot)^{\mathfrak{F}\uparrow\mathfrak{i}}$-closed sets are exactly the $(\cdot)^{\mathfrak{F}}$-closed sets respecting $\mathfrak{i}$. Assume A is $(\cdot)^{\mathfrak{F}}$-closed and respects $\mathfrak{i}$. From the latter, we can derive that A is also $(\cdot)^{\mathcal{F}_\mathfrak{i}}$-closed and hence it must be $(\cdot)^{\mathfrak{F}\uparrow\mathfrak{i}}$-closed. Next, assume B is $(\cdot)^{\mathfrak{F}\uparrow\mathfrak{i}}$-closed. Since $\mathfrak{F} \subseteq \mathfrak{F}{\uparrow}\mathfrak{i}$, we obtain that B must be $(\cdot)^{\mathfrak{F}}$-closed. On the other hand $\mathcal{F}_\mathfrak{i} \in \mathfrak{F}{\uparrow}\mathfrak{i}$ implies that B is $(\cdot)^{\mathcal{F}_\mathfrak{i}}$-closed, i.e., it respects $\mathfrak{i}$. □

4.5. Lattice Operations

Last but not least, we will examine succinctness of the diverse representation types when applying the lattice operations $\vee$ and $\wedge$ in the lattice of closure operators described in Section 2. We will distinguish between binary and n-ary application.
For contexts, $\wedge$ with arbitrary arity is very easy to compute and incurs no blowup whatsoever: one simply needs to concatenate all input contexts.

Proposition 61. *Given n contexts* $\mathcal{F}_1, \ldots, \mathcal{F}_n$*, we let* $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n$*. Then,* $(\cdot)^{\mathcal{F}_1} \wedge \ldots \wedge (\cdot)^{\mathcal{F}_n} \equiv (\cdot)^{\mathcal{F}}$ *and* $\#\mathcal{F} = \#\mathcal{F}_1 + \ldots + \#\mathcal{F}_n$*.*

Proof. Direct consequence from Proposition 17. □

On the other hand already the binary application of $\vee$ may result in exponential blowup, a result shown in the context of model-based reasoning (Eiter et al., 1998).

Proposition 62 (Eiter et al. 1998)**.** *There exist sequences* $(\varphi_n)_{n\in\mathbb{N}}$ *and* $(\psi_n)_{n\in\mathbb{N}}$ *of closure operators such that* $\#\mathcal{F}(\varphi_n \vee \psi_n) \in \Theta(2^n)$ *whereas* $\#\mathcal{F}(\varphi_n) = \#\mathcal{F}(\psi_n) \in \Theta(n^2)$*.*

We provide the construction used by (Eiter et al., 1998), but omit the proof. They let $M = \{1, \ldots, 4n\}$ and define φ_n via the context

$$\mathcal{F}_n = \{M \setminus (\{2n{+}1, \ldots, 3n\} \cup \{i, (i \bmod n) + 3n\}) \mid i \in \{1, \ldots, 2n\}\}$$

and ψ_n via the context

$$\mathcal{F}'_n = \{M \setminus (\{3n{+}1, \ldots, 4n\} \cup \{i, (i \bmod n) + 2n\}) \mid i \in \{1, \ldots, 2n\}\}.$$

For implications, conversely, $\vee$ is very easily computable by just taking the union of the implication sets.

Proposition 63. *Given n implication sets* $\mathfrak{I}_1, \ldots, \mathfrak{I}_n$*, we let* $\mathfrak{I} = \mathfrak{I}_1 \cup \ldots \cup \mathfrak{I}_n$*. Then,* $(\cdot)^{\mathfrak{I}_1} \vee \ldots \vee (\cdot)^{\mathfrak{I}_n} \equiv (\cdot)^{\mathfrak{I}}$ *and* $\#\mathfrak{I} = \#\mathfrak{I}_1 + \ldots + \#\mathfrak{I}_n$*.*

Proof. Direct consequence from Proposition 26. □

On the other hand, $\wedge$ may result in exponential blowup even if applied only binarily:

Proposition 64. *There exist sequences* $(\varphi_n)_{n\in\mathbb{N}}$ *and* $(\psi_n)_{n\in\mathbb{N}}$ *of closure operators such that* $\#\mathfrak{I}(\varphi_n \wedge \psi_n) \in \Theta(2^n)$ *whereas* $\#\mathfrak{I}(\varphi_n) \in \Theta(n^2)$ *and* $\#\mathfrak{I}(\psi_n) \in \Theta(n)$*.*

Proof. Let $M_n = \{a_i, b_i \mid 1 \leq i \leq n\} \cup \{c, d\}$, let φ_n be represented by $\mathfrak{I}_1$ containing the implications

$$\begin{array}{rcll} a_i & \to & b_i & 1 \leq i \leq n, \\ b_1, \ldots, b_n & \to & d, & \end{array}$$

and let ψ_n be represented by $\mathfrak{I}_2 = \{c \to d\}$. We now show that $\mathfrak{I}(\varphi_n \vee \psi_n)$ contains 2^n implications by showing that there are 2^n pseudo-closed sets. For every set $S \subseteq \{1, \ldots, n\}$ let $A_S := \{a_i \mid i \in S\} \cup \{b_i \mid i \notin S\} \cup \{c\}$. It can be easily verified that A_S is pseudo-closed, since it is not closed (as the closure must contain d) and it cannot not properly contain pseudo-closed sets since each of its subsets is closed. Clearly, there are 2^n distinct subsets of $\{1, \ldots, n\}$. On the other hand, every minimal implicational representation of $\varphi_n \vee \psi_n$ must contain at least as many implications as there are pseudo-closed sets (Guigues and Duquenne, 1986; Ganter and Wille, 1997). $\square$

Switching to extended implications improves the situation. Computing $\vee$ remains easy and can be done by taking the union of the implication sets. One just has to take care (possibly via a renaming) that the auxiliary attributes of the separate sets are disjoint. The quadratic blowup comes from the fact that both the number of implications and the auxiliary attribute sets add up.

Proposition 65. *Given n extended implication sets $\mathfrak{R}_1, \ldots, \mathfrak{R}_n$ on M with total attribute sets $M_1, \ldots, M_n$, let $\mathfrak{R} = \bigcup_{i=1}^n \operatorname{rename}(\mathfrak{R}_i, i)$. Then, $(\cdot)^{\mathfrak{R}_1}|_M \vee \ldots \vee (\cdot)^{\mathfrak{R}_n}|_M \equiv (\cdot)^{\mathfrak{R}}|_M$ and $\#\mathfrak{R} = \left(\sum_{i=1}^n |\mathfrak{R}_i|\right) \cdot \left(|M| + \sum_{i=1}^n |M_i \setminus M|\right)$.*

Proof. We show that every $(\cdot)^{\mathfrak{R}}|_M$-closed set is also $(\cdot)^{\mathfrak{R}_1}|_M \vee \ldots \vee (\cdot)^{\mathfrak{R}_n}|_M$-closed and vice versa. Assume B is $(\cdot)^{\mathfrak{R}}|_M$-closed but (toward a contradiction) not $(\cdot)^{\mathfrak{R}_i}|_M$-closed for some i. Consequently, there is some $b \in B^{\mathfrak{R}_i} \cap M \setminus B$. But then, since $\operatorname{rename}(\mathfrak{R}_i) \subseteq \mathfrak{R}$ we have $b \in B^{\operatorname{rename}(\mathfrak{R}_i)} \cap M \subseteq B^{\mathfrak{R}} \cap M = B$, a contradiction.
Next, assume B is $(\cdot)^{\mathfrak{R}_i}|_M$-closed for every i but (toward a contradiction) not $(\cdot)^{\mathfrak{R}}|_M$-closed. Consequently, there is some $b \in B^{\mathfrak{R}} \cap M \setminus B$. Then there must be a sequence $C_1 \to D_1, \ldots, C_k \to D_k$ of implications from $\mathfrak{R}$ and a sequence $B_0, \ldots, B_k$ of sets such that $B_0 = B$ and $b \in B_k$ as well as $C_i \subseteq B_{i-1}$ and $B_i = B_{i-1} \cup D_i$ for all $i \in \{1, \ldots, k\}$. Let b and this sequences be chosen such that k is minimal. Therefore, for all $i < k$, $B_i \cap M = B$. Then, by minimality and disjointness of the auxiliary element sets, all implications from the sequence must come from the same $\operatorname{rename}(\mathfrak{R}_i)$. But then B cannot be $(\cdot)^{\mathfrak{R}_i}|_M$-closed, a contradiction. $\square$

Computing $\wedge$ for extended implication sets is remarkably easier than for implication sets. The idea here is to introduce disjoint "copies" of all implication sets such that closure computation is done independently. Finally one has to add some "confluence rules" which make sure that a proper attribute is added to the closure if it is contained in each of the separate independently computed closures.

Definition 66. *Let* renameall *be the function that takes an extended implication set $\mathfrak{R}$ and a natural number i as input and returns the implication set with every (proper or auxiliary) attribute m in $\mathfrak{R}$ replaced by a new attribute denoted (m, i).*
Given n extended implication sets $\mathfrak{R}_1, \ldots, \mathfrak{R}_n$ on M, let

$$\bigwedge\{\mathfrak{R}_1, \ldots, \mathfrak{R}_n\} := \mathfrak{R}_{\text{in}} \cup \mathfrak{R}_{\text{out}} \cup \bigcup_{1 \leq i \leq n} \operatorname{renameall}(\mathfrak{R}_i, i),$$

define a new extended implication set on M where $\mathfrak{R}_{\text{in}} = \{m \to (m, i) \mid m \in M, 1 \leq i \leq n\}$ and $\mathfrak{R}_{\text{out}} = \{(m, 1), \ldots, (m, n) \to m \mid m \in M\}$.

Proposition 67. *Given n extended implication sets $\mathfrak{K}_1, \ldots, \mathfrak{K}_n$ on M, we let $\mathfrak{K} = \bigwedge\{\mathfrak{K}_1, \ldots, \mathfrak{K}_n\}$. Then, $(\cdot)^{\mathfrak{K}_1}|_M \wedge \ldots \wedge (\cdot)^{\mathfrak{K}_n}|_M \equiv (\cdot)^{\mathfrak{K}}|_M$ and $\#\mathfrak{K} = \left(|M| + \sum_{i=1}^{n}(|\mathfrak{K}_i| + |M|)\right) \cdot \left(|M| + \sum_{i=1}^{n}|M_i|\right)$*

Proof. We show that $B^{\mathfrak{K}} \cap M = (B^{\mathfrak{K}_1} \cap M) \cap \ldots \cap (B^{\mathfrak{K}_n} \cap M)$ holds for every $B \subseteq M$. By construction we have

$$B^{\mathfrak{K}} = B \cup \left(\bigcup_{1 \leq i \leq n} B^{\mathfrak{K}_i} \times \{i\}\right) \cup \bigcap_{1 \leq i \leq n} \{b \mid (b,i) \in (B^{\mathfrak{K}_i} \times \{i\})\} = \left(\bigcup_{1 \leq i \leq n} B^{\mathfrak{K}_i} \times \{i\}\right) \cup \bigcap_{1 \leq i \leq n} B^{\mathfrak{K}_i}$$

Therefore $B^{\mathfrak{K}} \cap M = M \cap \bigcap_{1 \leq i \leq n} B^{\mathfrak{K}_i} = (B^{\mathfrak{K}_1} \cap M) \cap \ldots \cap (B^{\mathfrak{K}_n} \cap M)$ as claimed. □

Finally, we turn to context families. Like for implications, $\vee$ is very easily computable by just taking the union of the separate context families.

Proposition 68. *Given n context families $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$, we let $\mathfrak{F}' = \mathfrak{F}_1 \cup \ldots \cup \mathfrak{F}_n$. Then, $(\cdot)^{\mathfrak{F}_1} \vee \ldots \vee (\cdot)^{\mathfrak{F}_n} \equiv (\cdot)^{\mathfrak{F}'}$ and $\#\mathfrak{F} = \#\mathfrak{F}_1 + \ldots + \#\mathfrak{F}_n$.*

Proof. Follows from the fact that for every context family $\mathfrak{F}$ holds $(\cdot)^{\mathfrak{F}} \equiv \bigvee_{\mathcal{F} \in \mathfrak{F}} (\cdot)^{\mathcal{F}}$. □

On the other hand, computing $\wedge$ turns out to require exponential blow-up, even if we compute it for only two context families.

Proposition 69. *There exist sequences $(\mathfrak{F}_n)_{n \in \mathbb{N}}$ and $(\mathfrak{F}'_n)_{n \in \mathbb{N}}$ of context families such that $\#\mathfrak{F}_n \in \Theta(n^3)$ and $\#\mathfrak{F}'_n \in \Theta(n)$ but for any sequence $(\mathfrak{F}''_n)_{n \in \mathbb{N}}$ of context families with $(\cdot)^{\mathfrak{F}''_n} = (\cdot)^{\mathfrak{F}_n} \wedge (\cdot)^{\mathfrak{F}'_n}$ holds $\#\mathfrak{F}''_n \in \Theta(2^n)$.*

Proof. This is a straightforward consequence from Proposition 50, using the fact that $(\cdot)^{\mathfrak{F}}\downarrow A \equiv (\cdot)^{\mathfrak{F}} \wedge (\cdot)^{\{\{A\}\}}$ □

5. Conclusion

In this paper we have investigated two archetypical and two more exotic representations of closure operators with respect to their mutual succinctness and their suitability for performing certain operations in terms of computation time and output size. The results are summarized in Table 2. Therein, for closure computation and comparison via $\leq$, upper bounds for the computation time are given in case poly-time algorithms exist, whereas "intractable" indicates coNP-hardness. For the other computations, the expressions give an upper bound on the output size in case a polynomial such bound exists (for all those cases, the computation time is linearly bounded by the output size), "exponential" denotes that exponential blow-up can be demonstrated. Note that for computation of n-ary $\bigwedge$ of context families, n must be considered fixed to ensure polynomiality.
There are many open questions left. On the theoretical side, central open questions are if – in the cases where an exponential blowup may occur – there are algorithms transforming one representation into another in *output polynomial time*, that is, if the time required for the computation is polynomially bounded by the size of the output. Note that a negative answer to this question would also disprove the existence of polynomial-delay algorithms.
On the practical side, coming back to our initial motivation, it should be experimentally investigated if variants of standard FCA algorithms can be improved by adding the option of working with alternative closure operator representations.

	context $\mathcal{F}$	implication set $\mathfrak{I}$	context family $\mathfrak{F}$	extended implication set $\mathfrak{K}$
closure	$\#\mathcal{F}$	$\#\mathfrak{I}$	$\#\mathfrak{F} \cdot \lvert M\rvert$	$\#\mathfrak{K}$
check $\leq$	$\#\mathcal{F} \cdot \#\mathcal{F}'/\lvert M\rvert$	$\#\mathfrak{I} \cdot \#\mathfrak{I}'/\lvert M\rvert$	intractable	intractable
project	$\#\mathcal{F}$	exponential	exponential	$\#\mathfrak{K}$
add implication	$(\#\mathcal{F})^2/\lvert M\rvert$	$\#\mathfrak{I} + \lvert M\rvert$	$\#\mathfrak{F} + \lvert M\rvert^3$	$\#\mathfrak{K} + \lvert N\rvert$
add closed set	$\#\mathcal{F} + \lvert M\rvert$	exponential	exponential	$2 \cdot \#\mathfrak{K} + \lvert\mathfrak{K}\rvert + 4\lvert N\rvert^2 + 2\lvert N\rvert$
n-ary $\bigwedge$	$\sum_i \#\mathcal{F}_i$	exponential n=2	exponential n=2	$(\sum_i \#\mathfrak{K}_i)^2$
n-ary $\bigvee$	exponential n=2	$\sum_i \#\mathfrak{I}_i$	$\sum_i \#\mathfrak{F}_i$	$(\sum_i \#\mathfrak{K}_i)^2$

Table 2: Upper bounds for computations with the four representation types.

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