On Composing Finite Forests with Modal Logics

BARTOSZ BEDNARCZYK, TU Dresden, Germany & University of Wrocław, Poland
STÉPHANE DEMRI, Université Paris-Saclay, ENS Paris-Saclay, CNRS, LMF, 91190, Gif-sur-Yvette, France
RAUL FERVARI, FAMAF, Universidad Nacional de Córdoba & CONICET, Argentina, & GTIIT, China
ALESSIO MANSUTTI, University of Oxford, UK & IMDEA Software Institute, Spain

We study the expressivity and complexity of two modal logics interpreted on finite forests and equipped with standard modalities to reason on submodels. The logic $\mathsf{ML}(|)$ extends the modal logic K with the composition operator $|$ from ambient logic, whereas $\mathsf{ML}(*)$ features the separating conjunction $*$ from separation logic. Both operators are second-order in nature. We show that $\mathsf{ML}(|)$ is as expressive as the graded modal logic GML (on trees) whereas $\mathsf{ML}(*)$ is strictly less expressive than GML. Moreover, we establish that the satisfiability problem is TOWER-complete for $\mathsf{ML}(*)$, whereas it is (only) $\mathrm{AExp}_{\mathrm{POL}}$-complete for $\mathsf{ML}(|)$, a result which is surprising given their relative expressivity. As by-products, we solve open problems related to sister logics such as static ambient logic and modal separation logic.

CCS Concepts: • **Theory of computation → Modal and temporal logics**.

Additional Key Words and Phrases: modal logic on trees, separation logic, static ambient logic, graded modal logic, expressive power, complexity

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1 INTRODUCTION

The ability to quantify over substructures to express properties of a model is often instrumental to perform modular and local reasoning. Two well-known examples are provided by separation logics [32, 41, 48], dedicated to reasoning on pointer programs, and ambient (or more generally, spatial) logics [11, 14, 16, 21], dedicated to reasoning on disjoint data structures. In the realm of modal logics dedicated to knowledge representation, submodel reasoning remains a key ingredient to express the dynamics of knowledge and belief, as done in the logics of public announcement [5, 37, 42], sabotage modal logics [4], refinement modal logics [13] and relation-changing logics [1–3]. Though the models may be of different nature (e.g. memory states for separation logics, epistemic models for logics of public announcement or finite edge-labelled trees for ambient logics), all those logics feature operators that enable to compose or decompose substructures in a very natural way.

From a technical point of view, reasoning about submodels requires a global analysis, unlike the local approach for classical modal and temporal logics (typically based on automata techniques [55, 56]). This makes the comparison between those formalisms quite challenging and often limited to

Authors' addresses: Bartosz Bednarczyk, TU Dresden, Germany & University of Wrocław, Poland, bartosz.bednarczyk@cs.uni.wroc.pl; Stéphane Demri, Université Paris-Saclay, ENS Paris-Saclay, CNRS, LMF, 91190, Gif-sur-Yvette, France, demri@lsv.fr; Raul Fervari, FAMAF, Universidad Nacional de Córdoba & CONICET, Argentina, & GTIIT, China, rfervari@unc.edu.ar; Alessio Mansutti, University of Oxford, UK & IMDEA Software Institute, Spain, alessio.mansutti@imdea.org.

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a superficial analysis on the different classes of models and composition operators. For instance, the composition operator $|$ in ambient logics decomposes a tree into two disjoint pieces such that once a node has been assigned to one submodel, all its descendants belong to the same submodel. Instead, the separating conjunction $*$ from separation logic decomposes the memory states into two disjoint memory states. Obviously, these and other well-known operators are closely related but no uniform framework investigates exhaustively their relationships in terms of expressive power.

Most of these logics can be easily encoded in monadic second-order logic MSO (or in second-order modal logics [27, 34]). Complexity-wise, if models are tree-like structures, we can then infer decidability thanks to the celebrated Rabin's theorem [46]. However, most likely, this does not produce the best decision procedures when it comes to solving simple reasoning tasks (e.g. the satisfiability problem of MSO is Tower-complete [49]). Thus, relying on MSO as a common umbrella to understand the differences between those logical formalisms is often not satisfactory.

Our motivations. Our intention in this work is to provide an in-depth comparison between the composition operator $|$ from static ambient logic [14] and the separating conjunction $*$ from separation logics [48] by identifying common ground in terms of logical languages and models. As a consequence, we are able to study the effects of having these operators as far as expressivity and complexity are concerned. We aim at defining two logics whose only differences rest on their use of $|$ and $*$ syntactically and semantically (by considering the adequate composition operation). To do so, we pick as our common class of models, the Kripke-style finite trees (actually finite forests, so that the class is closed under taking submodels), which provides a ubiquitous class of structures, intensively studied in computer science. For the underlying logical language (i.e. apart from $|$ or $*$), we advocate the use of the standard modal logic K (i.e. to have Boolean connectives and the modality $\Diamond$) so that the main operations on the models amount to quantifying over submodels or to moving along the edges. The generality of this framework enables us to take advantage of model theoretical tools from modal logics [6, 10, 22]. The benefits of settling common ground for comparison may lead to further comparisons with other logics and to new results.

Our contributions. We introduce $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$, two logics interpreted on Kripke-style forest models. The logic $\mathsf{ML}(|)$ features the standard modality $\Diamond$ and the composition operator $|$ from static ambient logic [14]; whereas $\mathsf{ML}(*)$ puts together the modality $\Diamond$ with the separating conjunction $*$ from separation logic [48]. Both logical formalisms can state non-trivial properties about submodels, but the binary modalities $|$ and $*$ operate differently: whereas $*$ is able to decompose the models at any depth, $|$ is much less permissive as the decomposition is completely determined by what happens at the level of the children of the current node. We study their expressive power and complexity, obtaining surprising results. We show that $\mathsf{ML}(|)$ is as expressive as the graded modal logic GML [6, 52] (Theorem 3.7) whereas $\mathsf{ML}(*)$ is strictly less expressive than GML (Theorem 5.6). Interestingly, this latter development partially reuses the result for $\mathsf{ML}(|)$, hence showing how our framework allows us to transpose results between the two logics. To show that GML is strictly more expressive than $\mathsf{ML}(*)$, we define Ehrenfeucht-Fraïssé games for $\mathsf{ML}(*)$. In terms of complexity, the satisfiability problem for $\mathsf{ML}(|)$ is shown $\mathrm{AExp}_{\mathrm{Pol}}$-complete[1] (Corollary 3.12), interestingly the same complexity as for the refinement modal logic RML [13] handling a quantifier over refinements (generalising the submodel construction). The $\mathrm{AExp}_{\mathrm{Pol}}$ upper bound follows from an exponential-size model property (Lemma 3.9), whereas the lower bound is by reducing the satisfiability problem for an $\mathrm{AExp}_{\mathrm{Pol}}$-complete team logic [30]. Much more surprisingly, although $\mathsf{ML}(*)$ is strictly less expressive than $\mathsf{ML}(|)$, its complexity is much higher (not even elementary). Precisely, we show

[1] Problems in $\mathrm{AExp}_{\mathrm{Pol}}$ are decidable by an alternating Turing machine working in exponential-time and using polynomially many alternations [12].

that the satisfiability problem for $\mathsf{ML}(*)$ is Tower-complete (Theorem 4.34). The Tower upper bound is a consequence of [46], as $\mathsf{ML}(*)$ is a fragment of MSO. Hardness is shown by reduction from a Tower-complete tiling problem, adapting substantially the Tower-hardness proof from [7] for second-order modal logic K on finite trees, see also a similar method used in [43]. To conclude, we get the best of our results on $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$ to solve several open problems. We relate $\mathsf{ML}(|)$ with an intensional fragment of static ambient logic $\mathsf{SAL}(|)$ from [14] by providing polynomial-time reductions between their satisfiability problems. Consequently, we establish $\textsc{AExp}_{\textsc{Pol}}$-completeness of $\mathsf{SAL}(|)$ (Corollary 6.6), refuting hints from [14, Section 6]. Similarly, we show that the modal separation logic $\mathsf{MSL}(\Diamond^{-1}, *)$ from [23] is Tower-complete (Corollary 7.3).

The following table states the main results of the paper, illustrating the relations in terms of expressivity and complexity between the logics for composing forests.

	$\mathsf{ML}(\mid)$	$\mathsf{ML}(*)$
Expressive Power	Graded Modal Logic (GML)	$<$ GML
Complexity (satisfiability problem)	$\textsc{AExp}_{\textsc{Pol}}$-complete	Tower-complete

This paper is a revised and completed version of the conference paper [8]. Omitted proofs can be found in the Electronic Appendix of the paper or in [9, 40].

2 PRELIMINARIES

In this section, we introduce the logics $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$ interpreted on tree-like structures equipped with operators to split the structure into disjoint pieces. Due to the presence of such operators, we are required to consider a class of models that is closed under submodels, which we call Kripke-style finite forests (or finite forests for short).

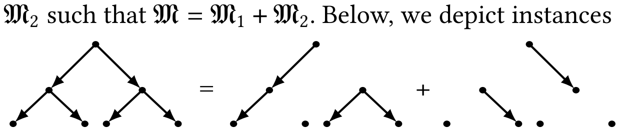
Let AP be a countably infinite set of *atomic propositions*. A *(Kripke-style) finite forest* is a triple $\mathfrak{M} = (W, R, V)$ where W is a non-empty finite set of *worlds*, $V : \mathrm{AP} \to \mathcal{P}(W)$ is a *valuation* and $R \subseteq W \times W$ is a binary relation whose inverse R^{-1} is functional and acyclic. In particular, the graph described by (W, R) is a finite collection of disjoint finite trees, where R encodes the child relation. We define $R(w) \overset{\text{def}}{=} \{w' \in W \mid (w, w') \in R\}$. Worlds in $R(w)$ are understood as *children* of w. We inductively define R^n as $R^0 \overset{\text{def}}{=} \{(w, w) \mid w \in W\}$ and $R^{n+1} \overset{\text{def}}{=} \{(w, w'') \mid \exists w'\, (w, w') \in R^n$ and $(w', w'') \in R\}$. Moreover, R^+ denotes the transitive closure of R.

We define operators that chop a finite forest. It should be noted that these operators, as well as the resulting logics, can be cast under the umbrella of the logic of bunched implications BI [28, 45], with the exception that we do not explicitly require them to have an identity element (as enforced on the multiplicative operators of BI, see [28]). Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}_i = (W_i, R_i, V_i)$ (for $i \in \{1, 2\}$) be three finite forests.

The separation logic composition. We introduce the binary operator $+$ that performs the disjoint union at the level of parent-child relation. Formally,

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 \overset{\text{def}}{\Leftrightarrow} R_1 \uplus R_2 = R,\ W_1 = W_2 = W,\ V_1 = V_2 = V.$$

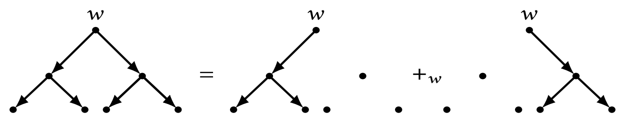
This is the composition used in separation logic [23, 48]. We say that $\mathfrak{M}_1$ is a *submodel* of $\mathfrak{M}$, written $\mathfrak{M}_1 \sqsubseteq \mathfrak{M}$, if there is $\mathfrak{M}_2$ such that $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$. Below, we depict instances for $\mathfrak{M}$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$.



The ambient logic composition. We introduce the operator $+_w$, where $w \in W$, refining $+$:

$$\mathfrak{M} = \mathfrak{M}_1 +_w \mathfrak{M}_2 \overset{\text{def}}{\Leftrightarrow} \mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 \text{ and, for all } i \in \{1, 2\} \text{ and } w' \in R_i(w),\ R_i^+(w') = R^+(w').$$

The finite forest $\mathfrak{M}$ decomposed with $+_w$ is understood as a disjoint union between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ except that, as soon as $w' \in R_i(w)$, the whole subtree of w' in R belongs to $\mathfrak{M}_i$, like the composition in ambient logic [14]. Below, we illustrate a finite forest decomposed with $+_w$.



Modal logics on trees. The logic $\mathsf{ML}(|)$ enriches the *basic modal logic* ML with a binary connective $|$, called *composition operator*, that admits submodel reasoning via the operator $+_w$. Similarly, $\mathsf{ML}(*)$ enriches ML with the connective $*$, called *separating conjunction* (or *star*) that admits submodel reasoning via the operator $+$. Both connectives $|$ and $*$ are understood as binary modalities. As we show throughout the paper, $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$ are strongly related to the graded modal logic GML [22]. For conciseness, let us define all these logics by considering formulae that contain all of their ingredients. These formulae are built from the grammar below:

$$\varphi := \top \mid p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \Diamond_{\geq k}\,\varphi \mid \varphi * \varphi \mid \varphi | \varphi,$$

where $p \in \mathrm{AP}$ and $k \in \mathbb{N}$ (encoded in unary). A *pointed forest* $(\mathfrak{M}, w)$ is a finite forest $\mathfrak{M} = (W, R, V)$ together with a world $w \in W$. The satisfaction relation $\models$ is defined as follows (standard clauses for $\wedge$, $\neg$ and $\top$ are omitted):

$\mathfrak{M}, w \models p \quad \Leftrightarrow w \in V(p)$;

$\mathfrak{M}, w \models \Diamond\varphi \quad \Leftrightarrow$ there is $w' \in R(w)$ such that $\mathfrak{M}, w' \models \varphi$;

$\mathfrak{M}, w \models \Diamond_{\geq k}\,\varphi \quad \Leftrightarrow |\{w' \in R(w) \mid \mathfrak{M}, w' \models \varphi\}| \geq k$;

$\mathfrak{M}, w \models \varphi_1 * \varphi_2 \quad \Leftrightarrow$ there are $\mathfrak{M}_1, \mathfrak{M}_2$ such that $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$, $\mathfrak{M}_1, w \models \varphi_1$ and $\mathfrak{M}_2, w \models \varphi_2$;

$\mathfrak{M}, w \models \varphi_1 | \varphi_2 \quad \Leftrightarrow$ there are $\mathfrak{M}_1, \mathfrak{M}_2$ such that $\mathfrak{M} = \mathfrak{M}_1 +_w \mathfrak{M}_2$, $\mathfrak{M}_1, w \models \varphi_1$ and $\mathfrak{M}_2, w \models \varphi_2$.

The formulae $\varphi \Rightarrow \psi$, $\varphi \vee \psi$ and $\perp$ are defined as usual. We use the following standard abbreviations: $\Box\varphi \stackrel{\text{def}}{=} \neg\Diamond\neg\varphi$, $\Diamond_{\leq k}\,\varphi \stackrel{\text{def}}{=} \neg\Diamond_{\geq k+1}\,\varphi$ and $\Diamond_{=k}\,\varphi \stackrel{\text{def}}{=} \Diamond_{\geq k}\,\varphi \wedge \Diamond_{\leq k}\,\varphi$. Notice that both $|$ and $*$ are associative operators (we will use this fact implicitely in the rest of the paper). We write $\mathsf{size}(\varphi)$ to denote the *size* of φ with a tree representation of formulae and with a reasonably succinct encoding of atomic formulae. Besides, we write $\mathsf{md}(\varphi)$ to denote the *modal degree* of φ understood as the maximal number of nested unary modalities (i.e. $\Diamond$ or $\Diamond_{\geq k}$) in φ. Similarly, the *graded rank* $\mathsf{gr}(\varphi)$ of φ is defined as $\max(\{k \mid \Diamond_{\geq k}\,\psi \in \mathsf{subf}(\varphi)\} \cup \{0\})$, where $\mathsf{subf}(\varphi)$ is the set of all the subformulae of φ.

Given the formulae φ and ψ, $\varphi \equiv \psi$ denotes that φ and ψ are *logically equivalent*; i.e., for every pointed forest $(\mathfrak{M}, w)$, $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}, w \models \psi$. For instance ($k \geq 1$ and $p \in \mathrm{AP}$):

(1). $\Diamond\varphi \equiv \Diamond_{\geq 1}\,\varphi$; (2). $(\Box\Box\perp | \Box\Box\perp) \not\equiv (\Box\Box\perp * \Box\Box\perp)$;

(3). $\Diamond_{\geq k}\,p \equiv \underbrace{\Diamond p * \cdots * \Diamond p}_{k \text{ times}}$; (4). $\Diamond_{\geq k}\,\varphi \equiv \underbrace{\Diamond\varphi | \cdots | \Diamond\varphi}_{k \text{ times}}$.

The modal logic ML is the logic restricted to formulae with the unique modality $\Diamond$ [10]. Similarly, the graded modal logic GML is restricted to the *graded modalities* $\Diamond_{\geq k}$ [22]. We introduce the modal logics $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$, which are restricted to the suites of modalities $(\Diamond, |)$ and $(\Diamond, *)$, respectively. The two equivalences (3) and (4) already shed some light on $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$: the two logics are similar when it comes to their formulae of modal degree one (as (3) does not generalise to arbitrary formulae).

Lemma 2.1. *Let φ be a formula in $\mathsf{ML}(|)$ with $\mathsf{md}(\varphi) \leq 1$. Then, $\varphi \equiv \varphi[| \leftarrow *]$ where $\varphi[| \leftarrow *]$ is the formula in $\mathsf{ML}(*)$ obtained from φ by replacing every occurrence of $|$ by $*$.*

The proof of Lemma 2.1 can be found in Appendix A. However, as shown by the non-equivalence (2) above, it is unclear how the two logics compare when it comes to formulae of modal degree greater than one. Indeed, since $\mathfrak{M} = \mathfrak{M}_1 +_w \mathfrak{M}_2$ implies $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$ (in formula, $\varphi \,|\, \psi \Rightarrow \varphi * \psi$ is valid) but not vice-versa, the separating conjunction $*$ is more permissive than the operator $|$. However, further connections between the two operators can be easily established. Let us introduce the auxiliary operator $\blacklozenge$ defined as $\blacklozenge\varphi \stackrel{\text{def}}{=} \varphi * \Box\bot$. Formally,

$$(W, R, V), w \models \blacklozenge\varphi \;\Leftrightarrow \text{there is } R' \subseteq R \text{ such that } R'(w) = R(w) \text{ and } (W, R', V), w \models \varphi.$$

Similar operators are studied in [2, 4, 13]. We show that $\blacklozenge$ and $|$ are sufficient to capture $*$ (essential property for Section 5).

Lemma 2.2. *Let $\varphi, \psi \in \mathsf{GML}$. We have $\varphi * \psi \equiv \blacklozenge(\varphi \,|\, \psi)$.*

The proof of Lemma 2.2 can be found in Appendix B. Unlike $|$, when $*$ splits a finite forest $\mathfrak{M}$ into $\mathfrak{M}_1$ and $\mathfrak{M}_2$, it may disconnect in both submodels worlds that are otherwise reachable, from the current world, in $\mathfrak{M}$. Applying $\blacklozenge$ before $|$ allows us to imitate this behaviour. Indeed, even though $|$ preserves reachability in either $\mathfrak{M}_1$ or $\mathfrak{M}_2$, $\blacklozenge$ deletes part of $\mathfrak{M}$, making some world inaccessible. This way of expressing the separating conjunction allows us to reuse some methods developed for $\mathsf{ML}(|)$ in order to study $\mathsf{ML}(*)$.

The logic QK^t. Both $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$ can be seen as fragments of the logic QK^t, which in turn is known to be a fragment of monadic second-order logic on trees [7]. The logic QK^t extends ML with second-order quantification and is interpreted on finite trees. Its formulae are defined according to the following grammar: $\varphi := p \mid \Diamond\varphi \mid \varphi \land \varphi \mid \neg\varphi \mid \exists p\, \varphi$. Given $\mathfrak{M} = (W, R, V)$ and $w \in W$, the satisfaction relation $\models$ of ML is extended as follows:

$$\mathfrak{M}, w \models \exists p\, \varphi \;\Leftrightarrow\; \text{there is } \exists W' \subseteq W \text{ such that } (W, R, V[p \leftarrow W']), w \models \varphi.$$

One can show logspace reductions from $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$ to QK^t, by simply reinterpreting the operators $*$ and $|$ as restrictive forms of second-order quantification, and by relativising $\Diamond$ to appropriate propositional symbols in order to capture the notion of submodel (details are omitted). Consequently, Tower-hardness of the satisfiability problem for $\mathsf{ML}(*)$ proved in Section 4 entails the Tower-hardness of QK^t, refining the proof for QK^t in [7].

Expressive power. Given two logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$, we say that $\mathfrak{L}_2$ is *at least as expressive as* $\mathfrak{L}_1$ (written $\mathfrak{L}_1 \preceq \mathfrak{L}_2$) whenever for every formula φ of $\mathfrak{L}_1$, there is a formula ψ of $\mathfrak{L}_2$ such that $\varphi \equiv \psi$. $\mathfrak{L}_1 \approx \mathfrak{L}_2$ denotes that $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are *equally expressive*, i.e. $\mathfrak{L}_1 \preceq \mathfrak{L}_2$ and $\mathfrak{L}_2 \preceq \mathfrak{L}_1$. Lastly, $\mathfrak{L}_1 \prec \mathfrak{L}_2$ denotes that $\mathfrak{L}_2$ is *strictly more expressive* than $\mathfrak{L}_1$, i.e. $\mathfrak{L}_1 \preceq \mathfrak{L}_2$ and $\mathfrak{L}_1 \not\approx \mathfrak{L}_2$. The equivalence (1) recalls us that $\mathsf{ML} \prec \mathsf{GML}$ [22]. From the equivalence (4), we get $\mathsf{GML} \preceq \mathsf{ML}(|)$.

Satisfiability problem. The *satisfiability problem* for a logic $\mathfrak{L}$, written $\mathrm{Sat}(\mathfrak{L})$, takes as input a formula φ in $\mathfrak{L}$ and checks whether there is a pointed forest $(\mathfrak{M}, w)$ such that $\mathfrak{M}, w \models \varphi$.

Note that any $\mathfrak{L}$ among ML, GML, $\mathsf{ML}(|)$ or $\mathsf{ML}(*)$ has the tree model property, i.e. any satisfiable formula is also satisfied in some tree structure. The problems $\mathrm{Sat}(\mathsf{ML})$ and $\mathrm{Sat}(\mathsf{GML})$ are known to be PSpace-complete, see e.g. [10, 25, 33, 50, 52], and therefore $\mathrm{Sat}(\mathsf{ML}(|))$ and $\mathrm{Sat}(\mathsf{ML}(*))$ are PSpace-hard. Note that $\mathrm{Sat}(\mathsf{GML})$ is PSpace-complete even when the numbers k appearing in graded modalities $\Diamond_{\geq k}$ are encoded in binary. However, we stress the fact that in this paper we consider k to be encoded in unary, as it better matches the definition of $\Diamond_{\geq k}$ in $\mathsf{ML}(|)$ given in (4). As an upper bound, by Rabin's theorem [46], the satisfiability problem for QK^t is decidable in Tower, which transfers directly to $\mathrm{Sat}(\mathsf{ML}(|))$ and $\mathrm{Sat}(\mathsf{ML}(*))$.

3 $\mathsf{ML}(\mathbin{|})$: EXPRESSIVENESS AND COMPLEXITY

In this section, we study the expressive power of $\mathsf{ML}(\mathbin{|})$ and the complexity of its satisfiability problem. We start by constructively showing that $\mathsf{ML}(\mathbin{|}) \preceq \mathsf{GML}$, hence proving $\mathsf{ML}(\mathbin{|}) \approx \mathsf{GML}$. Then, we study its computational complexity for which we establish that $\mathrm{Sat}(\mathsf{ML}(\mathbin{|}))$ is $\mathrm{AExp}_{\mathrm{Pol}}$-complete. We recall that $\mathrm{AExp}_{\mathrm{Pol}}$ denotes the complexity class of those problems decided by exponential-time bounded alternating Turing Machines using a polynomially bounded number of alternations. A problem P is $\mathrm{AExp}_{\mathrm{Pol}}$-complete if it is in $\mathrm{AExp}_{\mathrm{Pol}}$ and every problem in $\mathrm{AExp}_{\mathrm{Pol}}$ can be reduced to P under polynomial-time reductions.

The $\mathrm{AExp}_{\mathrm{Pol}}$ upper bound for $\mathsf{ML}(\mathbin{|})$ follows from an exponential-size model property. The lower bound is by reduction from the satisfiability problem for propositional team logic [30, Thm. 4.9].

3.1 A disjoint form for graded modal logic

The method for establishing $\mathsf{ML}(\mathbin{|}) \preceq \mathsf{GML}$ relies on the fact that GML is closed under the operator $\mathbin{|}$. We show that given two formulae φ_1 and φ_2 in GML, one can construct a formula ψ in GML such that $\varphi_1 \mathbin{|} \varphi_2 \equiv \psi$. For instance, a simple case analysis yields $(p \vee \Diamond_{\geq 3}\, r) \mathbin{|} (q \vee \Diamond_{\leq 5}\, q) \equiv (p \vee \Diamond_{\geq 3}\, r)$. With this closure property at hand, the general algorithm consists in iteratively replacing innermost subformulae of the form $\varphi_1 \mathbin{|} \varphi_2$ by a counterpart in GML, allowing us to eliminate all the occurrences of $\mathbin{|}$ and obtain an equivalent formula in GML. In order to establish the closure property, we first put the GML formulae φ_1 and φ_2 in a *disjoint form*, a normal form that is introduced in this section alongside other useful definitions.

Let φ be a formula in GML. We write $\mathtt{max}_{\mathtt{PC}}(\varphi)$ for the set of atomic propositions of φ that appear at least once outside the scope of a graded modality. Similarly, $\mathtt{max}_{\mathtt{GM}}(\varphi)$ denotes the set of subformulae ψ of φ such that ψ is of the form $\Diamond_{\geq k}\, \psi'$ and one of its occurrences in φ is not in the scope of any graded modality. For instance, given $\varphi = (p \vee \Diamond_{\geq 3}\, r) \wedge (q \vee \Diamond_{\geq 5}\, \Diamond_{\geq 2}\, q)$,

$$\mathtt{max}_{\mathtt{PC}}(\varphi) = \{p, q\} \qquad\qquad \mathtt{max}_{\mathtt{GM}}(\varphi) = \{\Diamond_{\geq 3}\, r, \Diamond_{\geq 5}\, \Diamond_{\geq 2}\, q\}.$$

Clearly, every formula φ in GML is a Boolean combination of formulae from $\mathtt{max}_{\mathtt{PC}}(\varphi) \cup \mathtt{max}_{\mathtt{GM}}(\varphi)$. Given a natural number $d \in \mathbb{N}$, we extend the notion of $\mathtt{max}_{\mathtt{GM}}(\varphi)$ and write $\mathtt{gm}(d, \varphi)$ to denote the set of subformulae of φ of the form $\Diamond_{\geq k}\, \psi$ occurring under the scope of exactly d nested graded modalities. Formally,

$$\mathtt{gm}(0, \varphi) \stackrel{\text{def}}{=} \mathtt{max}_{\mathtt{GM}}(\varphi), \qquad\qquad \mathtt{gm}(d+1, \varphi) \stackrel{\text{def}}{=} \bigcup_{\Diamond_{\geq k}\, \psi \in \mathtt{max}_{\mathtt{GM}}(\varphi)} \mathtt{gm}(d, \psi).$$

For simplicity, we also write $C_\wedge(\varphi_1, \ldots, \varphi_n) = \{\gamma_1 \wedge \cdots \wedge \gamma_n \mid \text{ for all } \ i \in [1, n], \gamma_i \in \{\varphi_i, \neg\varphi_i\}\}$ for the set of all complete conjunctions of (possibly negated) formulae $\varphi_1, \ldots, \varphi_n$. The disjoint form for formulae in GML is defined as follows.

Definition 3.1. A formula φ in GML is said to be in *disjoint form* if for every $d \in [0, \mathtt{md}(\varphi)]$ and all $\Diamond_{\geq k}\, \psi, \Diamond_{\geq k'}\, \psi' \in \mathtt{gm}(d, \varphi)$, either $\psi \equiv \psi'$ or the conjunction $\psi \wedge \psi'$ is unsatisfiable.

The lemma below leads to an inductive procedure to put every GML formula into disjoint form.

Lemma 3.2. *Let φ be a formula in GML and $\mathtt{max}_{\mathtt{GM}}(\varphi) \subseteq \{\Diamond_{\geq k_1}\, \psi_1, \ldots, \Diamond_{\geq k_n}\, \psi_n\}$ such that $\psi_1 \wedge \cdots \wedge \psi_n$ is in disjoint form. Let $\overline{k} = \max\{k_1, \ldots, k_n\}$. There is a GML formula φ' in disjoint form logically equivalent to φ and such that $\mathtt{max}_{\mathtt{GM}}(\varphi') \subseteq \{\Diamond_{\geq k}\, \chi \mid k \in [0, \overline{k}]$ and $\chi \in C_\wedge(\psi_1, \ldots, \psi_n)\}$ and $\mathtt{max}_{\mathtt{PC}}(\varphi') \subseteq \mathtt{max}_{\mathtt{PC}}(\varphi)$.*

Proof. The assumption that $\psi_1 \wedge \cdots \wedge \psi_n$ is in disjoint form implies that for every $d \in [1, \mathtt{md}(\varphi)]$ and every $\Diamond_{\geq k}\, \psi, \Diamond_{\geq k'}\, \psi' \in \mathtt{gm}(d, \varphi)$, either $\psi \equiv \psi'$ or the conjunction $\psi \wedge \psi'$ is unsatisfiable. Therefore, to construct φ' it is sufficient to manipulate the formulae of $\mathtt{gm}(0, \varphi) = \mathtt{max}_{\mathtt{GM}}(\varphi)$, without modifying the set $\mathtt{gm}(1, \varphi)$. We do so by using axioms from GML [6] as well as the equivalences:

$$\textbf{(guess)}\quad \Diamond_{\geq k}\,\varphi \equiv \Diamond_{\geq k}\,\big((\varphi \land \psi) \lor (\varphi \land \neg\psi)\big),$$
$$(\Diamond_{\geq k}\,\textbf{distr})\quad \text{if } \varphi \land \psi \text{ is unsatisfiable, } \Diamond_{\geq k}\,(\varphi \lor \psi) \equiv \textstyle\bigvee_{k=k_1+k_2}(\Diamond_{\geq k_1}\varphi \land \Diamond_{\geq k_2}\psi).$$

Notice that the two disjuncts $\varphi \land \psi$ and $\varphi \land \neg\psi$ in the right-hand side of (guess) are such that their conjunction is unsatisfiable, enabling us to use ($\Diamond_{\geq k}$ distr).

We manipulate each $\Diamond_{\geq k_j}\,\psi_j \in \mathtt{max}_{\mathtt{GM}}(\varphi)$ separately. Let $j \in [1, n]$. Consider the set of formulae $\mathcal{G} = C_\land(\psi_1, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_n)$. By propositional reasoning and by applying (guess) $n-1$ times:

$$\Diamond_{\geq k_j}\,\psi_j \equiv \Diamond_{\geq k_j}\,\textstyle\bigvee_{(\chi_1 \land \cdots \land \chi_{j-1} \land \chi_{j+1} \land \cdots \land \chi_n) \in \mathcal{G}}\big(\chi_1 \land \cdots \land \chi_{j-1} \land \psi_j \land \chi_{j+1} \land \cdots \land \chi_n\big).$$

Let $\mathcal{D}$ be the set of functions $d\colon \mathcal{G} \to [0, k_j]$ assigning to each formula of $\mathcal{G}$ a number in $[0, k_j]$, such that $k_j = \sum_{\gamma \in \mathcal{G}} d(\gamma)$. By relying on ($\Diamond_{\geq k}$ distr), we obtain $\Diamond_{\geq k_j}\,\psi_j \equiv \psi'_j$ where

$$\psi'_j \stackrel{\text{def}}{=} \bigvee_{d \in \mathcal{D}} \textstyle\bigwedge_{(\chi_1 \land \cdots \land \chi_{j-1} \land \chi_{j+1} \land \cdots \land \chi_n) = \gamma \in \mathcal{G}} \Diamond_{\geq d(\gamma)}\;\big(\chi_1 \land \ldots \chi_{j-1} \land \psi_j \land \chi_{j+1} \land \cdots \land \chi_n\big).$$

Let φ' be the formula obtained from φ by replacing with ψ'_j every occurrence of $\Diamond_{\geq k_j}\,\psi_j$ not appearing under the scope of graded modalities. By definition of $\mathcal{G}$ and $\mathcal{D}$, the formula φ' satisfies all the expected properties. □

Lemma 3.3. *Let φ in* GML. *There is a* GML *formula φ' in disjoint form such that $\varphi' \equiv \varphi$.*

Proof. Use Lemma 3.2 bottom-up, from formulae in $\mathtt{gm}(\mathtt{md}(\varphi)-1, \varphi)$ to formulae in $\mathtt{gm}(0, \varphi)$. □

When discussing the exponential-size model property for $\mathsf{ML}(|)$, we are interested in the size of the smallest pointed forest satisfying a GML formula already given in disjoint form. To this end, we need to introduce one last notion: the *branching degree* of a formula. Let φ be a formula GML, with $\mathtt{max}_{\mathtt{GM}}(\varphi) = \{\Diamond_{\geq k_1}\,\psi_1, \ldots, \Diamond_{\geq k_n}\,\psi_n\}$. We define $\mathtt{bd}(0, \varphi) \stackrel{\text{def}}{=} k_1 + \cdots + k_n$ and, for all $m \geq 0$, $\mathtt{bd}(m+1, \varphi) \stackrel{\text{def}}{=} \max\{\mathtt{bd}(m, \psi) \mid \Diamond_{\geq k}\,\psi \in \mathtt{max}_{\mathtt{GM}}(\varphi)\}$. Hence, $\mathtt{bd}(m, \varphi)$ can be understood as the maximal $\mathtt{bd}(0, \psi)$ for some subformula ψ occurring at the modal depth m within φ. We write $\mathtt{max}_{\mathtt{bd}}(\varphi) \stackrel{\text{def}}{=} \max\{\mathtt{bd}(m, \varphi) \mid m \in [0, \mathtt{md}(\varphi)]\}$ for the *branching degree* of φ.

Lemma 3.4. *Every satisfiable* GML *formula φ in disjoint form is satisfied by a pointed forest with at most $(\mathtt{max}_{\mathtt{bd}}(\varphi) + 1)^{\mathtt{md}(\varphi)}$ worlds.*

Proof. The proof follows with a straightforward induction on the modal degree of φ.

base case: $\mathtt{md}(\varphi) = 0$. In this case, φ is a Boolean combination of atomic propositions, and thus the satisfaction of φ can be witnessed on a pointed forest with one single world (i.e. the satisfaction of φ only depends on the atomic propositions satisfied by the current world).

induction step: $\mathtt{md}(\varphi) = d + 1$. By propositional reasoning, there is a GML formula φ' in disjoint form such that $\varphi \equiv \varphi'$ and φ' is a disjunction of conjunctions of possibly negated formulae from $\mathtt{max}_{\mathtt{GM}}(\varphi) \cup \mathtt{max}_{\mathtt{PC}}(\varphi)$. Since φ is satisfiable and $\varphi \equiv \varphi'$, one of the disjuncts of φ' must be satisfiable. Let χ be such a disjunct, which is a conjunction of the form:

$$\chi \;=\; \Diamond_{\geq k_1}\psi_1 \land \ldots \land \Diamond_{\geq k_n}\psi_n \land \neg\Diamond_{\geq j_1}\psi'_1 \land \ldots \land \neg\Diamond_{\geq j_m}\psi'_m \land L_1 \land \cdots \land L_r,$$

where $\{\Diamond_{\geq k_i}\psi_i \mid i \in [1, n]\} \cup \{\Diamond_{\geq j_i}\psi'_i \mid i \in [1, m]\} \subseteq \mathtt{max}_{\mathtt{GM}}(\varphi)$ and $L_1, \ldots, L_r$ are literals built upon $\mathtt{max}_{\mathtt{PC}}(\varphi)$. Since $\mathtt{max}_{\mathtt{GM}}(\chi) \subseteq \mathtt{max}_{\mathtt{GM}}(\varphi)$ we have $\mathtt{max}_{\mathtt{bd}}(\chi) \leq \mathtt{max}_{\mathtt{bd}}(\varphi)$, $\mathtt{md}(\chi) \leq \mathtt{md}(\varphi)$ and χ is in disjoint form. Without loss of generality, we can assume each k_i, with $i \in [1, n]$, to be at least 1. Indeed, formulae of the form $\Diamond_{\geq 0}\,\psi$ are valid and can be replaced with $\top$. From the satisfiability of χ, we conclude that for all $i \in [1, n]$ and $r \in [1, m]$ if $\psi_i \equiv \psi'_r$ then $k_i < j_r$. We consider a set $\mathcal{R} = \{\Diamond_{\geq \widetilde{k}_1}\gamma_1, \ldots, \Diamond_{\geq \widetilde{k}_q}\gamma_q\}$ of representative formulae for $\{\Diamond_{\geq k_1}\psi_1, \ldots, \Diamond_{\geq k_n}\psi_n\}$, i.e. $\mathcal{R}$ is a subset of $\{\Diamond_{\geq k_1}\psi_1, \ldots, \Diamond_{\geq k_n}\psi_n\}$ such that for every $i \in [1, n]$, there is exactly one $j \in [1, q]$ such that $\psi_i \equiv \gamma_j$, and in that case $\widetilde{k}_j \geq k_i$. Since χ is in disjoint form and satisfiable and each k_i ($i \in [1,n]$) is assumed to be at least 1, we conclude that every formula in $\mathcal{R}$ is satisfiable, and for all $i \neq j \in [1,q]$, $\gamma_i \wedge \gamma_j$ is unsatisfiable. Then, constructing a model for χ becomes straightforward: by induction hypothesis, for every $i \in [1,q]$ there is a pointed forest $(\mathfrak{M}_i, w_i)$ with at most $(\mathtt{max}_{\mathtt{bd}}(\gamma_i)+1)^{\mathtt{md}(\gamma_i)}$ worlds that satisfy γ_i. Let us pick $\widetilde{k}_i$ copies $(\mathfrak{M}_{1,i}, w_{1,i}), \ldots, (\mathfrak{M}_{\widetilde{k}_i,i}, w_{\widetilde{k}_i,i})$ of the pointed forest $(\mathfrak{M}_i, w_i)$, constructed over distinct sets of worlds. For all $i \in [1,m]$ and $c \in [1,\widetilde{k}_i]$, let $\mathfrak{M}_{c,i} = (W_{c,i}, R_{c,i}, V_{c,i})$. Let us consider the finite forest $\mathfrak{M} = (W,R,V)$ defined as

- $W \stackrel{\text{def}}{=} \{w\} \cup \bigcup_{i \in [1,q]} \bigcup_{c \in [1,\widetilde{k}_i]} W_{c,i}$, where w is a fresh world not appearing in any $W_{c,i}$,
- $R = \{(w, w_{c,i}) \mid i \in [1,m],\ c \in [1,\widetilde{k}_i]\} \cup \bigcup_{i \in [1,q]} \bigcup_{c \in [1,\widetilde{k}_i]} R_{c,i}$,
- for every atomic proposition p appearing in φ, for every $i \in [1,q]$, $c \in [1,\widetilde{k}_i]$ and $w' \in W_{c,i}$, $w' \in V(p)$ if and only if $w' \in V_{c,i}(p)$,
- for every $p \in \mathtt{max}_{\mathtt{PC}}(\varphi)$, $w \in V(p)$ if and only if p occurs positively in $L_1 \wedge \cdots \wedge L_r$.

We have $\mathfrak{M}, w \models \chi$. Indeed, $\mathfrak{M}, w \models L_1 \wedge \cdots \wedge L_r$ holds by definition of V, whereas $\mathfrak{M}, w \models \Diamond_{\geq k_1}\psi_1 \wedge \cdots \wedge \Diamond_{\geq k_n}\psi_n$ holds directly from the definition of $\mathcal{R}$ together with the definition of the various $(\mathfrak{M}_{c,i}, w_{c,i})$ with $i \in [1,q]$ and $c \in [1,\widetilde{k}_i]$. Similarly, $\mathfrak{M}, w \models \neg\Diamond_{\geq j_1}\psi'_1 \wedge \ldots \wedge \neg\Diamond_{\geq j_m}\psi'_m$ holds by definition of $\mathcal{R}$ together with the satisfiability of χ, which implies that for all $i \in [1,n]$ and $r \in [1,m]$ if $\psi_i \equiv \psi'_r$ then $k_i < j_r$.
Space-wise, by definition of $\mathcal{R}$, $\sum_{i=1}^{q} \widetilde{k}_i \leq \sum_{i=1}^{n} k_i \leq \mathtt{bd}(0,\chi) \leq \mathtt{max}_{\mathtt{bd}}(\varphi)$. Let $|W_i|$ be the number of worlds in $\mathfrak{M}_i$. The number of worlds in W is

$$\begin{aligned} |W| = 1 + \textstyle\sum_{i=1}^{q} \widetilde{k}_q \cdot |W_i| &\leq 1 + \textstyle\sum_{i=1}^{q} \widetilde{k}_i \cdot (\mathtt{max}_{\mathtt{bd}}(\gamma_i) + 1)^{\mathtt{md}(\chi_i)} \\ &\leq 1 + (\mathtt{max}_{\mathtt{bd}}(\varphi) + 1)^{\mathtt{md}(\varphi)-1} \cdot \textstyle\sum_{i=1}^{q} \widetilde{k}_i \\ &\leq 1 + (\mathtt{max}_{\mathtt{bd}}(\varphi) + 1)^{\mathtt{md}(\varphi)-1} \cdot \mathtt{max}_{\mathtt{bd}}(\varphi) \leq (\mathtt{max}_{\mathtt{bd}}(\varphi) + 1)^{\mathtt{md}(\varphi)} \end{aligned}$$ □

3.2 ML(|) is as expressive as GML

Let φ_1, φ_2 be GML formulae such that $\varphi_1 \wedge \varphi_2$ is in disjoint form. We show that there is a GML formula ψ such that $\varphi_1 | \varphi_2 \equiv \psi$. To do so, we take a slight detour through Presburger arithmetic interpreted on the set of natural numbers $\mathbb{N}$, see e.g., [29, 44] for details. We characterise the formula $\varphi_1 | \varphi_2$ by using linear arithmetic constraints for the number of successors. Then, we take advantage of basic properties of Presburger arithmetic to eliminate quantifiers, and obtain a GML formula. Below, the variables $\mathsf{x}, \mathsf{y}, \mathsf{z}, \ldots$, possibly decorated and occurring in formulae, are from Presburger arithmetic and therefore they are interpreted by natural numbers. We write $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ for a formula in Presburger arithmetic χ with free variables $\mathsf{x}_1, \ldots, \mathsf{x}_n$.

Let φ be in GML such that $\mathtt{max}_{\mathtt{PC}}(\varphi) \subseteq \{p_1, \ldots, p_m\}$ and $\{\psi \mid \Diamond_{\geq k}\,\psi \in \mathtt{max}_{\mathtt{GM}}(\varphi)\} \subseteq \{\psi_1, \ldots, \psi_n\}$. We define formulae in Presburger arithmetic that state constraints about the number of children satisfying a formula ψ_j ($j \in [1,n]$), as well as the polarity of the atomic propositions p_j ($j \in [1,m]$) not appearing under the scope of graded modalities. In this respect, the variable x_j is intended to be interpreted as the number of children satisfying ψ_j, whereas with some abuse of notation we see p_j directly as a variable. Whenever non-zero, the variable p_j shall encode the fact that the homonymous atomic proposition is satisfied. We write $\varphi^{\mathsf{PA}}(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$ to denote the quantifier-free formula of Presburger arithmetic obtained from φ by replacing with $\mathsf{x}_j \geq k$ (resp. $p_j \geq 1$) every occurrence of $\Diamond_{\geq k}\,\psi_j$ (resp. p_j) that it is not in the scope of a graded modality. For instance, assuming that $\varphi = \neg p \wedge (\Diamond_{\geq 5}\,(p \wedge q) \vee \neg\Diamond_{\geq 4}\,\neg p)$, the expression $\varphi^{\mathsf{PA}}(\mathsf{x}_1, \mathsf{x}_2)$ denotes the formula $\neg p \geq 1 \wedge (\mathsf{x}_1 \geq 5 \vee \neg(\mathsf{x}_2 \geq 4))$.

Consider now formulae φ_1 and φ_2 in GML, such that the conjunction $\varphi_1 \wedge \varphi_2$ is in disjoint form, $\mathtt{max}_{\mathtt{PC}}(\varphi_1 \wedge \varphi_2) \subseteq \{p_1, \ldots, p_m\}$ and $\{\psi \mid \Diamond_{\geq k}\,\psi \in \mathtt{max}_{\mathtt{GM}}(\varphi_1 \wedge \varphi_2)\} \subseteq \{\psi_1, \ldots, \psi_n\}$. We consider the formula $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$ of Presburger arithmetic defined below:

$$\exists\, \mathsf{y}^1_1, \mathsf{y}^2_1, \ldots, \mathsf{y}^1_n, \mathsf{y}^2_n \ (\textstyle\bigwedge_{j=1}^{n} \mathsf{x}_j = \mathsf{y}^1_j + \mathsf{y}^2_j) \wedge \varphi_1^{\mathsf{PA}}(\mathsf{y}^1_1, \ldots, \mathsf{y}^1_n, p_1, \ldots, p_m) \wedge \varphi_2^{\mathsf{PA}}(\mathsf{y}^2_1, \ldots, \mathsf{y}^2_n, p_1, \ldots, p_m).$$

This formula states that there is a way to divide the children in two distinct sets and each set allows to satisfy φ_1^{PA} or φ_2^{PA}, respectively. As Presburger arithmetic admits quantifier elimination [18, 44, 47], there is a quantifier-free formula $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$ equivalent to the formula $[\varphi_1, \varphi_2]^{\mathsf{PA}}$. In the next lemma, we show that thanks to the shape of the formula $[\varphi_1, \varphi_2]^{\mathsf{PA}}$, the atomic formulae appearing in χ are of the form $\mathsf{x}_j \geq k$ and $p_j \geq 1$, i.e. the quantifier elimination step does not introduce 'modulo constraints' or constraints of the form $\sum a_j \mathsf{y}_j \geq k$.

Lemma 3.5. *Let $\varphi_1, \varphi_2 \in \mathsf{GML}$ s.t. $\varphi_1 \wedge \varphi_2$ is in disjoint form. Then $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$ is equivalent to a quantifier-free formula $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$ of Presburger arithmetic, whose atomic formulae are only of the form $\mathsf{x}_j \geq k$ ($j \in [1, n]$), with $k \leq \mathrm{gr}(\varphi_1) + \mathrm{gr}(\varphi_2)$, or $p_j \geq 1$ ($j \in [1, m]$).*

Proof. Notice that if either φ_1^{PA} or φ_2^{PA} is inconsistent, then χ can be defined as $\perp$. In the sequel, we assume that both φ_1^{PA} and φ_2^{PA} are consistent. For each $i \in \{1, 2\}$, it is straightforward to establish that there is an arithmetical formula $\varphi'_i(\mathsf{y}^i_1, \ldots, \mathsf{y}^i_n, p_1, \ldots, p_m)$ in disjunctive normal form that is logically equivalent to the formula $\varphi_i^{\mathsf{PA}}(\mathsf{y}^i_1, \ldots, \mathsf{y}^i_n, p_1, \ldots, p_m)$, and where in each disjunct of φ'_i, every variable y^i_j ($j \in [1, n]$) occurs in at most two literals with the following three options:

- y^i_j occurs in a unique literal of the form $\mathsf{y}^i_j \geq k$,
- y^i_j occurs in a unique (negative) literal of the form $\neg(\mathsf{y}^i_j \geq k)$, or
- y^i_j occurs in two literals whose conjunction is $\mathsf{y}^i_j \geq k \wedge \neg(\mathsf{y}^i_j \geq k')$ and, $k' > k$.

Above, we can guarantee that $k, k' \leq \mathrm{gr}(\varphi_i)$. Moreover, in each disjunct of φ'_i, every variable p_j ($j \in [1, m]$) occurs exactly once, in a (possibly negated) atomic proposition of the form $p_j \geq 1$. Using propositional reasoning and the fact that disjunction distributes over existential first-order quantification and that the variables p_j are free, the formula $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ is therefore logically equivalent to a formula of the form

$$\bigvee_{\alpha, \beta} P^1_\alpha \wedge P^2_\beta \wedge \exists\, \mathsf{y}^1_1, \mathsf{y}^2_1, \ldots, \mathsf{y}^1_n, \mathsf{y}^2_n \ \big(C^1_\alpha \wedge C^2_\beta \wedge \bigwedge_{j=1}^{n} \mathsf{x}_j = \mathsf{y}^1_j + \mathsf{y}^2_j\big)$$

where $P^1_\alpha \wedge C^1_\alpha$ (resp. $P^2_\alpha \wedge C^2_\beta$) is a conjunction from φ'_1 (resp. from φ'_2) and, for $i \in 1, 2$, P^i_α is written with variables from $\{p_1, \ldots, p_m\}$ whereas C^i_α is written with variables from $\{\mathsf{y}^i_1, \ldots, \mathsf{y}^i_n\}$. In order to build $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$ from $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$, we take advantage of quantifier elimination in PA and we explain below how this can be done. It is sufficient to explain how to eliminate quantifiers for subformulae of the form

$$\Psi \ = \ \exists\, \mathsf{y}^1_1, \mathsf{y}^2_1, \ldots, \mathsf{y}^1_n, \mathsf{y}^2_n \ (\bigwedge_{j=1}^{n} \mathsf{x}_j = \mathsf{y}^1_j + \mathsf{y}^2_j) \wedge C^1_\alpha \wedge C^2_\beta.$$

Inductively, let $j \in [1, n]$ and suppose that by performing quantifier elimination on the quantifier prefix $\exists\, \mathsf{y}^1_{j+1}, \mathsf{y}^2_{j+1}, \ldots, \mathsf{y}^1_n, \mathsf{y}^2_n$, the formula Ψ is shown equivalent to $\exists\, \mathsf{y}^1_1, \mathsf{y}^2_1, \ldots, \mathsf{y}^1_j, \mathsf{y}^2_j\ \Psi_{j+1}$, with $\Psi_{n+1} = (\bigwedge_{j=1}^{n} \mathsf{x}_j = \mathsf{y}^1_j + \mathsf{y}^2_j) \wedge C^1_\alpha \wedge C^2_\beta$, and the following properties hold:

1. Ψ_{j+1} is quantifier-free with no occurrences of the variables $\mathsf{y}^1_{j+1}, \mathsf{y}^2_{j+1}, \ldots, \mathsf{y}^1_n, \mathsf{y}^2_n$,
2. Ψ_{j+1} is of the form $(\bigwedge_{a \in [1,j]} \mathsf{x}_a = \mathsf{y}^1_a + \mathsf{y}^2_a) \wedge D \wedge C'_1 \wedge C'_2$, where
   (a) D is a conjunction of literals built from constraints of the form $\mathsf{x}_{j'} \geq k$ with $j' \in [j, n]$,
   (b) for each $i \in \{1, 2\}$, C'_i a conjunction such that for each $j' \in [1, j]$, $\mathsf{y}^i_{j'}$ is in at most two literals with the following three options:

- $\mathsf{y}^i_{j'}$ occurs in a unique literal of the form $\mathsf{y}^i_{j'} \geq k$,
- $\mathsf{y}^i_{j'}$ occurs in a unique (negative) literal of the form $\neg(\mathsf{y}^i_{j'} \geq k)$,
- $\mathsf{y}^i_{j'}$ occurs in two literals whose conjunction is $\mathsf{y}^i_{j'} \geq k_1 \wedge \neg(\mathsf{y}^i_{j'} \geq k_2)$ and $k_2 > k_1$.

Now, let us show how to perform quantifier elimination of $\exists\, \mathsf{y}^1_j\ \exists\, \mathsf{y}^2_j\ \Psi_{j+1}$ to preserve the property for $j-1$. First note that $\exists\, \mathsf{y}^1_j\ \exists\, \mathsf{y}^2_j\ \Psi_{j+1}$ is logically equivalent to

$$(\bigwedge_{a=1}^{j-1} \mathsf{x}_a = \mathsf{y}^1_a + \mathsf{y}^2_a) \wedge D \wedge C''_1 \wedge C''_2 \wedge\ \exists\, \mathsf{y}^1_j\ \exists\, \mathsf{y}^2_j\ (\mathsf{x}_j = \mathsf{y}^1_j + \mathsf{y}^2_j \wedge D_1 \wedge D_2),$$

where $C'_1 = C''_1 \wedge D_1$ (assuming abusively that $A \wedge \top = A$), $C'_2 = C''_2 \wedge D_2$ and each variable y^i_j does not occur in C''_i, and each D_i is either $\top$, or contains at most 2 literals involving the variable y^i_j. It is then easy to eliminate quantifiers in $\exists\, \mathsf{y}^1_j\ \exists\, \mathsf{y}^2_j\ (\mathsf{x}_j = \mathsf{y}^1_j + \mathsf{y}^2_j) \wedge D_1 \wedge D_2$. Below we treat all the cases, depending on the value for $D_1 \wedge D_2$ leading to the formula D_{12} (we omit the symmetrical cases):
case $\top \wedge \top$ **or** $\neg(\mathsf{y}^1_j \geq k) \wedge \top$**:** $D_{12} \stackrel{\text{def}}{=} \top$,
case $(\mathsf{y}^1_j \geq k) \wedge \top$ **or** $((\mathsf{y}^1_j \geq k) \wedge \neg(\mathsf{y}^1_j \geq k')) \wedge \top$**:** $D_{12} \stackrel{\text{def}}{=} (\mathsf{x}_j \geq k)$,
case $\neg(\mathsf{y}^1_j \geq k) \wedge (\mathsf{y}^2_j \geq k'')$**:** $D_{12} \stackrel{\text{def}}{=} (\mathsf{x}_j \geq k'')$,
case $(\mathsf{y}^1_j \geq k) \wedge (\mathsf{y}^2_j \geq k'')$ **or** $((\mathsf{y}^1_j \geq k) \wedge \neg(\mathsf{y}^1_j \geq k')) \wedge (\mathsf{y}^2_j \geq k'')$**:** $D_{12} \stackrel{\text{def}}{=} (\mathsf{x}_j \geq k + k'')$,
case $((\mathsf{y}^1_j \geq k) \wedge \neg(\mathsf{y}^1_j \geq k')) \wedge ((\mathsf{y}^2_j \geq k'') \wedge \neg(\mathsf{y}^2_j \geq k'''))$**:** $D_{12} \stackrel{\text{def}}{=} (\mathsf{x}_j \geq k+k'') \wedge \neg(\mathsf{x}_j \geq k'+k''')$.
It is now easy to check that the formula

$$\exists\, \mathsf{y}^1_1, \mathsf{y}^2_1, \ldots, \mathsf{y}^1_{j-1}, \mathsf{y}^2_{j-1}\ (\bigwedge_{a=1}^{j-1} \mathsf{x}_a = \mathsf{y}^1_a + \mathsf{y}^2_a) \wedge (D \wedge D_{12}) \wedge C''_1 \wedge C''_2,$$

satisfies the conditions for Ψ_j. By iterating the process of quantifier elimination, we get the desired formula $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$. From the case analysis above, notice that all the atomic formulae of the form $\mathsf{x}_j \geq k$ appearing in $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ are such that $k \leq \mathsf{gr}(\varphi_1) + \mathsf{gr}(\varphi_2)$. □

From the formula $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$, we derive the GML formula χ^{GML} by replacing every occurrence of $\mathsf{x}_j \geq k$ by $\Diamond_{\geq k}\ \psi_j$, and every occurrence of $p_j \geq 1$ by p_j. We show that $\varphi_1 \,|\, \varphi_2 \equiv \chi^{\mathsf{GML}}$.

Lemma 3.6. *Given φ_1 and φ_2 GML formulae in disjoint form, there is a GML formula χ^{GML} in disjoint form such that $\chi^{\mathsf{GML}} \equiv \varphi_1 | \varphi_2$, $\mathsf{gr}(\chi^{\mathsf{GML}}) \leq \mathsf{gr}(\varphi_1) + \mathsf{gr}(\varphi_2)$, $\mathtt{max}_{\mathsf{PC}}(\chi^{\mathsf{GML}}) \subseteq \mathtt{max}_{\mathsf{PC}}(\varphi_1 \wedge \varphi_2)$ and $\{\psi \mid \Diamond_{\geq k}\ \psi \in \mathtt{max}_{\mathsf{GM}}(\chi^{\mathsf{GML}})\} \subseteq \{\psi \mid \Diamond_{\geq k}\ \psi \in \mathtt{max}_{\mathsf{GM}}(\varphi_1 \wedge \varphi_2)\}$.*

The assumption that $\varphi_1 \wedge \varphi_2$ is in disjoint form is essential to obtain $\varphi_1 | \varphi_2 \equiv \chi^{\mathsf{GML}}$. Here is a simple counter-example. The formula $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\mathsf{x}_1, \mathsf{x}_2)$ obtained from $\Diamond_{\geq 1}\ p \,|\, \Diamond_{\geq 1}\ q$ is defined as $\exists\, \mathsf{y}^1_1, \mathsf{y}^2_1, \mathsf{y}^1_2, \mathsf{y}^2_2\ (\mathsf{x}_1 = \mathsf{y}^1_1 + \mathsf{y}^2_1) \wedge (\mathsf{x}_2 = \mathsf{y}^1_2 + \mathsf{y}^2_2) \wedge (\mathsf{y}^1_1 \geq 1) \wedge (\mathsf{y}^2_2 \geq 1)$. Obviously, $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\mathsf{x}_1, \mathsf{x}_2)$ is arithmetically equivalent to $(\mathsf{x}_1 \geq 1) \wedge (\mathsf{x}_2 \geq 1)$ but $\Diamond_{\geq 1}\ p \,|\, \Diamond_{\geq 1}\ q \not\equiv \Diamond_{\geq 1}\ p \wedge \Diamond_{\geq 1}\ q$. Indeed, when $\mathfrak{M}, w \models \Diamond_{\geq 1}\ p \wedge \Diamond_{\geq 1}\ q$ and w has a unique child satisfying $p \wedge q$, $\mathfrak{M}, w \not\models \Diamond_{\geq 1}\ p \,|\, \Diamond_{\geq 1}\ q$.

Proof. Let $\mathtt{max}_{\mathsf{PC}}(\varphi_1 \wedge \varphi_2) = \{p_1, \ldots, p_m\}$ and $\{\psi_1, \ldots, \psi_n\} = \{\psi \mid \Diamond_{\geq k}\ \psi \in \mathtt{max}_{\mathsf{GM}}(\varphi_1 \wedge \varphi_2)\}$. Consider the formula $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$, equivalent to $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$, from Lemma 3.5. Let χ^{GML} be the formula obtained from $\chi(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$ by replacing every occurrence of $\mathsf{x}_j \geq k$ with $\Diamond_{\geq k}\ \psi_j$, and every occurrence of $p_j \geq 1$ with p_j. The formula χ^{GML} enjoys the following properties: $\mathsf{gr}(\chi^{\mathsf{GML}}) \leq \mathsf{gr}(\varphi_1) + \mathsf{gr}(\varphi_2)$, $\mathtt{max}_{\mathsf{PC}}(\chi^{\mathsf{GML}}) \subseteq \mathtt{max}_{\mathsf{PC}}(\varphi_1 \wedge \varphi_2)$ and $\{\psi \mid \Diamond_{\geq k}\ \psi \in \mathtt{max}_{\mathsf{GM}}(\chi^{\mathsf{GML}})\} \subseteq \{\psi \mid \Diamond_{\geq k}\ \psi \in \mathtt{max}_{\mathsf{GM}}(\varphi_1 \wedge \varphi_2)\}$. As $\varphi_1 \wedge \varphi_2$ is in disjoint form, the last inclusion implies that χ^{GML} is in disjoint form.

To grasp the relationship between φ_i and its arithmetical counterpart $\varphi_i^{\mathsf{PA}}(\mathsf{x}_1, \ldots, \mathsf{x}_n, p_1, \ldots, p_m)$, consider a finite forest $\mathfrak{M}_i = (W_i, R_i, V_i)$, $w \in W_i$, and

- for all $j \in [1, n]$, let $\beta^i_j = |\{w' \in W_i \mid \mathfrak{M}_i, w' \models \psi_j \text{ and } (w, w') \in R_i\}|$,
- for all $j \in [1, m]$, if $w \in V_i(p_j)$ then let c^i_j be an arbitrary number greater than 0, else let $c^i_j = 0$.

We have the following equivalence

$$\mathfrak{M}_i, w \models \varphi_i \text{ if and only if } \varphi_i^{\mathsf{PA}}(\beta^i_1, \dots, \beta^i_n, c^i_1, \dots, c^i_m) \text{ is valid,} \tag{1}$$

where $\varphi_i^{\mathsf{PA}}(\beta^i_1, \dots, \beta^i_n, c^i_1, \dots, c^i_m)$ is the sentence from Presburger arithmetic obtained by replacing each variable x_j (resp. p_j) with the natural number β^i_j (resp. c^i_j).

Let us show that $\varphi_1 \,|\, \varphi_2 \equiv \chi^{\mathsf{GML}}$. We start by showing that $\varphi_1 \,|\, \varphi_2 \Rightarrow \chi^{\mathsf{GML}}$ is valid. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$, such that $\mathfrak{M}, w \models \varphi_1 \,|\, \varphi_2$. By definition of $\models$, there are $\mathfrak{M}_1$, $\mathfrak{M}_2$ such that $\mathfrak{M} = \mathfrak{M}_1 +_w \mathfrak{M}_2$, $\mathfrak{M}_1, w \models \varphi_1$ and $\mathfrak{M}_2, w \models \varphi_2$. Let us keep the definition of the β^i_j's and c^i_j's from above, and for each $j \in [1, n]$, let $\alpha_j = |\{w' \in W \mid \mathfrak{M}, w' \models \psi_j \text{ and } (w, w') \in R\}|$. Since V is shared between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, $c^1_j \geq 1$ holds if and only if $c^2_j \geq 1$. Let $c_j = \max(c^1_j, c^2_j)$. By (1) and as $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$ holds too, we have the following:

$$\text{for all } j \in [1, n] \; \alpha_j = \beta^1_j + \beta^2_j, \qquad \text{for all } i \in \{1, 2\} \; \varphi_1^{\mathsf{PA}}(\beta^i_1, \dots, \beta^i_n, c_1, \dots, c_m) \text{ is valid,}$$

which implies the validity of $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\alpha_1, \dots, \alpha_n, c_1, \dots, c_m)$. Hence, $\chi(\alpha_1, \dots, \alpha_n, c_1, \dots, c_m)$ is valid. By definition of χ^{GML} together with the definitions of α_j and c_j, $\mathfrak{M}, w \models \chi^{\mathsf{GML}}$.

Now, we show that $\chi^{\mathsf{GML}} \Rightarrow \varphi_1 \,|\, \varphi_2$ is valid. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$ such that $\mathfrak{M}, w \models \chi^{\mathsf{GML}}$. As above,

- for each $j \in [1, n]$, let $\alpha_j = |\{w' \in W \mid \mathfrak{M}, w' \models \psi_j \text{ and } (w, w') \in R\}|$.
- for all $j \in [1, m]$, if $w \in V(p_j)$ then let c_j be an arbitrary number greater than 0, else let $c_j = 0$.

Similarly to (1), we get that $\chi(\alpha_1, \dots, \alpha_n, c_1, \dots, c_m)$ is valid, and so $[\varphi_1, \varphi_2]^{\mathsf{PA}}(\alpha_1, \dots, \alpha_n, c_1, \dots, c_m)$ is valid. From the semantics of the formula $[\varphi_1, \varphi_2]^{\mathsf{PA}}$, there are $\beta^1_1, \beta^2_1, \dots, \beta^1_n, \beta^2_n \in \mathbb{N}$ such that

$$\text{for all } j \in [1, n] \; \alpha_j = \beta^1_j + \beta^2_j, \qquad \text{for all } i \in \{1, 2\} \; \varphi_1^{\mathsf{PA}}(\beta^i_1, \dots, \beta^i_n, c_1, \dots, c_m) \text{ is valid.}$$

For each $i \in \{1, 2\}$ let us build $\mathfrak{M}_i$ such that for all $j \in [1, n]$, w has β^i_j children in $\mathfrak{M}_i$, and by construction for each such a child, its whole subtree in (W, R) is present in (W, R_i) too. Such a division is possible because, if a child of w contributes to the value α_j in $\mathfrak{M}$ (and therefore it satisfies ψ_j), it cannot contribute to any value $\alpha_{j'}$ with $j' \neq j$, thanks to the assumption that $\psi_j \wedge \psi_{j'}$ is unsatisfiable, given by the disjoint form of $\varphi_1 \wedge \varphi_2$. Hence, by construction, $\mathfrak{M} = \mathfrak{M}_1 +_w \mathfrak{M}_2$. Moreover, for any child w' of w in $\mathfrak{M}_i$, we have $\mathfrak{M}_i, w' \models \psi_j$ if and only if $\mathfrak{M}, w' \models \psi_j$ (for all $j \in [1, n]$) as the whole subtree of w' in $\mathfrak{M}$ is present in $\mathfrak{M}_i$. For $i \in \{1, 2\}$, the validity of $\varphi_i^{\mathsf{PA}}(\beta^i_1, \dots, \beta^i_n, c_1, \dots, c_m)$ entails, by (1), $\mathfrak{M}_i, w \models \varphi_i$. Consequently, we get $\mathfrak{M}, w \models \varphi_1 \,|\, \varphi_2$. □

The bound on $\mathrm{gr}(\chi^{\mathsf{GML}})$ stated in this key lemma is essential to obtain an exponential bound on the smallest model satisfying a formula in $\mathsf{ML}(|)$ (see Section 3.3). Combining Lemma 3.3 and Lemma 3.6, we conclude that GML is closed under the operator $|$.

Theorem 3.7. $\mathsf{ML}(|) \preceq \mathsf{GML}$. *Therefore,* $\mathsf{ML}(|) \approx \mathsf{GML}$.

Proof. Let φ be a formula in $\mathsf{ML}(|)$. As $\Diamond\psi \equiv \Diamond_{\geq 1} \psi$, we can assume that the only modalities in φ are of the form $\Diamond_{\geq 1}$ or $|$. If φ has no occurrence of $|$, we are done. Otherwise, let ψ be a subformula of φ whose outermost connective is $|$ and the arguments are in GML, say $\psi = \varphi_1 \,|\, \varphi_2$. By Lemma 3.3 there are GML formulae φ_1' and φ_2' in disjoint form such that $\varphi_1' \equiv \varphi_1$ and $\varphi_2' \equiv \varphi_2$. Hence, $\varphi_1' \,|\, \varphi_2' \equiv \psi$. We apply Lemma 3.6 on $\varphi_1' \,|\, \varphi_2'$, obtaining a formula ψ' in GML that is equivalent to ψ. We have $\varphi \equiv \varphi[\psi \leftarrow \psi']$, where $\varphi[\psi \leftarrow \psi']$ is obtained from φ by replacing every occurrence of ψ by ψ'. Note that the number of occurrences of $|$ in $\varphi[\psi \leftarrow \psi']$ is strictly less than the number of occurrences of $|$ in φ. By repeating such a type of replacement, eventually we obtain a formula φ' in GML such that $\varphi \equiv \varphi'$. □

3.3 The satisfiability problem of $\mathsf{ML}(\ast)$ is AExp_{Pol}-complete

First, we will prove the upper bound, i.e., that Sat($\mathsf{ML}(\ast)$) is in AExp_{Pol}. To do so, the main ingredient is to show that given a formula φ in $\mathsf{ML}(\ast)$, we build φ' in GML such that $\varphi' \equiv \varphi$ and the models for φ' (if any) do not require a number of children per node more than exponential in size(φ). The proof of Theorem 3.7 needs to be refined to improve the way φ' is computed. In particular, this requires a more "global" strategy that does not require to put subformulae in disjoint form multiple times. Aiming for an inductive argument on the line of Lemmata 3.2 and 3.3, we first consider the logic $\mathcal{L}$, which is a variant of $\mathsf{ML}(\ast)$ given by the grammar below:

$$\varphi := \Diamond_{\geq k}\,\psi \mid p \mid \varphi \ast \varphi \mid \varphi \wedge \varphi \mid \neg\varphi,$$

where $p \in \mathrm{AP}$ and $\Diamond_{\geq k}\,\psi$ is a formula in GML (abusively assumed to be in $\mathsf{ML}(\ast)$ but we know $\mathsf{GML} \preceq \mathsf{ML}(\ast)$). Given φ in $\mathsf{ML}(\ast)$ or in $\mathcal{L}$, we write $\mathtt{cd}(\varphi)$ to denote its *composition degree*, i.e. the number of $\ast$ appearing in φ. We extend the notion of $\mathtt{max}_{\mathsf{GM}}(.)$ to formulae in $\mathcal{L}$, so that $\mathtt{max}_{\mathsf{GM}}(\varphi) \stackrel{\text{def}}{=} \mathtt{max}_{\mathsf{GM}}(\varphi[\ast \leftarrow \wedge])$, where $\varphi[\ast \leftarrow \wedge]$ is the formula obtained from φ by replacing every occurrence of $\ast$ by $\wedge$. Similarly, $\mathtt{gm}(d,\varphi) \stackrel{\text{def}}{=} \mathtt{gm}(d,\varphi[\ast \leftarrow \wedge])$. We say that φ in $\mathcal{L}$ is in disjoint form if so is $\varphi[\ast \leftarrow \wedge]$. Alternatively, this means that given $\mathtt{max}_{\mathsf{GM}}(\varphi) = \{\Diamond_{\geq k_1}\,\psi_1,\ldots,\Diamond_{\geq k_n}\,\psi_n\}$, $\widehat{k} = \max\{k_1,\ldots,k_n\}$, the GML formula $\Diamond_{\geq k_1}\psi_1 \wedge \cdots \wedge \Diamond_{\geq k_n}\psi_n$ is in disjoint form.

We start by extending Lemma 3.6 for formulae of the fragment $\mathcal{L}$ in disjoint form.

Lemma 3.8. *Let φ be a formula of the fragment $\mathcal{L}$ such that $\mathtt{max}_{\mathsf{GM}}(\varphi) = \{\Diamond_{\geq k_1}\psi_1,\ldots,\Diamond_{\geq k_n}\psi_n\}$ and φ is in disjoint form. There is a GML formula ψ in disjoint form such that $\varphi \equiv \psi$, $\mathtt{max}_{\mathsf{PC}}(\psi) \subseteq \mathtt{max}_{\mathsf{PC}}(\varphi)$ and $\mathtt{max}_{\mathsf{GM}}(\psi) \subseteq \{\Diamond_{\geq j}\psi_i \mid j \in [0,(\mathtt{cd}(\varphi)+1)\cdot\mathtt{gr}(\varphi)]$ and $i \in [1,n]\}$.*

Proof. By induction on $\mathtt{cd}(\varphi)$. If $\mathtt{cd}(\varphi) = 0$, then $\psi = \varphi$. Otherwise, let Φ be the set of subformulae of the form $\varphi_1 \ast \varphi_2$ of φ appearing not in scope of a modality $\ast$. Fix $\varphi_1 \ast \varphi_2$ in Φ. As $\mathtt{cd}(\varphi_1)+\mathtt{cd}(\varphi_2) < \mathtt{cd}(\varphi)$, by induction hypothesis, there are GML formulae φ_1', φ_2' in disjoint form such that, for all $i \in \{1,2\}$, $\varphi_i \equiv \varphi_i'$ and $\mathtt{max}_{\mathsf{GM}}(\varphi_i') \subseteq \{\Diamond_{\geq j}\psi_i \mid j \leq (\mathtt{cd}(\varphi_i)+1)\cdot\mathtt{gr}(\varphi_i)$ and $i \in [1,n]\}$ and $\mathtt{max}_{\mathsf{PC}}(\varphi_i') \subseteq \mathtt{max}_{\mathsf{PC}}(\varphi_i)$. Notice that $\mathtt{gr}(\varphi_i') \leq (\mathtt{cd}(\varphi_i)+1)\cdot\mathtt{gr}(\varphi_i) \leq (\mathtt{cd}(\varphi_i)+1)\cdot\mathtt{gr}(\varphi)$. By Lemma 3.6, there is a formula χ in disjoint form such that $\chi \equiv \varphi_1' \ast \varphi_2'$, $\mathtt{max}_{\mathsf{PC}}(\chi) \subseteq \mathtt{max}_{\mathsf{PC}}(\varphi_1' \wedge \varphi_2')$, $\mathtt{gr}(\chi) \leq \mathtt{gr}(\varphi_1') + \mathtt{gr}(\varphi_2')$, and $\{\psi \mid \Diamond_{\geq k}\,\psi \in \mathtt{max}_{\mathsf{GM}}(\chi)\} \subseteq \{\psi \mid \Diamond_{\geq k}\,\psi \in \mathtt{max}_{\mathsf{GM}}(\varphi_1' \wedge \varphi_2')\}$. Let $\Diamond_{\geq j}\,\gamma \in \mathtt{max}_{\mathsf{GM}}(\chi)$. By definition, $\gamma \in \{\psi_1,\ldots,\psi_n\}$ and $j \leq \mathtt{gr}(\chi) \leq \mathtt{gr}(\varphi_1') + \mathtt{gr}(\varphi_2') \leq (\mathtt{cd}(\varphi_1) + \mathtt{cd}(\varphi_2) + 2)\cdot\mathtt{gr}(\varphi) \leq (\mathtt{cd}(\varphi)+1)\cdot\mathtt{gr}(\varphi)$.

Let ψ be the formula obtained from φ by replacing every occurrence of $\varphi_1 \ast \varphi_2$ not appearing under the scope of a modality $\ast$ with the equivalent formula χ, for every formula $\varphi_1 \ast \varphi_2$ in Φ. The formula ψ satisfies the required properties. Indeed, by definition it is equivalent to φ, and since every χ is in disjoint form, so is ψ. Clearly, $\mathtt{max}_{\mathsf{PC}}(\psi) \subseteq \mathtt{max}_{\mathsf{PC}}(\varphi)$. Lastly, the satisfaction of $\mathtt{max}_{\mathsf{GM}}(\psi) \subseteq \{\Diamond_{\geq j}\psi_i \mid j \leq [0,(\mathtt{cd}(\varphi)+1)\cdot\widehat{k}]$ and $i \in [1,n]\}$ stems from the fact that all the formulae χ equivalent to some formula in Φ satisfy this same property. ☐

Applying adequately the transformation from Lemma 3.8 to a formula in $\mathsf{ML}(\ast)$, i.e. by considering maximal subformulae of the fragment $\mathcal{L}$, allows us to get a logically equivalent GML formula having exponential size models by Lemma 3.4. We extend the notion of *branching degree* to formulae in $\mathcal{L}$, so that $\mathtt{bd}(m,\varphi) \stackrel{\text{def}}{=} \mathtt{bd}(m,\varphi[\ast \leftarrow \wedge])$.

Lemma 3.9. *Every satisfiable φ in $\mathsf{ML}(\ast)$ is satisfied by a pointed forest of size in $2^{O(\mathtt{size}(\varphi))}$.*

Proof. Let φ be a formula in $\mathsf{ML}(\ast)$. During the proof, we see $\Diamond$ as $\Diamond_{\geq 1}$ and assume that every subformula of φ without occurrences of the graded modalities is a Boolean combination of atomic propositions. This assumption is without loss of generality. Indeed, a formula ψ of $\mathsf{ML}(\ast)$ without graded modalities (thus without $\Diamond$) is a formula built upon Boolean connectives, the composition operator $|$ and atomic propositions, and is thus equivalent to $\psi[| \leftarrow \land]$.

Let $\overline{m} = \mathsf{md}(\varphi)$, $\overline{k} = \mathsf{gr}(\varphi)$, $\overline{c} = \mathsf{cd}(\varphi)$ and $\overline{n} = \max\{|\mathsf{gm}(j,\varphi)| \mid j \in [0, \mathsf{md}(\varphi)]\}$. We reason inductively, building a chain of equivalent formulae $\varphi_0, \dots, \varphi_{\overline{m}}$ where $\varphi_0 = \varphi$ and, for $i \in [0, \overline{m}]$,

(1) $\mathsf{md}(\varphi_i) \leq \overline{m}$, $\mathsf{cd}(\varphi_i) \leq \overline{c}$, all the atomic propositions in φ_i are from φ, and all subformulae of φ_i appearing under the scope of $\overline{m} - i$ graded modalities belong to GML,
(2) for all $j \in [0, i]$ and $\Diamond_{\geq k}\ \psi, \Diamond_{\geq k'}\ \psi' \in \mathsf{gm}(\overline{m} - j, \varphi_i)$, either $\psi \equiv \psi'$ or the formula $\psi \land \psi'$ is unsatisfiable (equivalently, the conjunction of all formulae in $\mathsf{gm}(\overline{m} - i, \varphi_i)$ is in disjoint form),
(3) for all $j \in [i + 1, \overline{m}]$, $|\mathsf{gm}(\overline{m} - j, \varphi_i)| \leq |\mathsf{gm}(\overline{m} - j, \varphi)|$ and $\mathsf{bd}(\overline{m} - j, \varphi_i) \leq \mathsf{bd}(\overline{m} - j, \varphi)$,
(4) for every $j \in [0, i]$, $|\mathsf{gm}(\overline{m} - j, \varphi_i)| \leq 2^{\overline{n}} \cdot ((\overline{c} + 1) \cdot \overline{k} + 1)$ and $\mathsf{bd}(\overline{m} - j, \varphi_i) \leq 2^{\overline{n}} \cdot ((\overline{c} + 1) \cdot \overline{k})^2$.

Properties (1) and (2) above guarantee that each step on the chain of equivalences are in the proper shape, i.e., without violating any syntactic condition. On the other hand, properties (3) and (4) ensure that on each step the bounds in the formula obtained grow in a way that lead us to the lemma's statement, via the application of Lemma 3.4.

Precisely, the numbers $\overline{m}$, $\overline{k}$, $\overline{c}$ and $\overline{n}$ are all bounded by $\mathtt{size}(\varphi)$ (recall that we consider the numbers appearing in graded modalities to be encoded in unary). Based on the properties above, the formula $\varphi_{\overline{m}}$ that we obtain at the end is a GML formula in disjoint form such that $\mathsf{max}_{\mathsf{bd}}(\varphi_{\overline{m}}) \leq 2^{\mathtt{size}(\varphi)} \cdot ((\mathtt{size}(\varphi) + 1) \cdot \mathtt{size}(\varphi))^2$, $\mathsf{md}(\varphi_{\overline{m}}) \leq \mathtt{size}(\varphi)$, and therefore $\mathsf{max}_{\mathsf{bd}}(\varphi_{\overline{m}})$ is in $2^{O(\mathtt{size}(\varphi))}$. As $\varphi \equiv \varphi_{\overline{m}}$, the fact that φ is satisfied by a pointed forest of size in $2^{O(\mathtt{size}(\varphi))}$ then follows directly from Lemma 3.4. Moreover, since GML is a fragment of $\mathsf{ML}(|)$, the construction of $\varphi_{\overline{m}}$ actually reproves Lemma 3.3, but this time with precise bounds on the size of the equivalent GML formula in disjoint form.

Clearly, for $i = 0$, the formula $\varphi_0 = \varphi$ satisfies all the expected properties (note that $\mathsf{gm}(\overline{m}, \varphi) = \emptyset$ and that $\mathsf{bd}(\varphi) \leq \mathtt{size}(\varphi)$). So, below suppose $i \geq 1$ and assume that we are provided with the formula $\varphi_{i-1} \equiv \varphi$, satisfying

(1_{i-1}) $\mathsf{md}(\varphi_{i-1}) \leq \overline{m}$, $\mathsf{cd}(\varphi_{i-1}) \leq \overline{c}$, all atomic propositions in φ_{i-1} are from φ, and all subformulae of φ_{i-1} appearing under the scope of $\overline{m} - (i - 1)$ graded modalities belong to GML,
(2_{i-1}) for all $j \in [0, i-1]$ and $\Diamond_{\geq k}\ \psi, \Diamond_{\geq k'}\ \psi' \in \mathsf{gm}(\overline{m}-j, \varphi_{i-1})$, either $\psi \land \psi'$ is unsatisfiable or $\psi \equiv \psi'$,
(3_{i-1}) for all $j \in [i, \overline{m}]$, $|\mathsf{gm}(\overline{m} - j, \varphi_{i-1})| \leq |\mathsf{gm}(\overline{m} - j, \varphi)|$ and $\mathsf{bd}(\overline{m} - j, \varphi_{i-1}) \leq \mathsf{bd}(\overline{m} - j, \varphi)$,
(4_{i-1}) for every $j \in [0, i-1]$, $|\mathsf{gm}(\overline{m}-j, \varphi_{i-1})| \leq 2^{\overline{n}} \cdot ((\overline{c}+1) \cdot \overline{k}+1)$ and $\mathsf{bd}(\overline{m}-j, \varphi_{i-1}) \leq 2^{\overline{n}} \cdot ((\overline{c}+1) \cdot \overline{k})^2$.

Let us explain how we define φ_i. Consider the set $\Phi = \{\chi_1, \dots, \chi_p\}$ of maximal subformulae of φ_{i-1} appearing under the scope of exactly $\overline{m} - i$ graded modalities. Note that if $\overline{m} - i = 0$ then $\Phi = \{\varphi_{i-1}\}$, and otherwise we have $\mathsf{gm}(\overline{m} - (i + 1), \varphi_{i-1}) = \{\Diamond_{\geq j_1}\ \chi_1, \dots, \Diamond_{\geq j_p}\ \chi_p\}$. From the property ($1_{i-1}$), all the formulae in Φ belong to the fragment $\mathcal{L}$ of $\mathsf{ML}(|)$. Notice that $\mathsf{max}_{\mathsf{GM}}(\chi_1 \land \dots \land \chi_p) = \mathsf{gm}(\overline{m} - i, \varphi_{i-1})$. Let $\mathsf{gm}(\overline{m} - i, \varphi_{i-1}) = \{\Diamond_{\geq k_1}\ \psi_1, \dots, \Diamond_{\geq k_n}\ \psi_n\}$. From property ($2_{i-1}$), $\psi_1 \land \dots \land \psi_n$ is in disjoint form. From property (3_{i-1}), $n \leq |\mathsf{gm}(\overline{m} - i, \varphi)| \leq \overline{n}$ and $\mathsf{bd}(\overline{m} - i, \varphi_{i-1}) \leq \mathsf{bd}(\overline{m} - i, \varphi)$. Let us consider each $\Diamond_{\geq k_j}\ \psi_j$ separately. Let $j \in [1, n]$. Since $\psi_1 \land \dots \land \psi_n$ is in disjoint form, so is $\Diamond_{\geq k_j}\ \psi_j$. Hence, applying Lemma 3.2, we conclude that $\Diamond_{\geq k_j}\ \psi_j \equiv \psi'_j$, for some GML formula ψ'_j in disjoint form such that $\mathsf{max}_{\mathsf{GM}}(\psi'_j) \subseteq \{\Diamond_{\geq k}\ \chi \mid k \in [0, \overline{k}] \text{ and } \chi \in C_\land(\psi_1, \dots, \psi_n)\}$. For every $\ell \in [1, p]$, let χ'_ℓ be the formula obtained from χ_ℓ by substituting with ψ'_j each occurrence of $\Diamond_{\geq k_j}\ \psi_j$ not appearing under the scope of graded modalities, for all $j \in [1, n]$. The formula χ'_ℓ belongs to $\mathcal{L}$; moreover, $\chi'_\ell \equiv \chi_\ell$, and $\mathsf{max}_{\mathsf{GM}}(\chi'_\ell) \subseteq \{\Diamond_{\geq k}\ \gamma \mid k \in [0, \overline{k}] \text{ and } \gamma \in C_\land(\psi_1, \dots, \psi_n)\}$. The latter implies that χ'_ℓ is in disjoint form. Applying Lemma 3.8, there is a GML formula χ''_ℓ in disjoint form such that $\chi''_\ell \equiv \chi'_\ell$, $\mathsf{max}_{\mathsf{GM}}(\chi''_\ell) \subseteq \{\Diamond_{\geq j}\gamma \mid j \in [0, (\overline{c} + 1) \cdot \overline{k}] \text{ and } \gamma \in C_\land(\psi_1, \dots, \psi_n)\}$ and $\mathsf{max}_{\mathsf{PC}}(\chi''_\ell) \subseteq \mathsf{max}_{\mathsf{PC}}(\chi'_\ell)$.

Let φ_i be the formula obtained from φ_{i-1} by replacing with χ''_ℓ every occurrence of χ_ℓ appearing under the scope of $\overline{m} - i$ graded modalities, for every $\ell \in [1, p]$. Let us analyse φ_i. First of all, since φ_i is obtained from φ_{i-1} by only substituting formulae χ_ℓ appearing under the scope of $\overline{m} - i$ graded modalities with equivalent formulae χ''_ℓ from GML, such that $\mathsf{md}(\chi''_\ell) \leq \mathsf{md}(\chi_\ell)$, the properties (1) and (3) hold directly from the properties (1_{i-1}) and (3_{i-1}). By definition of φ_i,

$$\mathsf{gm}(\overline{m} - i, \varphi_i) = \mathsf{max}_{\mathsf{GM}}(\chi''_1 \wedge \cdots \wedge \chi''_p) \subseteq \{\Diamond_{\geq j}\gamma \mid j \in [0, (\overline{c} + 1) \cdot \overline{k}] \text{ and } \gamma \in C_\wedge(\psi_1, \ldots, \psi_n)\}. \quad (\dagger)$$

As $\psi_1 \wedge \cdots \wedge \psi_n$ is in disjoint form, (†) implies that $\chi''_1 \wedge \cdots \wedge \chi''_p$ is in disjoint form. Hence, property (2) holds. Lastly, let us look at property (4). From (†), together with property (4_{i-1}), we conclude that for every $j \in [0, i-1]$, $|\mathsf{gm}(\overline{m} - j, \varphi_{i-1})| \leq 2^{\overline{n}} \cdot ((\overline{c}+1) \cdot \overline{k} + 1)$ and $\mathsf{bd}(\overline{m} - j, \varphi_{i-1}) \leq 2^{\overline{n}} \cdot ((\overline{c}+1) \cdot \overline{k})^2$. So, to establish (4), it is sufficient to treat the case $j = i$. Again by (†),

$$\begin{aligned} |\mathsf{gm}(\overline{m} - i, \varphi_i)| &\leq |C_\wedge(\psi_1, \ldots, \psi_n)| \cdot ((\overline{c} + 1) \cdot \overline{k} + 1) \leq 2^{\overline{n}} \cdot ((\overline{c} + 1) \cdot \overline{k} + 1) \\ \mathsf{bd}(\overline{m} - i, \varphi_i) &\leq |C_\wedge(\psi_1, \ldots, \psi_n)| \cdot \textstyle\sum_{j=0}^{(\overline{c}+1)\cdot\overline{k}} j \leq 2^{\overline{n}} \cdot ((\overline{c} + 1) \cdot \overline{k})^2. \end{aligned}$$

□

The exponential-size model property derived in Lemma 3.9 directly leads to an AExp$_{\text{Pol}}$ upper bound for Sat(ML($|$)). The proof of the theorem is rather standard and sketched below.

Theorem 3.10. Sat(ML($|$)) *is in AExp$_{Pol}$.*

Proof. (sketch) Let φ be in ML($|$). Here we present an algorithm running in exponential-time on $\mathsf{size}(\varphi)$ with an alternating Turing machine using only polynomially many alternations to decide the satisfiability status of φ.

(1) Guess a pointed forest $\mathfrak{M} = (W, R, V)$ with root $w \in W$, whose depth is bounded by $\mathsf{md}(\varphi)$ and of exponential size thanks to Lemma 3.9.
(2) Return the result of checking $\mathfrak{M}, w \models \varphi$. This can be done in exponential-time using an alternating Turing machine with a linear amount of alternations (between universal states and existential states). To do so, one can use a standard model-checking algorithm by viewing ML($|$) as a fragment of MSO. Recall that the standard model-checking algorithm for MSO runs in alternating polynomial time in the size of the structure (which, in our case, has size exponential in $\mathsf{size}(\varphi)$), and uses a number of alternations that is linear in the number of negations appearing in φ. □

It remains to establish AExp$_{\text{Pol}}$-hardness. We provide a logspace reduction from the satisfiability problem for the team logic PL[~] shown AExp$_{\text{Pol}}$-complete in [30, Thm. 4.9].

PL[~] formulae are defined by the following grammar:

$$\varphi := p \mid \dot{\neg} p \mid \varphi \wedge \varphi \mid {\sim}\varphi \mid \varphi \dot{\vee} \varphi,$$

where $p \in \mathrm{AP}$ and the connectives $\dot{\neg}$ and $\dot{\vee}$ are dotted to avoid confusion with those of ML($|$). PL[~] is interpreted on sets of (Boolean) propositional valuations over a finite subset of AP. They are called *teams* and are denoted by $\mathfrak{T}, \mathfrak{T}_1, \ldots$. A model for φ is a team $\mathfrak{T}$ over a set of propositional variables including those occurring in φ and such that $\mathfrak{T} \models \varphi$ with:

$$\begin{aligned} &\mathfrak{T} \models p && \Leftrightarrow \text{for all } \mathfrak{v} \in \mathfrak{T}, \text{ we have } \mathfrak{v}(p) = \top; \\ &\mathfrak{T} \models \dot{\neg} p && \Leftrightarrow \text{for all } \mathfrak{v} \in \mathfrak{T}, \text{ we have } \mathfrak{v}(p) = \bot; \\ &\mathfrak{T} \models \varphi_1 \dot{\vee} \varphi_2 && \Leftrightarrow \text{there are } \mathfrak{T}_1, \mathfrak{T}_2 \text{ such that } \mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2,\ \mathfrak{T}_1 \models \varphi_1 \text{ and } \mathfrak{T}_2 \models \varphi_2. \end{aligned}$$

The connectives ~ and $\wedge$ are interpreted as the classical negation and conjunction, respectively. Notice that, in the clause for $\dot{\vee}$, the teams $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are not necessarily disjoint.

Let us discuss the reduction from Sat(PL[~]) to Sat(ML($|$)). A direct encoding of a team $\mathfrak{T}$ into a pointed forest $(\mathfrak{M}, w)$ consists in having a correspondence between the propositional valuations in $\mathfrak{T}$ and the propositional valuations of the children of w. This would work fine if there were no mismatch between the semantics for $\mid$ (disjointness of the children) and the one for $\dot\vee$ (disjointness not required). To handle this issue, when checking the satisfaction of φ in PL[~] with n occurrences of $\dot\vee$, we impose that if a propositional valuation occurs among the children of w, then it occurs in least $n+1$ children. This property must be maintained after applying $\dot\vee$ several times, always with respect to the number of occurrences of $\dot\vee$ in the subformula of φ that is evaluated. Non-disjointness of the teams is encoded by carefully separating the children of w having identical valuations.

We now formalise the reduction. Assume that we wish to translate φ from PL[~], written with atomic propositions in $\mathsf{P} = \{p_1, \ldots, p_m\}$ and containing at most n occurrences of the operator $\dot\vee$. We introduce a set $\mathsf{Q} = \{q_1, \ldots, q_{n+1}\}$ of auxiliary propositions disjoint from P. The elements of Q are used to distinguish different copies of the same propositional valuation of a team. Thus, with respect to a pointed forest $(\mathfrak{M}, w)$, we require each child of w to satisfy exactly one element of Q. This can be done with the formula

$$\mathtt{uni}(\mathsf{Q}) \stackrel{\text{def}}{=} \Box \bigvee_{i=1}^{n+1} \big(q_i \land \bigwedge_{j=1}^{i-1} \neg q_j \land \bigwedge_{j=i+1}^{n+1} \neg q_j\big).$$

We require that if a child of w satisfies a propositional valuation over (elements in) P, then there are $n+1$ children satisfying that valuation over P, each of them satisfying a distinct symbol in Q. So, every valuation over P occurring in some child of w, occurs at least in $n+1$ children of w. However, as the translation of the operator $\dot\vee$ modifies the set of copies of a propositional valuation, this property must be extended to arbitrary subsets of Q. Given $\emptyset \neq X \subseteq [1, n+1]$, we require that for all $k \neq k' \in X$, if a children of w satisfies q_k, then there is a child satisfying $q_{k'}$ with the same valuation over P. The formula $\mathtt{cp}(X)$ below does the job:

$$\mathtt{cp}(X) \stackrel{\text{def}}{=} \bigwedge_{k \neq k' \in X} \neg\big(\Box q_k \mid (\Diamond_{=1}\, q_k \land \neg(\top \mid \Diamond_{=1}\, q_k \land \Diamond_{=1}\, q_{k'} \land \bigwedge_{j=1}^{m} (\Diamond p_j \Rightarrow \Box p_j)))\big).$$

Lastly, before defining the translation map τ, we describe how different copies of the same propositional valuation are split. We introduce two auxiliary choice functions $\mathfrak{c}_1$ and $\mathfrak{c}_2$ that take as arguments $X \subseteq [1, n+1]$, and $n_1, n_2 \in \mathbb{N}$ with $|X| \geq n_1 + n_2$ such that for each $i \in \{1, 2\}$, we have $\mathfrak{c}_i(X, n_1, n_2) \subseteq X$, $|\mathfrak{c}_i(X, n_1, n_2)| \geq n_i$. Moreover $\mathfrak{c}_1(X, n_1, n_2) \uplus \mathfrak{c}_2(X, n_1, n_2) = X$. The maps $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are instrumental to decide how to split X into two disjoint subsets respecting basic cardinality constraints. The translation map τ is designed as follows ($\emptyset \neq X \subseteq [1, n+1]$):

$$\begin{aligned}
\tau(p, X) &\stackrel{\text{def}}{=} \Box((\textstyle\bigvee_{j \in X} q_j) \Rightarrow p);\\
\tau(\dot\neg p, X) &\stackrel{\text{def}}{=} \Box((\textstyle\bigvee_{j \in X} q_j) \Rightarrow \neg p);\\
\tau({\sim}\varphi, X) &\stackrel{\text{def}}{=} \neg\tau(\varphi, X);\\
\tau(\varphi_1 \land \varphi_2, X) &\stackrel{\text{def}}{=} \tau(\varphi_1, X) \land \tau(\varphi_2, X);\\
\tau(\varphi_1 \dot\vee \varphi_2, X) &\stackrel{\text{def}}{=} (\tau(\varphi_1, X_1) \land \mathtt{cp}(X_1)) \mid (\tau(\varphi_2, X_2) \land \mathtt{cp}(X_2)),
\end{aligned}$$

where ***(i)*** $|X|$ is greater or equal to the number of occurrences of $\dot\vee$ in $\varphi_1 \dot\vee \varphi_2$ plus one; ***(ii)*** given n_1, n_2 such that n_1 (resp. n_2) is the number of occurrences of $\dot\vee$ in φ_1 (resp. φ_2) plus one, for each $i \in \{1, 2\}$, we have $\mathfrak{c}_i(X, n_1, n_2) = X_i$.

Lemma 3.11 below guarantees that starting with a linear number of children with the same propositional valuation is sufficient to encode $\dot\vee$ within $\mathsf{ML}(\mid)$, hence solving the mismatch between the two operators $\dot\vee$ and $\mid$.

Lemma 3.11. *Let φ be in PL[~] with n occurrences of $\dot\vee$ and built upon $p_1, \ldots, p_m$. Then, φ is satisfiable if and only if so is* $\mathtt{uni}(q_1, \ldots, q_{n+1}) \land \mathtt{cp}([1, n+1]) \land \tau(\varphi, [1, n+1])$.

The proof of Lemma 3.11 can be found in Appendix C. The $\mathsf{ML}(\,|\,)$ formula involved in Lemma 3.11 has modal depth one and can be computed in logspace in the size of φ. Hence, $\mathrm{Sat}(\mathsf{ML}(\,|\,))$ is already AExp$_{\text{Pol}}$-hard when restricted to formulae of modal depth at most one. Together with Theorem 3.10, this concludes the complexity analysis of $\mathrm{Sat}(\mathsf{ML}(\,|\,))$.

THEOREM 3.12. $\mathrm{Sat}(\mathsf{ML}(\,|\,))$ *is AExp$_{\text{Pol}}$-complete.*

As we show in the next section, the complexity of $\mathsf{ML}(*)$ does not collapse to modal depth one: $\mathrm{Sat}(\mathsf{ML}(*))$ restricted to formulae of modal depth k is exponentially easier than $\mathrm{Sat}(\mathsf{ML}(*))$ restricted to formulae of modal depth $k+1$.

4 $\mathsf{ML}(*)$ IS TOWER-COMPLETE

This section is devoted to show that $\mathrm{Sat}(\mathsf{ML}(*))$ is Tower-complete; i.e., it is complete for the class of all problems of time complexity bounded by a tower of exponentials whose height is an elementary function [49]. Given $k, n \geq 0$, we inductively define the tetration function $\mathfrak{t}$ as $\mathfrak{t}(0,n) \stackrel{\text{def}}{=} n$ and $\mathfrak{t}(k+1,n) = 2^{\mathfrak{t}(k,n)}$. Intuitively, $\mathfrak{t}(k,n)$ defines a tower of exponentials of height k. By k-NExpTime, we denote the class of all problems decidable with a nondeterministic Turing machine (NTM) of working time $O(\mathfrak{t}(k,p(n)))$ for some polynomial $p(.)$, on each input of length n. To show Tower-hardness, we design a uniform elementary reduction allowing us to get k-NExpTime-hardness for all k greater than a certain (fixed) integer. In our case, we achieve an exponential-space reduction from the k-NExpTime variant of the tiling problem, for all $k \geq 2$.

The tiling problem $\mathtt{Tile}_k$ takes as input a triple $\mathscr{T} = (\mathcal{T}, \mathcal{H}, \mathcal{V})$ where $\mathcal{T}$ is a finite set of tile types and $\mathcal{H} \subseteq \mathcal{T} \times \mathcal{T}$ (resp. $\mathcal{V} \subseteq \mathcal{T} \times \mathcal{T}$) represents the horizontal (resp. vertical) matching relation, and an initial tile type $\mathsf{c} \in \mathcal{T}$. A solution for the instance $(\mathscr{T}, c)$ of the problem $\mathtt{Tile}_k$ is a mapping $\tau : [0, \mathfrak{t}(k,n)-1] \times [0, \mathfrak{t}(k,n)-1] \to \mathcal{T}$ such that

(first) $\tau(0,0) = \mathsf{c}$, and
(hor&vert) for all $i \in [0, \mathfrak{t}(k,n)-1]$ and $j \in [0, \mathfrak{t}(k,n)-2]$,
$$(\tau(j,i), \tau(j+1,i)) \in \mathcal{H} \text{ and } (\tau(i,j), \tau(i,j+1)) \in \mathcal{V}.$$
The problem of checking whether an instance of $\mathtt{Tile}_k$ has a solution is known to be k-NExpTime-complete (see [51, 54]).

The reduction below from $\mathtt{Tile}_k$ to $\mathrm{Sat}(\mathsf{ML}(*))$ recycles ideas from [7], where $\mathtt{Tile}_k$ is reduced to $\mathrm{Sat}(\mathrm{QK}^t)$ (see also a similar construction in [43]). Actually, in [7] the presentation uses mainly quantified CTL over trees restricted to the next-time modality EX. To provide the adequate adaptation for $\mathsf{ML}(*)$, we need to solve two major issues. First, QK^t admits second-order quantification, whereas in $\mathsf{ML}(*)$, the second-order features are limited to the separating conjunction $*$. Second, the second-order quantification of QK^t essentially colours the nodes in the tree-like Kripke-style structures without changing the frame (W,R). By contrast, the operator $*$ modifies the accessibility relation, possibly making worlds that were reachable from the current world, completely unreachable in submodels. The Tower-hardness proof for $\mathrm{Sat}(\mathsf{ML}(*))$ becomes then much more challenging. We would like to characterise the position on the grid encoded by a world w by exploiting some properties of its descendants (as done for QK^t). At the same time, we need to be careful and only consider submodels where the world w keeps encoding the same position. In a sense, our encoding is robust: when the operator $*$ is used to reason on submodels, we can enforce that no world changes the position of the grid that it encodes.

4.1 Principles for enforcing $\mathfrak{t}(j,n)$ children

In what follows, let $\mathfrak{M} = (W,R,V)$ be a finite forest. We consider two disjoint sets of atomic propositions $\mathsf{P} = \{p_1, \ldots, p_n, \mathtt{val}\}$ and $\mathtt{Aux} = \{\mathsf{x}, \mathsf{y}, \mathsf{l}, \mathsf{s}, \mathsf{r}\}$ (whose respective role is later defined).

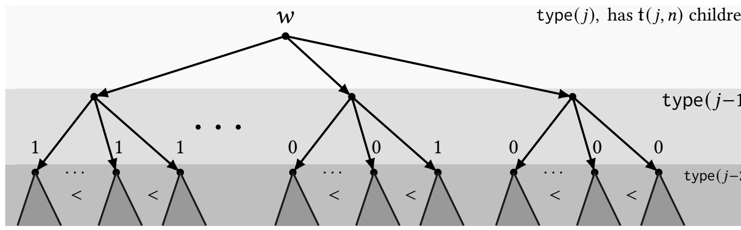


Fig. 1. Schema of a model satisfying $\mathtt{type}(j)$ (for $j \geq 2$).

Elements from $\mathtt{Aux}$ are understood as *auxiliary* propositions. We call $\mathtt{ax}$-*node* (resp. $\mathtt{Aux}$-*node*) a world satisfying the proposition $\mathtt{ax} \in \mathtt{Aux}$ (resp. satisfying some proposition in $\mathtt{Aux}$). We call *t-node* a world that satisfies the formula $\mathtt{t} \stackrel{\text{def}}{=} \bigwedge_{\mathtt{ax} \in \mathtt{Aux}} \neg \mathtt{ax}$. Every world of $\mathfrak{M}$ is either a t-node or an $\mathtt{Aux}$-node. We say that w' is a t-child of $w \in W$ if $w' \in R(w)$ and w' is a t-node. We define the concepts of $\mathtt{Aux}$-child and $\mathtt{ax}$-child analogously. The set of t-nodes is intended to form a tree with large numbers of children per node and to be well-balanced admitting some regularity properties on its structure. As expected, $\mathtt{Aux}$-nodes are auxiliary nodes for which removing incoming edges simulates propositional quantification.

The key development of our reduction is given by the definition of a formula, of exponential size in $j \geq 1$ and polynomial size in $n \geq 1$, that when satisfied by $(\mathfrak{M}, w)$ forces every t-node in $R^i(w)$, where $0 \leq i < j$, to have exactly $\mathfrak{t}(j-i,n)$ t-children, each of them encoding a different number in $[0, \mathfrak{t}(j-i,n)-1]$. As we impose that w is a t-node, it must have $\mathfrak{t}(j,n)$ t-children. We assume n to be fixed throughout the section and denote this formula by $\mathtt{type}(j)$. From the property above, if $\mathfrak{M}, w \models \mathtt{type}(j)$ then for all $i \in [1, j{-}1]$ and all t-nodes $w' \in R^i(w)$ we have $\mathfrak{M}, w' \models \mathtt{type}(j{-}i)$.

First, let us informally describe how numbers are encoded in the model $(\mathfrak{M}, w)$ satisfying $\mathtt{type}(j)$. Let $i \in [1, j]$. Given a t-node $w' \in R^i(w)$, $\mathbf{n}_i(w')$ denotes the number encoded by w'. We omit the subscript i when it is clear from the context. When $i = j$, we represent $\mathbf{n}(w')$ by using the truth values of the atomic propositions $p_1, \ldots, p_n$. The proposition p_b is responsible for the b-th bit of the number, with the least significant bit being encoded by p_1. For example, for $n = 3$, we have $\mathfrak{M}, w' \models p_3 \land p_2 \land \neg p_1$ whenever $\mathbf{n}(w') = 6$ (in binary, 110). The formula $\mathtt{type}(1)$ forces the parent of w' (i.e. is a t-node in $R^{j-1}(w)$) to have exactly 2^n t-children by requiring one t-child for each possible valuation upon $p_1, \ldots, p_n$. Otherwise, for $i < j$ (and therefore $j \geq 2$), the number $\mathbf{n}_i(w')$ is represented by the binary encoding of the truth values of $\mathtt{val}$ on the t-children of w' which, since $(\mathfrak{M}, w') \models \mathtt{type}(j-i)$, are $\mathfrak{t}(j-i,n)$ children implicitly ordered by the number they, in turn, encode. The essential property of $\mathtt{type}(j)$ is therefore the following: the numbers encoded by the t-children of a t-node $w'' \in R^i(w)$, represent positions in the binary representation of the number $\mathbf{n}_i(w'')$. Thanks to this property, the formula $\mathtt{type}(j)$ forces w to have exactly $\mathfrak{t}(j,n)$ children, all encoding different numbers in $[0, \mathfrak{t}(j,n)-1]$. This is roughly represented in Figure 1, where "1" stands for $\mathtt{val}$ being true whereas "0" stands for $\mathtt{val}$ being false. To characterise these trees in $\mathsf{ML}(*)$, we simulate second-order quantification by using $\mathtt{Aux}$-nodes. Informally, we require a pointed forest $(\mathfrak{M}, w)$ satisfying $\mathtt{type}(j)$ to be such that

(i) every t-node $w' \in R(w)$ has exactly one $\mathtt{x}$-child, and one (different) $\mathtt{y}$-child. These nodes do not satisfy any other auxiliary proposition;

(ii) for every $i \geq 2$, every t-node $w' \in R^i(w)$ has exactly five $\mathtt{Aux}$-children, one for each $\mathtt{ax} \in \mathtt{Aux}$.

We can simulate second-order existential quantification on t-nodes with respect to the symbol $\mathtt{ax} \in \mathtt{Aux}$ by using the operator $*$ in order to remove edges leading to $\mathtt{ax}$-nodes. Then, we evaluate whether a property holds on the resulting model where a t-node "satisfies" $\mathtt{ax} \in \mathtt{Aux}$ if it has a child satisfying ax. To better emphasise the need to move along t-nodes, given a formula φ, we write $\langle t\rangle\varphi$ for the formula $\Diamond(t \wedge \varphi)$. This formula is a relativised version of $\Diamond$ that only considers t-nodes. Dually, $[t]\varphi \stackrel{\text{def}}{=} \Box(t \Rightarrow \varphi)$. $\langle t\rangle^i$ and $[t]^i$ are also defined as expected.

Let us start to formalise this encoding. Let $j \geq 1$. First, we restrict ourselves to models where every t-node reachable in at most j steps does not have two Aux-children satisfying the same proposition. Moreover, these Aux-nodes have no children and only satisfy exactly one $\mathsf{ax} \in \mathsf{Aux}$. We express this condition with the formula $\mathtt{init}(j)$ below:

$$\mathtt{init}(j) \stackrel{\text{def}}{=} \boxplus^j \bigwedge_{\mathsf{ax}\in\mathsf{Aux}} \Big(\big(\mathsf{t} \Rightarrow \neg(\Diamond\mathsf{ax} * \Diamond\mathsf{ax})\big) \wedge \Box\big(\mathsf{ax} \Rightarrow \Box\bot \wedge \bigwedge_{\mathsf{bx}\in\mathsf{Aux}\setminus\{\mathsf{ax}\}} \neg\mathsf{bx}\big)\Big),$$

where $\boxplus^0\varphi \stackrel{\text{def}}{=} \varphi$ and $\boxplus^{m+1}\varphi \stackrel{\text{def}}{=} \varphi \wedge \Box \boxplus^m (\varphi)$.

In the following statements and proofs, let $\mathfrak{M} = (W, R, V)$ be a finite forest, $w \in W$ and $j \geq 1$.

Lemma 4.1. *$\mathfrak{M}, w \models \mathtt{init}(j)$ if and only if for every $0 \leq i \leq j$, $w' \in R^i(w)$ and $\mathsf{ax} \in \mathsf{Aux}$,*

(1) if $\mathfrak{M}, w' \models t$ then for all $w'_1, w'_2 \in R(w')$, if $\mathfrak{M}, w'_1 \models \mathsf{ax}$ and $\mathfrak{M}, w'_2 \models \mathsf{ax}$ then $w'_1 = w'_2$ (i.e. at most one child of w' satisfies ax);

(2) for every $w'' \in R(w')$, if $\mathfrak{M}, w'' \models \mathsf{ax}$, then $R(w'') = \emptyset$ (i.e. w'' does not have children) and it cannot be that $\mathfrak{M}, w'' \models \mathsf{bx}$ for some $\mathsf{bx} \in \mathsf{Aux}$ syntactically different from ax (i.e. among the propositions in Aux, w'' only satisfies ax).

Moreover, given $\mathfrak{M}' \sqsubseteq \mathfrak{M}$, $\mathfrak{M}', w \models \mathtt{init}(j)$.

Proof. The proof is straightforward (and hence here only sketched). Indeed, the statement "for every $0 \leq i \leq j$, every $w' \in R^i(w)$ and every $\mathsf{ax} \in \mathsf{Aux}$" is captured by the prefix $\boxplus^j \bigwedge_{\mathsf{ax}\in\mathsf{Aux}}$ of $\mathtt{init}(j)$. Then, (1) corresponds to the conjunct $t \Rightarrow \neg(\Diamond\mathsf{ax} * \Diamond\mathsf{ax})$ whereas (2) corresponds to the conjunct $\Box\big(\mathsf{ax} \Rightarrow \Box\bot \wedge \bigwedge_{\mathsf{bx}\in\mathsf{Aux}\setminus\{\mathsf{ax}\}} \neg\mathsf{bx}\big)$. ◻

Among the models $((W, R, V), w)$ satisfying $\mathtt{init}(j)$, we define the ones satisfying $\mathtt{type}(j)$ described below (see similar conditions in [7, Section IV]):

(sub$_j$) every t-node in $R(w)$ satisfies $\mathtt{type}(j-1)$;
(zero$_j$) there is a t-node $\tilde{w} \in R(w)$ such that $\mathbf{n}(\tilde{w}) = 0$;
(uniq$_j$) distinct t-nodes in $R(w)$ encode different numbers;
(compl$_j$) for every t-node $w_1 \in R(w)$ with $\mathbf{n}(w_1) < \mathfrak{t}(j, n) - 1$, there is a t-node $w_2 \in R(w)$ such that $\mathbf{n}(w_2) = \mathbf{n}(w_1) + 1$;
(aux) w is a t-node, every t-node in $R(w)$ has one x-child and one y-child, and every t-node in $R^2(w)$ has three children satisfying l, r and s, respectively.

We define $\mathtt{type}(0) \stackrel{\text{def}}{=} \top$, and for $j \geq 1$, $\mathtt{type}(j)$ is defined as

$$\mathtt{type}(j) \stackrel{\text{def}}{=} \mathtt{sub}(j) \wedge \mathtt{zero}(j) \wedge \mathtt{uniq}(j) \wedge \mathtt{compl}(j) \wedge \mathtt{aux},$$

where each conjunct expresses its homonymous property. The formulae $\mathtt{sub}(j)$, $\mathtt{aux}$ and $\mathtt{zero}(j)$ are defined as

$$\begin{aligned} \mathtt{sub}(j) &\stackrel{\text{def}}{=} [t]\mathtt{type}(j-1); \\ \mathtt{aux} &\stackrel{\text{def}}{=} t \wedge [t](\Diamond\mathsf{x} * \Diamond\mathsf{y}) \wedge [t]^2(\Diamond\mathsf{l} * \Diamond\mathsf{s} * \Diamond\mathsf{r}); \\ \mathtt{zero}(1) &\stackrel{\text{def}}{=} \langle t\rangle \textstyle\bigwedge_{b\in[1,n]} \neg p_b; \\ \mathtt{zero}(j+1) &\stackrel{\text{def}}{=} \langle t\rangle[t]\neg\mathtt{val}. \end{aligned}$$

The challenge is therefore how to express $\mathtt{uniq}(j)$ and $\mathtt{compl}(j)$, in order to guarantee that the numbers encoded by the children of w span all over $[0, \mathfrak{t}(j, n) - 1]$. The structural properties expressed by $\mathtt{type}(j)$ lead to strong constraints, which permits to control the effects of the separating conjunction $*$ when submodels are built. This is a key point in designing $\mathtt{type}(j)$ as it helps us to control which edges are lost when taking a submodel.

4.2 Nominals, forks and number comparisons

In order to define $\mathtt{uniq}(j)$ and $\mathtt{compl}(j)$ (completing the definition of $\mathtt{type}(j)$), we introduce auxiliary formulae, characterising classes of models that emerge naturally when trying to capture the semantics of (uniq$_j$) and (compl$_j$).

Let us consider a finite forest $\mathfrak{M} = (W, R, V)$ and $w \in W$. A first ingredient is given by the concept of *local nominals*, borrowed from [7]. We say that $\mathsf{ax} \in \mathsf{Aux}$ is a (local) nominal for the depth $i \geq 1$ if there is exactly one t-node $w' \in R^i(w)$ having an ax-child. In this case, w' is said to be the world that corresponds to the local nominal ax. The following formula states that ax is a local nominal for the depth i:

$$nom_i(ax) \stackrel{\text{def}}{=} \langle t\rangle^i \Diamond \mathsf{ax} \wedge \bigwedge_{k=0}^{i-1} [t]^k \neg\big(\langle t\rangle^{i-k} \Diamond \mathsf{ax} * \langle t\rangle^{i-k} \Diamond \mathsf{ax}\big).$$

Lemma 4.2. *Let* $\mathsf{ax} \in \mathsf{Aux}$ *and* $0 < i \leq j \in \mathbb{N}$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j)$. *Then,* $\mathfrak{M}, w \models \mathsf{nom}_i(\mathsf{ax})$ *if and only if* ax *is a local nominal for the depth* i.

The proof is direct by applying the semantics of the formula $\mathsf{nom}_i(\mathsf{ax})$, and is given in Appendix D. We define the formula:

$$@^i_{ax}\varphi \stackrel{\text{def}}{=} \langle t\rangle^i (\Diamond \mathsf{ax} \wedge \varphi),$$

which, under the hypothesis that ax is a local nominal for the depth i, states that φ holds on the t-node that corresponds to ax.

Lemma 4.3. *Let* $\mathsf{ax} \in \mathsf{Aux}$ *and* $0 < i \leq j \in \mathbb{N}$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j) \wedge \mathsf{nom}_i(\mathsf{ax})$. *Then,* $\mathfrak{M}, w \models @^i_{\mathsf{ax}}\varphi$ *iff* $\mathfrak{M}, w' \models \varphi$, *where* w' *is the world corresponding to the nominal* ax *for the depth* i.

Proof. Both directions are straightforward. As we are working under the hypothesis that $\mathfrak{M}, w \models \mathtt{init}(j) \wedge \mathsf{nom}_i(\mathsf{ax})$, by Lemma 4.2, ax is a nominal for the depth i. In the following, let w' be the world in $R^i(w)$ corresponding to the nominal ax (i.e. w' has an ax-child).

($\Rightarrow$): Suppose $\mathfrak{M}, w \models @^i_{\mathsf{ax}}\varphi$. By definition, there is $w'' \in R^i(w)$ such that $\mathfrak{M}, w'' \models \Diamond \mathsf{ax} \wedge \varphi$. Since ax is a nominal for the depth i, we conclude that $w' = w''$ and hence $\mathfrak{M}, w'' \models \varphi$.

($\Leftarrow$): Suppose that w' is such that $\mathfrak{M}, w' \models \varphi$. By definition, w' is the world corresponding to the nominal ax (for the depth i). Hence $\mathfrak{M}, w' \models \Diamond \mathsf{ax}$. Since $w' \in R^i(w)$, by $\mathfrak{M}, w \models \mathtt{init}(j)$ we conclude that there is a path of t-nodes from w to w', of length i. Thus, $\mathfrak{M}, w \models \langle t\rangle^i(\Diamond \mathsf{ax} \wedge \varphi)$. □

Moreover, we define $\mathsf{nom}_i(\mathsf{ax} \neq \mathsf{bx}) \stackrel{\text{def}}{=} \mathsf{nom}_i(\mathsf{ax}) \wedge \mathsf{nom}_i(\mathsf{bx}) \wedge \neg @^i_{\mathsf{ax}} \Diamond \mathsf{bx}$, which states that ax and bx are two nominals for the depth i with respect to two distinct t-nodes.

Lemma 4.4. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $0 < i \leq j \in \mathbb{N}$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j)$. *Then,* $\mathfrak{M}, w \models \mathsf{nom}_i(\mathsf{ax} \neq \mathsf{bx})$ *iff* ax *and* bx *are nominals for the depth* i, *corresponding to two different worlds.*

Proof. ($\Rightarrow$): Suppose $\mathfrak{M}, w \models \mathsf{nom}_i(\mathsf{ax} \neq \mathsf{bx})$. By Lemma 4.2, ax and bx are nominals for depth i. Let w_{ax} (resp. w_{bx}) be the world in $R^i(w)$ corresponding to the nominal ax (resp. bx). Note that $\mathfrak{M}, w_{\mathsf{bx}} \models \Diamond \mathsf{bx}$. By $\mathfrak{M}, w \models \neg @^i_{\mathsf{ax}} \Diamond \mathsf{bx}$ and Lemma 4.3, we get $\mathfrak{M}, w_{\mathsf{ax}} \not\models \Diamond \mathsf{bx}$. Thus, $w_{\mathsf{ax}} \neq w_{\mathsf{bx}}$.

($\Leftarrow$): This direction is analogous and simply relies on Lemmata 4.2 and 4.3. □

As a second ingredient, we introduce the notion of *fork* that is a specific type of models naturally emerging when trying to compare the numbers $\mathbf{n}(w_1)$ and $\mathbf{n}(w_2)$ of two worlds $w_1, w_2 \in R^i(w)$ (e.g. when checking whether $\mathbf{n}(w_1) = \mathbf{n}(w_2)$ or $\mathbf{n}(w_2) = \mathbf{n}(w_1) + 1$ holds). Given $j \geq i \geq 1$ we introduce the formula $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$ that is satisfied by $(\mathfrak{M}, w)$ if and only if:

- ax and bx are nominals for the depth i.
- w has exactly two t-children, say w_U and w_D.
- For every $k \in [1, i-1]$, both $R^k(w_U)$ and $R^k(w_D)$ contain exactly one t-child.

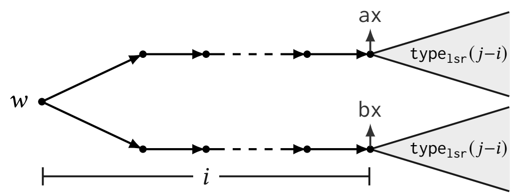


Fig. 2. Schema of a pointed forest $(\mathfrak{M}, w)$ satisfying $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$.

- The only t-node in $R^{i-1}(w_U)$, say w_{ax}, corresponds to the nominal ax. The only t-node in $R^{i-1}(w_D)$, say w_{bx}, corresponds to the nominal bx.
- If $i < j$, then $(\mathfrak{M}, w_{\mathsf{ax}})$ and $(\mathfrak{M}, w_{\mathsf{bx}})$ satisfy

$$\mathtt{type}_{\mathtt{lsr}}(j-i) \stackrel{\text{def}}{=} \mathtt{type}(j-i) \wedge [t](\Diamond \mathtt{l} \wedge \Diamond \mathtt{s} \wedge \Diamond \mathtt{r}).$$

It should be noted that, whenever $(\mathfrak{M}, w)$ satisfies the formula $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$, we witness two paths of length i, both starting at w and leading to w_{ax} and w_{bx}, respectively. Worlds in this path may have Aux-children. Figure 2 schematises a model satisfying $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$.
Since the definition of $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$ is recursive on i and j (due to $\mathtt{type}(j-i)$), we postpone its formal definition to the next two sections where we treat the base cases for $i = j$ and the inductive case for $j > i$ separately.

The last auxiliary formulae are $[\mathsf{ax} < \mathsf{bx}]^i_j$ and $[\mathsf{bx} = \mathsf{ax}+1]_j$. Under the hypothesis that $(\mathfrak{M}, w)$ satisfies $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$, the formula $[\mathsf{ax} < \mathsf{bx}]^i_j$ is satisfied whenever the two (distinct) worlds $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^i(w)$ corresponding to the nominals ax and bx are such that $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$. Similarly, under the hypothesis that $(\mathfrak{M}, w)$ satisfies $\mathtt{fork}^1_j(\mathsf{ax}, \mathsf{bx})$, the formula $[\mathsf{bx} = \mathsf{ax}+1]_j$ is satisfied whenever $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$ holds. Both formulae are recursively defined, with base cases for $i = j$ and $j = 1$, respectively.

For the base case, we define the formulae $\mathtt{fork}^j_j(\mathsf{ax}, \mathsf{bx})$ and $[\mathsf{ax} < \mathsf{bx}]^j_j$ (for arbitrary j), as well as $[\mathsf{bx} = \mathsf{ax}+1]_1$. From these formulae, we are then able to define $\mathtt{uniq}(1)$ and $\mathtt{compl}(1)$, which completes the characterisation of $\mathtt{type}(1)$ and $\mathtt{type}_{\mathtt{lsr}}(1)$. Afterwards, we consider the case $1 \leq i < j$ and $j \geq 2$, and define $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$, $[\mathsf{ax} < \mathsf{bx}]^i_j$, $[\mathsf{bx} = \mathsf{ax}+1]_j$, as well as $\mathtt{uniq}(j)$ and $\mathtt{compl}(j)$, by only relying on formulae that are already defined (by inductive reasoning).

4.3 Formal semantics of the inductively defined formulae used for $\mathtt{type}(j)$

Let us summarise the expected semantics of the formulae introduced to define $\mathtt{type}(j)$, and whose definition is inductive. Let $\mathfrak{M} = (W, R, V)$ be a finite forest, $w \in W$, $1 \leq i \leq j$ and $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$.

Formula $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$: Suppose $\mathfrak{M}, w \models \mathtt{init}(j)$.

$\mathfrak{M}, w \models \mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$ if and only if

(i) w has exactly two t-children and exactly two paths of t-nodes, both of length i;
(ii) one of these two paths ends on a world (say w_{ax}) corresponding to the nominal ax whereas the other ends on a world (say w_{bx}) corresponding to the nominal bx;
(iii) if $i < j$ then $(\mathfrak{M}, w_{\mathsf{ax}})$ and $(\mathfrak{M}, w_{\mathsf{bx}})$ satisfy $\mathtt{type}_{\mathtt{lsr}}(j{-}i) \stackrel{\text{def}}{=} \mathtt{type}(j{-}i) \wedge [t](\Diamond \mathtt{l} \wedge \Diamond \mathtt{s} \wedge \Diamond \mathtt{r})$.

Formula $[\mathsf{ax} < \mathsf{bx}]^i_j$: Suppose $\mathfrak{M}, w \models \mathtt{init}(j) \wedge \mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$.

$\mathfrak{M}, w \models [\mathsf{ax} < \mathsf{bx}]^i_j$ if and only if there are two distinct t-nodes $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^i(w)$ such that w_{ax} corresponds to the nominal ax, w_{bx} corresponds to the nominal bx and $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$.

Formula $[\mathsf{bx} = \mathsf{ax}+1]_j$: Suppose $\mathfrak{M}, w \models \mathtt{init}(j) \wedge \mathtt{fork}^1_j(\mathsf{ax}, \mathsf{bx})$.

$\mathfrak{M}, w \models [\mathsf{bx} = \mathsf{ax}+1]_j$ if and only if there are two distinct t-nodes $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$ s.t. w_{ax} corresponds to the nominal ax, w_{bx} corresponds to the nominal bx and $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$.

Formula $\texttt{uniq}(j)$**:** Suppose $\mathfrak{M}, w \models \texttt{init}(j) \land \texttt{sub}(j) \land \texttt{aux}$.
$\mathfrak{M}, w \models \texttt{uniq}(j)$ if and only if $(\mathfrak{M}, w)$ satisfies (uniq$_j$), i.e. distinct t-nodes in $R(w)$ encode different numbers.

Formula $\texttt{compl}(j)$**:** Suppose $\mathfrak{M}, w \models \texttt{init}(j) \land \texttt{sub}(j) \land \texttt{aux}$.
$\mathfrak{M}, w \models \texttt{compl}(j)$ if and only if $(\mathfrak{M}, w)$ satisfies (compl$_j$), i.e. for every t-node $w_1 \in R(w)$, if $\mathbf{n}(w_1) < \mathfrak{t}(j,n) - 1$ then $\mathbf{n}(w_2) = \mathbf{n}(w_1) + 1$ for some t-node $w_2 \in R(w)$.

Formula $\texttt{type}(j)$**:** Suppose $\mathfrak{M}, w \models \texttt{init}(j)$.
$\mathfrak{M}, w \models \texttt{type}(j)$ if and only if $(\mathfrak{M}, w)$ satisfies (sub$_j$), (zero$_j$), (uniq$_j$), (compl$_j$) and (aux).

The formulae $\texttt{sub}(j)$, $\texttt{aux}$ and $\texttt{zero}(j)$ ($j \geq 1$) are also required in order to define correctly $\texttt{type}(j)$. However their definition and proof of correctness are straightforward. Hence we omit the proofs, and simply state the expected semantics of these formulae. It should be noted that a formal proof of $\texttt{zero}(j)$ relies on $\texttt{type}(j-1)$, which (as we will see multiple times in the next sections), we can assume to be correctly defined by inductive hypothesis (on j).

Lemma 4.5. *Let $j \geq 1$. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$.*

- *$\mathfrak{M}, \mathfrak{w} \models \texttt{sub}(j)$ iff $(\mathfrak{M}, w)$ satisfies (sub$_j$), i.e. every t-node in $R(w)$ satisfies $\texttt{type}(j-1)$.*
- *$\mathfrak{M}, \mathfrak{w} \models \texttt{aux}$ iff $(\mathfrak{M}, w)$ satisfies (aux), i.e. w is a t-node, every t-node in $R(w)$ has one $\texttt{x}$-child and one $\texttt{y}$-child, and every t-node in $R^2(w)$ has three children satisfying $\texttt{l}$, $\texttt{r}$ and $\texttt{s}$, respectively.*
- *Suppose $\mathfrak{M}, \mathfrak{w} \models \texttt{sub}(j)$. $\mathfrak{M}, w \models \texttt{zero}(j)$ iff $(\mathfrak{M}, w)$ satisfies (zero$_j$), i.e. there is a t-node $\tilde{w} \in R(w)$ s.t. $\mathbf{n}(\tilde{w}) = 0$.*

We now prove the correctness of the formulae listed before Lemma 4.5, starting from the base case where $j = 1$ or $i = j$, to then show the proof for $1 \leq i < j$.

4.4 Base cases: $i = j$ or $j = 1$

In what follows, we consider a finite forest $\mathfrak{M} = (W, R, V)$ and a world w. Following its informal description, we have

$$fork^j_j(ax, bx) \stackrel{\text{def}}{=} \Diamond_{=2} t \land [t] \boxplus^{j-2} (t \Rightarrow \Diamond_{=1} t) \land \texttt{nom}_j(\texttt{ax} \neq \texttt{bx}),$$

where $\boxplus^j \varphi \stackrel{\text{def}}{=} \top$ for $j < 0$. We recall that t and $\Diamond_{=2} t$ are defined as

$$t = \bigwedge\nolimits_{\texttt{ax} \in \texttt{Aux}} \neg\texttt{ax}, \quad \Diamond_{=1} t = \Diamond t \land \neg(\Diamond t * \Diamond t), \qquad \Diamond_{=2} t = (\Diamond t * \Diamond t) \land \neg(\Diamond t * \Diamond t * \Diamond t).$$

Lemma 4.6. *Let $\texttt{ax} \neq \texttt{bx} \in \texttt{Aux}$, $j \geq 1$. Suppose $\mathfrak{M}, w \models \texttt{init}(j)$. Then, $\mathfrak{M}, w \models \texttt{fork}^j_j(\texttt{ax}, \texttt{bx})$ iff*

(1) w has exactly two t-children and exactly two paths of t-nodes, both of length j, ending in two t-nodes (say w_1 and w_2);

(2) w_1 (resp. w_2) corresponds to the nominal $\texttt{ax}$ (resp. $\texttt{bx}$) for the depth j.

Proof. ($\Rightarrow$): Suppose $\mathfrak{M}, w \models \texttt{fork}^j_j(\texttt{ax}, \texttt{bx})$. By $\mathfrak{M}, w \models \Diamond_{=2} t$, w has exactly two t-children (let us say w'_1 and w'_2). Then, by $\mathfrak{M}, w \models [t] \boxplus^{j-2} (t \Rightarrow \Diamond_{=1} t)$, it is easy to show that there is exactly one path of t-nodes of length $j-1$, starting in w'_1 (resp. w'_2) and ending in a t-node $w_1 \in R^j(w)$ (resp. $w_2 \in R^j(w)$). Then, the property (1) of the statement is verified. The property (2) of the statement follows by simply applying Lemma 4.4.

($\Leftarrow$): This direction is straightforward. In short, from (1), $\mathfrak{M}, w \models \Diamond_{=2} t \land [t] \boxplus^{j-2} (t \Rightarrow \Diamond_{=1} t)$, whereas from (2) together with Lemma 4.4 we have $\mathfrak{M}, w \models \texttt{nom}_j(\texttt{ax} \neq \texttt{bx})$. □

As previously explained, in the base case, the number $\mathbf{n}(w')$ encoded by a t-node $w' \in R^j(w)$ is represented by the truth values of $p_1, \ldots, p_n$. Then, the formula $[\texttt{ax} < \texttt{bx}]^j_j$ is defined as

$$[\texttt{ax} < \texttt{bx}]^j_j \stackrel{\text{def}}{=} \bigvee_{u=1}^{n} \big(@^j_{\texttt{ax}} \neg p_u \land @^j_{\texttt{bx}} p_u \land \bigwedge_{v=u+1}^{n} (@^j_{\texttt{ax}} p_v \Leftrightarrow @^j_{\texttt{bx}} p_v)\big).$$

The satisfaction of $(\mathfrak{M}, w) \models \mathtt{fork}_j^j(\mathsf{ax}, \mathsf{bx})$ enforces that the distinct t-nodes $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^j(w)$ corresponding to ax and bx satisfy $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$, which can be shown by using standard properties about bit vectors. Intuitively, the formula states that there is a bit (encoded by p_u) which is set to 0 in the binary encoding of $\mathbf{n}(w_{\mathsf{ax}})$ but is set to 1 in the binary encoding of $\mathbf{n}(w_{\mathsf{bx}})$, whereas every successive bit (encoded by p_v with $v > u$) is set to 1 in $\mathbf{n}(w_{\mathsf{ax}})$ iff it is set to 1 also in $\mathbf{n}(w_{\mathsf{bx}})$.

Lemma 4.7. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $j \geq 1$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathtt{fork}_j^j(\mathsf{ax}, \mathsf{bx})$. *Then,* $\mathfrak{M}, w \models [\mathsf{ax} < \mathsf{bx}]_j^j$ *if and only if there are two distinct t-nodes* $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^j(w)$ *such that* w_{ax} *corresponds to the nominal* ax, w_{bx} *corresponds to the nominal* bx *and* $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$.

Proof. Let x, y be natural numbers represented in binary by using n bits. Let us denote with x_i (resp. y_i) the i-th bit of the binary representation of x (resp. y). We have that $x < y$ if and only if

(A) there is a position $i \in [1, n]$ such that $x_i = 0$ and $y_i = 1$;
(B) for every position $j > i$, $x_j = 1 \Leftrightarrow y_j = 1$.

The formula $[\mathsf{ax} < \mathsf{bx}]_j^j$ uses exactly this characterisation in order to state that $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$.

In the following, since we are working under the hypothesis that $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathtt{fork}_j^j(\mathsf{ax}, \mathsf{bx})$, let w_{ax} (resp. w_{bx}) be the world corresponding to the nominal ax (resp. bx), w.r.t. the depth j.

($\Rightarrow$): Suppose $\mathfrak{M}, w \models [\mathsf{ax} < \mathsf{bx}]_j^j$. Then there is $u \in [1, n]$ such that

$$\mathfrak{M}, w \models @_{\mathsf{ax}}^j \neg p_u \land @_{\mathsf{bx}}^j\, p_u \land \bigwedge_{v=u+1}^{n} (@_{\mathsf{ax}}^j\, p_v \Leftrightarrow @_{\mathsf{bx}}^j\, p_v).$$

By Lemma 4.3 and $\mathfrak{M}, w \models @_{\mathsf{ax}}^j \neg p_u \land @_{\mathsf{bx}}^j\, p_u$, we conclude that $\mathfrak{M}, w_{\mathsf{ax}} \models \neg p_u$ and $\mathfrak{M}, w_{\mathsf{bx}} \models p_u$. Hence, the u-th bit is 0 in the number encoded by w_{ax}, whereas it is 1 in the number encoded by w_{bx}, as required by (A). Similarly, by Lemma 4.3 and $\mathfrak{M}, w \models \bigwedge_{v \in [u+1,n]} (@_{\mathsf{ax}}^j\, p_v \Leftrightarrow @_{\mathsf{bx}}^j\, p_v)$, we conclude that for every $v \in [u+1, n]$, $\mathfrak{M}, w_{\mathsf{ax}} \models p_v$ if and only if $\mathfrak{M}, w_{\mathsf{bx}} \models p_v$. This corresponds to the property (B) above, leading to $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$.

($\Leftarrow$): This direction follows similar arguments (backwards). ☐

The formula $[\mathsf{bx} = \mathsf{ax}{+}1]_1$ uses similar arithmetical properties. It is defined as

$$[\mathsf{bx} = \mathsf{ax}{+}1]_1 \stackrel{\text{def}}{=} \bigvee_{u=1}^{n} \Big(@_{\mathsf{ax}}^1(\neg p_u \land \bigwedge_{v=1}^{u-1} p_v) \land @_{\mathsf{bx}}^1(p_u \land \bigwedge_{v=1}^{u-1} \neg p_v) \land \bigwedge_{v=u+1}^{n} (@_{\mathsf{ax}}^1 p_v \Leftrightarrow @_{\mathsf{bx}}^1 p_v)\Big).$$

Assuming $(\mathfrak{M}, w) \models \mathtt{fork}_1^1(\mathsf{ax}, \mathsf{bx})$, this formula states that the two distinct t-nodes $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$ corresponding to ax and bx are such that $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$. As done for $[\mathsf{ax} < \mathsf{bx}]_j^j$, this formula states that there must be a bit (encoded by p_u) which is set to 0 in the binary encoding of $\mathbf{n}(w_{\mathsf{ax}})$ but is set to 1 in the binary encoding of $\mathbf{n}(w_{\mathsf{bx}})$; and that every successive bit (encoded by p_v with $v > u$) is set to 1 in $\mathbf{n}(w_{\mathsf{ax}})$ if and only if it is set to 1 also in $\mathbf{n}(w_{\mathsf{bx}})$. However, differently from $[\mathsf{ax} < \mathsf{bx}]_j^j$, this formula also requires that every bit before p_u (encoded by p_v with $v < u$) is set to 1 in the binary encoding of $\mathbf{n}(w_{\mathsf{ax}})$ but is set to 0 in the binary encoding of $\mathbf{n}(w_{\mathsf{bx}})$.

Lemma 4.8. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $\mathfrak{M}, w \models \mathtt{init}(1) \land \mathtt{fork}_1^1(\mathsf{ax}, \mathsf{bx})$. *Then,* $\mathfrak{M}, w \models [\mathsf{bx} = \mathsf{ax}{+}1]_1$ *if and only if there are two distinct t-nodes* $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$ *such that* w_{ax} *corresponds to the nominal* ax, w_{bx} *corresponds to the nominal* bx *and* $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$.

Proof. The proof uses standard properties of numbers encoded in binary. Let x, y be two natural numbers that can be represented in binary by using n bits. Let us denote with x_i (resp. y_i) the i-th bit of the binary representation of x (resp. y). We have that $y = x + 1$ if and only if

(A) there is a position $i \in [1, n]$ such that $x_i = 0$ and $y_i = 1$;

(B) for every position $j > i$, $x_j = 1 \Leftrightarrow y_j = 1$;
(C) for every position $j < i$, $x_j = 1$ and $y_j = 0$.

Notice that (A) and (B) are as in the characterisation of $x < y$ given in Lemma 4.7. The formula $[\mathsf{bx} = \mathsf{ax}{+}1]_1$ uses exactly this characterisation in order to state that $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$.

Since we are working under the hypothesis that $\mathfrak{M}, w \models \mathtt{init}(1) \wedge \mathtt{fork}_1^1(\mathsf{ax}, \mathsf{bx})$, there are two distinct worlds w_{ax} and w_{bx} corresponding to the two nominals ax and bx for the depth 1, respectively. Then, the proof of this lemma follows closely the proof of Lemma 4.7, and enforcing (C) by means of the subformula $@^1_{\mathsf{ax}}(\neg p_u \wedge \bigwedge_{v \in [1,u-1]} p_v) \wedge @^1_{\mathsf{bx}}(p_u \wedge \bigwedge_{v \in [1,u-1]} \neg p_v)$. □

To define $\mathtt{uniq}(1)$, we first recall that a model satisfying $\mathtt{type}(1)$ satisfies the formula aux and hence every t-node in $R(w)$ has two children, one x-node and one y-node. The idea is to use these two Aux-children and to take advantage of $*$ in order to state that it is not possible to find a submodel of $\mathfrak{M}$ such that w has only two distinct children w_{x} and w_{y} corresponding to the nominals x and y, respectively, and such that $\mathbf{n}(w_{\mathsf{x}}) = \mathbf{n}(w_{\mathsf{y}})$. In a sense, the operator $*$ simulates a second-order quantification on x and y. Let $[\mathsf{x} = \mathsf{y}]^1_1 \stackrel{\text{def}}{=} \neg([\mathsf{x} < \mathsf{y}]^1_1 \vee [\mathsf{y} < \mathsf{x}]^1_1)$. The corresponding formula is

$$\mathtt{uniq}(1) \stackrel{\text{def}}{=} \neg\big(\top * (\mathtt{fork}^1_1(\mathsf{x},\mathsf{y}) \wedge [\mathsf{x} = \mathsf{y}]^1_1)\big).$$

Lemma 4.9. *Suppose $\mathfrak{M}, w \models \mathtt{init}(1) \wedge \mathsf{aux}$. Then, $\mathfrak{M}, w \models \mathtt{uniq}(1)$ if and only if $(\mathfrak{M}, w)$ satisfies (uniq$_1$), i.e. distinct t-nodes in $R(w)$ encode different numbers.*

Proof. ($\Rightarrow$): Contrapositively, suppose that there are two distinct t-nodes w_{x} and w_{y} encoding the same number. Since $\mathfrak{M}, w \models \mathtt{init}(1) \wedge \mathsf{aux}$, every world in $R(w)$ has exactly one child satisfying x and exactly one (different) child satisfying y. Let us then consider the submodel $\mathfrak{M}' = (W, R_1, V)$ where $R_1(w) = \{w_{\mathsf{x}}, w_{\mathsf{y}}\}$, $R_1(w_{\mathsf{x}}) = \{w_1\}$ and $R_1(w_{\mathsf{y}}) = \{w_2\}$, so that w_1 satisfies x whereas w_2 satisfies y. By Lemma 4.6, $\mathfrak{M}', w \models \mathtt{fork}^1_1(\mathsf{x},\mathsf{y})$. By hypothesis, $\mathbf{n}(w_{\mathsf{x}}) = \mathbf{n}(w_{\mathsf{y}})$ and therefore we also have $\mathfrak{M}', w \models [\mathsf{x} = \mathsf{y}]^1_1$. Thus, by definition, $\mathfrak{M}, w \not\models \mathtt{uniq}(1)$.

($\Leftarrow$): Again contrapositively, suppose $\mathfrak{M}, w \not\models \mathtt{uniq}(1)$ and so $\mathfrak{M}, w \models \top * (\mathtt{fork}^1_1(\mathsf{x},\mathsf{y}) \wedge [\mathsf{x} = \mathsf{y}]^1_1)$. Then, there is a submodel $\mathfrak{M}' = (W, R_1, V)$ of $\mathfrak{M}$ such that $\mathfrak{M}', w \models \mathtt{fork}^1_1(\mathsf{x},\mathsf{y}) \wedge [\mathsf{x} = \mathsf{y}]^1_1$. Moreover, since the satisfaction of $\mathtt{init}(1)$ is preserved under submodels, we have $\mathfrak{M}', w \models \mathtt{init}(1)$. We can then apply Lemmata 4.6 and 4.7 in order to conclude that there are two distinct worlds w_{x} and w_{y} in $R'(w)$ such that $\mathbf{n}(w_{\mathsf{x}}) = \mathbf{n}(w_{\mathsf{y}})$. Since the encoding of a number (for $j = 1$) only depends on the satisfaction of the propositional symbols $p_1, \dots, p_n$ on a certain world, we conclude that the same property holds for $\mathfrak{M}$: the two worlds w_{x} and w_{y} in $R(w)$ are such that $\mathbf{n}(w_{\mathsf{x}}) = \mathbf{n}(w_{\mathsf{y}})$. Therefore, $(\mathfrak{M}, w)$ does not satisfy (uniq$_1$). □

Let us now consider $\mathtt{compl}(1)$. As done for $\mathtt{uniq}(1)$, we rely on the auxiliary propositions x and y and use the separating conjunction $*$ in order to simulate a second-order quantification. We need to state that it is not possible to find a submodel of $\mathfrak{M}$ that looses x-nodes from $R^2(w)$, keeps all y-nodes, and is such that

(i) x is a local nominal for the depth 1, corresponding to a world w_{x} encoding $\mathbf{n}(w_{\mathsf{x}}) < 2^n - 1$;
(ii) there is no submodel where w has two t-children, w_{x} and a second world w_{y}, such that w_{y} corresponds to the nominal y and $\mathbf{n}(w_{\mathsf{y}}) = \mathbf{n}(w_{\mathsf{x}}){+}1$.

Thus, $\mathtt{compl}(1)$ is defined as:

$$compl(1) \stackrel{\text{def}}{=} \neg\Big(\Box\bot * \big([t]\Diamond\mathsf{y} \wedge @^1_{\mathsf{x}}\neg 1_1 \wedge \neg(\top * (\mathtt{fork}^1_1(\mathsf{x},\mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1))\big)\Big).$$

The subscript "1" in the formula 1_1 refers to the fact that we are treating the base case of $\mathtt{compl}(j)$ with $j = 1$. We have $1_1 \stackrel{\text{def}}{=} \bigwedge_{i \in [1,n]} p_i$, reflecting the encoding of $2^n - 1$.

Lemma 4.10. *Suppose $\mathfrak{M}, w \models \mathtt{init}(1) \wedge \mathsf{aux}$. Then, $\mathfrak{M}, w \models \mathtt{compl}(1)$ iff $(\mathfrak{M}, w)$ satisfies (compl$_1$), i.e. for every t-node $w_1 \in R(w)$, if $\mathbf{n}(w_1) < 2^n - 1$ then $\mathbf{n}(w_2) = \mathbf{n}(w_1) + 1$ for some t-node $w_2 \in R(w)$.*

Proof. ($\Rightarrow$): Suppose $\mathfrak{M}, w \models \mathtt{compl}(1)$. By definition of $\models$, this implies that for any $\mathfrak{M}' = (W, R', V)$ submodel of $\mathfrak{M}$ such that $R'(w) = R(w)$, if $\mathfrak{M}', w \models [t]\Diamond \mathsf{y} \wedge @^1_{\mathsf{x}}\neg 1_1$, then $\mathfrak{M}', w \models \top * (\mathtt{fork}^1_1(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1)$. Then, let us pick a t-node $w_{\mathsf{x}} \in R'(w) = R(w)$ such that $\mathbf{n}(w_{\mathsf{x}}) < 2^n - 1$. We show that there must be a world $w_{\mathsf{y}} \in R'(w)$ such that $\mathbf{n}(w_{\mathsf{y}}) = \mathbf{n}(w_{\mathsf{x}}) + 1$. Let us consider the submodel $\mathfrak{M}'' = (W, R', V)$ of $\mathfrak{M}$ such that for every $\overline{w} \in W$, if $\overline{w} \neq w_{\mathsf{x}}$ then $R'(\overline{w}) = R(\overline{w})$ and otherwise $R'(w_{\mathsf{x}}) = \{w_1\}$ where w_1 is the only Aux-child of w_{x} (w.r.t. R) satisfying x. Notice that w_1 exists and it is unique by $\mathfrak{M}, w \models \mathtt{init}(1) \wedge \mathtt{aux}$. Moreover, w_{x} corresponds in $\mathfrak{M}'$ to the nominal x for the depth 1. Again by $\mathfrak{M}, w \models \mathtt{init}(1) \wedge \mathtt{aux}$, we conclude that $\mathfrak{M}', w \models [t]\Diamond \mathsf{y}$. Moreover, since $\mathbf{n}(w_{\mathsf{x}}) < 2^n - 1$, by Lemma 4.3 we have $\mathfrak{M}', w \models @^1_{\mathsf{x}}\neg 1_1$. Hence by hypothesis, $\mathfrak{M}', w \models \top * (\mathtt{fork}^1_1(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1)$. Then, let $\mathfrak{M}'' = (W, R'', V) \sqsubseteq \mathfrak{M}'$ be such that $\mathfrak{M}'', w \models \mathtt{fork}^1_1(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1$. By Lemmata 4.6 and 4.8, there is $w_{\mathsf{y}} \in R''(w)$ such that $\mathbf{n}(w_{\mathsf{y}}) = \mathbf{n}(w_{\mathsf{x}}) + 1$. Since the encoding of a number (for $j = 1$) only depends on the satisfaction of the propositional symbols $p_1, \ldots, p_n$ on a certain world, we conclude that the same property holds for $\mathfrak{M}$. Thus, $(\mathfrak{M}, w)$ satisfies (compl$_1$).

($\Leftarrow$): Suppose that $(\mathfrak{M}, w)$ satisfies (compl$_1$), and *ad absurdum* assume that $\mathfrak{M}, w \not\models \mathtt{compl}(1)$, hence $\mathfrak{M}, w \models \Box\bot * \big([t]\Diamond \mathsf{y} \wedge @^1_{\mathsf{x}}\neg 1_1 \wedge \neg(\top * (\mathtt{fork}^1_1(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1))\big)$. Then, there is a submodel $\mathfrak{M}' = (W, R', V)$ of $\mathfrak{M}$ such that $R'(w) = R(w)$ and $\mathfrak{M}', w \models [t]\Diamond \mathsf{y} \wedge @^1_{\mathsf{x}}\neg 1_1 \wedge \neg(\top * (\mathtt{fork}^1_1(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1))$. Notice that this formula does not enforce x to be a nominal for the depth 1, however from $\mathfrak{M}', w \models @^1_{\mathsf{x}}\neg 1_1$ we deduce that there is at least one t-node w_{x} such that $\mathfrak{M}', w_{\mathsf{x}} \models \Diamond \mathsf{x} \wedge \neg 1_1$. Then, $\mathbf{n}(w_{\mathsf{x}}) < 2^n - 1$ and by hypothesis there is a t-node w_{y} such that $\mathbf{n}(w_{\mathsf{y}}) = \mathbf{n}(w_{\mathsf{x}}) + 1$. Let us consider now the submodel $\mathfrak{M}'' = (W, R'', V)$ of $\mathfrak{M}'$ where $R''(w) = \{w_{\mathsf{x}}, w_{\mathsf{y}}\}$, $R''(w_{\mathsf{x}}) = \{w_1\}$ and $R''(w_{\mathsf{y}}) = \{w_2\}$, where w_1 (resp. w_2) is the only Aux-child of w_{x} (resp. w_{y}) that satisfies x (resp. y). The existence of w_1 and w_2 is guaranteed by $\mathfrak{M}', w_{\mathsf{x}} \models \Diamond \mathsf{x} \wedge \neg 1_1$ and $\mathfrak{M}', w \models [t]\Diamond \mathsf{y}$. By Lemma 4.6, $\mathfrak{M}'', w \models \mathtt{fork}^1_1(\mathsf{x}, \mathsf{y})$. Moreover, as the encoding of a number (for $j = 1$) only depends on the satisfaction of the propositional symbols $p_1, \ldots, p_n$ on a certain world, $\mathfrak{M}'', w \models [\mathsf{y} = \mathsf{x}{+}1]_1$. Then, we conclude that $\mathfrak{M}', w \models \top * (\mathtt{fork}^1_1(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1)$, in contradiction with $\mathfrak{M}', w \models [t]\Diamond \mathsf{y} \wedge @^1_{\mathsf{x}}\neg 1_1 \wedge \neg(\top * (\mathtt{fork}^1_1(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} = \mathsf{x}{+}1]_1))$. Thus, $\mathfrak{M}, w \models \mathtt{compl}(1)$. □

With all these definitions at hand, we conclude the definition of $\mathtt{type}(1)$ (and $\mathtt{type}_{\mathtt{1sr}}(1)$), which is established correct with respect to its specification.

Lemma 4.11. *Let $\mathfrak{M}, w \models \mathtt{init}(1)$. We have $\mathfrak{M}, w \models \mathtt{type}(1)$ if and only if $(\mathfrak{M}, w)$ satisfies (sub$_1$), (zero$_1$), (uniq$_1$), (compl$_1$) and (aux).*

The proof of Lemma 4.11 then follows directly from Lemmata 4.5, 4.9 and 4.10. Let us show the satisfiability of $\mathtt{type}(1)$. A quick check of $\mathtt{init}(1)$ and the conditions (sub$_1$), (zero$_1$), (uniq$_1$), (compl$_1$) and (aux) should convince the reader that they are simultaneously satisfiable, leading to $\mathtt{init}(1) \wedge \mathtt{type}(1)$ being satisfiable. However, in the following we provide an explicit model satisfiying this formula.

Lemma 4.12. *The formula $\mathtt{init}(1) \wedge \mathtt{type}(1)$ is satisfiable.*

Proof. Consider the finite forest $\mathfrak{M} = (W, R, V)$ and a world w such that

(1) R is the minimal set of pairs such that $R(w) = \{w_0, \ldots, w_{2^n-1}\}$ (where $w_0, \ldots, w_{2^n-1}$ are all distinct worlds), and for every $i \in [0, 2^n - 1]$, $R(w_i) = \{w^{\mathsf{x}}_i, w^{\mathsf{y}}_i\}$ (again, $w^{\mathsf{x}}_i, w^{\mathsf{y}}_i$ are distinct);
(2) $W = \{w\} \cup R(w) \cup \bigcup_{w' \in R(w)} R(w')$;
(3) $V(\mathsf{x}) = \{w^{\mathsf{x}}_0, \ldots, w^{\mathsf{x}}_{2^n-1}\}$, $V(\mathsf{y}) = \{w^{\mathsf{y}}_0, \ldots, w^{\mathsf{y}}_{2^n-1}\}$ and for every $i \in [0, 2^n - 1]$ and $j \in [1, n]$, $w_i \in V(p_j)$ if and only if the j-th bit in the binary encoding of i is 1.

It is easy to check that $(\mathfrak{M}, w)$ satisfies $\mathtt{init}(1)$ as well as (sub$_1$), (zero$_1$), (uniq$_1$), (compl$_1$) and (aux). Thus, by Lemma 4.11 $\mathfrak{M}, w \models \mathtt{init}(1) \wedge \mathtt{type}(1)$. □

4.5 Inductive case: $1 \leq i < j$

We now need to define the inductive cases for the corresponding formulae, and prove their correctness. As an implicit inductive hypothesis used to prove that the formulae are well-defined, we assume that $[\mathtt{bx} = \mathtt{ax{+}1}]_{j'}$ and $\mathtt{type}(j')$ are already defined for every $j' < j$, whereas $\mathtt{fork}^{i'}_{j'}(\mathtt{ax},\mathtt{bx})$, and $[\mathtt{ax} < \mathtt{bx}]^{i'}_{j'}$ are already defined for all $1 \leq i' \leq j'$ such that $j' - i' < j - i$. Therefore, we define:

$$\mathtt{fork}^{i}_{j}(\mathtt{ax},\mathtt{bx}) \stackrel{\text{def}}{=} \mathtt{fork}^{i}_{i}(\mathtt{ax},\mathtt{bx}) \land [t]^{i}\mathtt{type}_{\mathtt{lsr}}(j-i).$$

It is easy to see that this formula is well-defined: $\mathtt{fork}^{i}_{i}(\mathtt{ax},\mathtt{bx})$ is from the base case, whereas $\mathtt{type}_{\mathtt{lsr}}(j{-}i)$ is defined by inductive hypothesis, since we have $j - i < j$.

Assuming that $\mathtt{type}(j)$ is correctly defined, with semantics as in Section 4.3, the following result roughly states that the encoding of numbers is preserved under submodels.

Lemma 4.13. *Let $0 \leq i \leq j$ with $j \geq 2$. Let $\mathfrak{M} = (W, R, V)$ and $w \in W$ be such that $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathtt{type}(j)$. Consider a world $w' \in R^{i}(w)$ and a number $m \in [0, \mathfrak{t}(j-i,n)-1]$. Lastly, suppose $\mathfrak{M}' \sqsubseteq \mathfrak{M}$ such that $\mathfrak{M}', w' \models \mathtt{type}(j-i)$. Then,*

$$\mathbf{n}_{j-i}(w') = m \text{ w.r.t. } (\mathfrak{M}, w') \text{ if and only if } \mathbf{n}_{j-i}(w') = m \text{ w.r.t. } (\mathfrak{M}', w').$$

Proof. The proof is rather straightforward. From the semantics of $\mathtt{type}(j)$, with respect to any of the two models $(\mathfrak{M}, w')$ or $(\mathfrak{M}', w')$, $\mathbf{n}_{j-i}(w')$ is encoded by using

(1) the t-nodes reachable from w' in at most $j - i$ steps;
(2) the $\{\mathtt{x},\mathtt{y}\}$-nodes reachable from w' in exactly 2 steps;
(3) the $\mathtt{Aux}$-nodes reachable from w' in at least 3 steps and at most $j - i + 1$ steps.

Let $\mathfrak{M}' = (W, R_1, V)$. From $\mathfrak{M}', w' \models \mathtt{type}(j-i)$ we can show that the accessibility to all these nodes is preserved between $(\mathfrak{M}, w')$ and $(\mathfrak{M}', w')$, leading to the result (or rather, that losing the accessibility to any of these nodes leads to a model not satisfying $\mathtt{type}(j-i)$). Indeed,

(1) suppose that there is a t-node $\overline{w} \in R^{k}(w')$, with $k \in [1, j-i]$, not in $R^{k}_{1}(w')$. Let $\overline{w}_1$ be the parent of $\overline{w}$ in R. Then in particular, $\overline{w}_1 \in R^{k-1}(w')$ and $(\overline{w}_1, \overline{w}) \in R$. Since $\overline{w} \notin R^{k}_{1}(w')$, we conclude that $(\mathfrak{M}', \overline{w}_1)$ does not satisfy (compl$_j$) and therefore $\mathfrak{M}', \overline{w}_1 \not\models \mathtt{type}(j-i-k)$. Then, $(\mathfrak{M}', w')$ cannot satisfy (sub$_j$), in contradiction with $\mathfrak{M}', w' \models \mathtt{type}(j-i)$;
(2) suppose that one $\{\mathtt{x},\mathtt{y}\}$-node in $R^{2}(w')$ is not in $R^{2}_{1}(w')$. Then trivially $(\mathfrak{M}', w')$ cannot satisfy (aux), in contradiction with $\mathfrak{M}', w' \models \mathtt{type}(j)$;
(3) similarly, suppose that one $\mathtt{Aux}$-node in $R^{k}(w')$, where $k \in [3, j-i+1]$, is not in $R^{2}_{1}(w')$. Then again $(\mathfrak{M}', w')$ cannot satisfy (aux), in contradiction with $\mathfrak{M}', w' \models \mathtt{type}(j)$. ◻

With this technical lemma at hand, we are now able to show the correctness of $\mathtt{fork}^{i}_{j}(\mathtt{ax},\mathtt{bx})$.

Lemma 4.14. *Let $\mathtt{ax} \neq \mathtt{bx} \in \mathtt{Aux}$, $1 \leq i < j$, and $\mathfrak{M}, w \models \mathtt{init}(j)$. Then, $\mathfrak{M}, w \models \mathtt{fork}^{i}_{j}(\mathtt{ax},\mathtt{bx})$ if and only if the conditions below hold:*

(i) w has exactly two t-children and exactly two paths of t-nodes, both of length i;
(ii) one of these two paths ends on a world (say $w_{\mathtt{ax}}$) corresponding to the nominal $\mathtt{ax}$ whereas the other ends on a world (say $w_{\mathtt{bx}}$) corresponding to the nominal $\mathtt{bx}$;
(iii) $(\mathfrak{M}, w_{\mathtt{ax}})$ and $(\mathfrak{M}, w_{\mathtt{bx}})$ satisfy $\mathtt{type}_{\mathtt{lsr}}(j-i) \stackrel{\text{def}}{=} \mathtt{type}(j-i) \land [t](\Diamond \mathtt{l} \land \Diamond \mathtt{s} \land \Diamond \mathtt{r})$.

Proof. Recall that $\mathtt{fork}^{i}_{j}(\mathtt{ax},\mathtt{bx})$ is defined as $\mathtt{fork}^{i}_{i}(\mathtt{ax},\mathtt{bx}) \land [t]^{i}\mathtt{type}_{\mathtt{lsr}}(j-i)$. We have:

- $\mathfrak{M}, w \models \mathtt{fork}^{i}_{i}(\mathtt{ax},\mathtt{bx})$ if and only if (by Lemma 4.6)
  (i) w has exactly two t-children and exactly two paths of t-nodes, both of length j;
  (ii) one of these two paths ends on a world corresponding to the nominal $\mathtt{ax}$ whereas the other ends on a world corresponding to the nominal $\mathtt{bx}$.

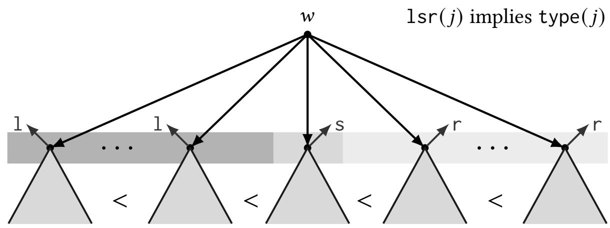


Fig. 3. Schema of a model satisfying $\mathtt{lsr}(j)$.

- Let $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^i(w)$, since $\mathfrak{M}, w \models [t]^i \mathtt{type}_{\mathtt{lsr}}(j-i)$ we get $\mathfrak{M}, w' \models \mathtt{type}_{\mathtt{lsr}}(j-i)$, for $w' \in \{w_{\mathsf{ax}}, w_{\mathsf{bx}}\}$, which proves *(iii), concluding the proof.* ◻

Consider now $[\mathsf{ax} < \mathsf{bx}]^i_j$. Assuming $\mathfrak{M}, w \models \mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$, we wish to express $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$ for the two distinct worlds $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^i(w)$ corresponding to the nominals ax and bx, respectively. As $i < j$, $\mathbf{n}(w_{\mathsf{ax}})$ (resp. $\mathbf{n}(w_{\mathsf{bx}})$) is encoded using the truth value of $\mathtt{val}$ on the t-children of w_{ax} (resp. w_{bx}). To rely on arithmetical properties of binary numbers used to define $[\mathsf{ax} < \mathsf{bx}]^j_j$, we need to find two partitions $P_{\mathsf{ax}} = \{L_{\mathsf{ax}}, S_{\mathsf{ax}}, R_{\mathsf{ax}}\}$ and $P_{\mathsf{bx}} = \{L_{\mathsf{bx}}, S_{\mathsf{bx}}, R_{\mathsf{bx}}\}$, one for the t-children of w_{ax} and another one for those of w_{bx} such that:

(LSR): Given $b \in \{\mathsf{ax}, \mathsf{bx}\}$, P_b splits the t-children as follows:
- there is a t-child s_b of w_b such that $S_b = \{s_b\}$;
- $\mathbf{n}(l) > \mathbf{n}(s_b) > \mathbf{n}(r)$, for every $r \in R_b$ and $l \in L_b$.

(LESS): P_{ax} and P_{bx} are constrained so that the intended relation $<$ between the two numbers can be satisfied:
- $\mathbf{n}(s_{\mathsf{ax}}) = \mathbf{n}(s_{\mathsf{bx}})$, $\mathfrak{M}, s_{\mathsf{ax}} \models \neg\mathtt{val}$ and $\mathfrak{M}, s_{\mathsf{bx}} \models \mathtt{val}$;
- for every $l_{\mathsf{ax}} \in L_{\mathsf{ax}}$ and $l_{\mathsf{bx}} \in L_{\mathsf{bx}}$, if $\mathbf{n}(l_{\mathsf{ax}}) = \mathbf{n}(l_{\mathsf{bx}})$ then $\mathfrak{M}, l_{\mathsf{ax}} \models \mathtt{val}$ iff $\mathfrak{M}, l_{\mathsf{bx}} \models \mathtt{val}$.

Above, 'L' stands for 'left', 'R' stands for 'right' and 'S' stands for 'selected bit'. As the numbers are encoded in binary with the least significant bit on the right, by way of example, the numbers associated to nodes in R_{ax} are strictly smaller than the number associated to the unique node in S_{ax}.

It is important to notice that these conditions essentially revolve around the numbers encoded by t-children, which will be compared using the already defined (by inductive reasoning) formulae $[\mathsf{ax} < \mathsf{bx}]^{i'}_{j'}$, where $j' - i' < j - i$. Since the semantics of $[\mathsf{ax} < \mathsf{bx}]^i_j$ is given under the hypothesis that $\mathfrak{M}, w \models \mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$, we can assume that every child of w_{ax} and w_{bx} has all the possible Aux-children. Then, we rely on the auxiliary propositions in $\{\mathtt{l}, \mathtt{s}, \mathtt{r}\}$ in order to mimic the reasoning done in (LSR) and (LESS).

We start by considering the constraints involved in (LSR) and we express them with the formula $\mathtt{lsr}(j)$ to be defined, which is satisfied by a pointed forest $(\mathfrak{M} = (W, R, V), w)$ whenever:

- $(\mathfrak{M}, w)$ satisfies $\mathtt{type}(j)$.
- Every t-child of w has exactly one $\{\mathtt{l}, \mathtt{s}, \mathtt{r}\}$-child, and only one of these t-children (say w') has an $\mathtt{s}$-child.
- Every t-child of w that has an $\mathtt{l}$-child (resp. $\mathtt{r}$-child) encodes a number greater (resp. smaller) than $\mathbf{n}(w')$.

Despite this formula being defined in terms of $\mathtt{type}(j)$, we only rely on $\mathtt{lsr}(j-i)$ (which is defined by inductive reasoning) in order to define $[\mathsf{ax} < \mathsf{bx}]^i_j$. Figure 3 sketches a model satisfying $\mathtt{lsr}(j)$. The definition of $\mathtt{lsr}(j)$ follows closely its specification:

$$\texttt{lsr}(j) \overset{\text{def}}{=} \texttt{type}(j) \land [t]\Diamond_{=1}(\texttt{l} \lor \texttt{s} \lor \texttt{r}) \land \texttt{nom}_1(\texttt{s}) \land \neg(\top * (\texttt{fork}^1_j(\texttt{s},\texttt{l}) \land \neg[\texttt{s} < \texttt{l}]^1_j)) \\ \land \neg(\top * (\texttt{fork}^1_j(\texttt{s},\texttt{r}) \land \neg[\texttt{r} < \texttt{s}]^1_j)).$$

Lemma 4.15. *Let* $1 \le i < j$. *Suppose* $\mathfrak{M}, w \models \texttt{init}(j)$. *Then,* $\mathfrak{M}, w \models \texttt{lsr}(j-i)$ *if and only if*

(1) $\mathfrak{M}, w \models \texttt{type}(j-i)$*;*
(2) every t*-node in* $R(w)$ *has exactly one* Aux*-child satisfying an atomic proposition from* $\{\texttt{l},\texttt{s},\texttt{r}\}$*;*
(3) exactly one t*-node in* $R(w)$ *(say* $w_{\texttt{s}}$*) has an* Aux*-child satisfying* $\texttt{s}$*;*
(4) given $w' \in R(w)$*,* w' *has an* Aux*-child satisfying* $\texttt{l}$ *if and only if* $\mathbf{n}(w') > \mathbf{n}(w_{\texttt{s}})$*;*
(5) given $w' \in R(w)$*,* w' *has an* Aux*-child satisfying* $\texttt{r}$ *if and only if* $\mathbf{n}(w') < \mathbf{n}(w_{\texttt{s}})$*.*

Proof. This proof is rather straightforward. The definition of $\texttt{lsr}(j-i)$ is reproduced below:

$$\texttt{type}(j-i) \land [t]\Diamond_{=1}(\texttt{l} \lor \texttt{s} \lor \texttt{r}) \land \texttt{nom}_1(\texttt{s}) \land$$

$$\neg(\top * (\texttt{fork}^1_{j-i}(\texttt{s},\texttt{l}) \land \neg[\texttt{s} < \texttt{l}]^1_{j-i})) \land \neg(\top * (\texttt{fork}^1_{j-i}(\texttt{s},\texttt{r}) \land \neg[\texttt{r} < \texttt{s}]^1_{j-i})).$$

Then, we provide the following analysis.

- The first, second and third conjuncts of $\texttt{lsr}(j-i)$ directly realise requirements (1), (2) and (3).
- The fourth conjunct of $\texttt{lsr}(j-i)$ realises the requirement (4). Indeed, suppose $\mathfrak{M}, w \models \neg(\top * (\texttt{fork}^1_{j-i}(\texttt{s},\texttt{l}) \land \neg[\texttt{s} < \texttt{l}]^1_{j-i}))$. Then, for all submodels $\mathfrak{M}' \sqsubseteq \mathfrak{M}$, if $\mathfrak{M}', w \models \texttt{fork}^1_{j-i}(\texttt{s},\texttt{l})$ then $\mathfrak{M}', w \models [\texttt{s} < \texttt{l}]^1_{j-i}$. Let $w' \in R(w)$ be such that w' has an Aux-child satisfying $\texttt{l}$. Then by Lemma 4.14, $\mathfrak{M}, w \models \texttt{fork}^1_{j-1}(\texttt{s},\texttt{l})$ and as a consequence $\mathfrak{M}, w \models [\texttt{s} < \texttt{l}]^1_{j-i}$. Let us consider $\mathfrak{M}' = (W, R', W)$ obtained from $\mathfrak{M}$ by removing from R every pair $(w_1, w_2) \in R$ such that
  - w_1 and w_2 are t-nodes;
  - (w_1, w_2) does not belong to the path from w to $w_{\texttt{s}}$, nor to the path from w to w';
  - (w_1, w_2) does not belong to any path starting from $w_{\texttt{s}}$ or w'.

  Then, we can show that $\mathfrak{M}', w \models \texttt{fork}^1_{j-i}(\texttt{s},\texttt{l})$ and thus, by hypothesis, $\mathfrak{M}', w \models [\texttt{s} < \texttt{l}]^1_{j-i}$. By the induction hypothesis, from $[\texttt{s} < \texttt{l}]^1_{j-i}$ we conclude that $\mathbf{n}(w') > \mathbf{n}(w_{\texttt{s}})$ with respect to $(\mathfrak{M}', w)$. Now, from $\mathfrak{M}', w \models \texttt{fork}^1_{j-i}(\texttt{s},\texttt{l})$ we also conclude that $\mathfrak{M}', w_{\texttt{s}} \models \texttt{type}(j-i)$ and $\mathfrak{M}', w' \models \texttt{type}(j-i)$. Then, by Lemma 4.13, $\mathbf{n}(w') > \mathbf{n}(w_{\texttt{s}})$ also holds with respect to $(\mathfrak{M}, w)$. The other direction is analogous.
- The fifth conjunct of $\texttt{lsr}(j-i)$ realises the requirement (5). The proof is similar to the one for the requirement (4), just above. □

Then, we have the ingredients to define the formula $[\texttt{ax} < \texttt{bx}]^i_j$ as follows:

$$[\texttt{ax} < \texttt{bx}]^i_j \overset{\text{def}}{=} \top * \big(\texttt{nom}_i(\texttt{ax} \neq \texttt{bx}) \land [t]^i \texttt{lsr}(j-i) \land \texttt{S}^i_j(\texttt{ax},\texttt{bx}) \land \texttt{L}^i_j(\texttt{ax},\texttt{bx})\big),$$

where $\texttt{S}^i_j(\texttt{ax},\texttt{bx})$ and $\texttt{L}^i_j(\texttt{ax},\texttt{bx})$ check the first and second condition in (LESS), respectively. In particular, by defining $[\texttt{ax} = \texttt{bx}]^i_j \overset{\text{def}}{=} \neg([\texttt{ax} < \texttt{bx}]^i_j \lor [\texttt{bx} < \texttt{ax}]^i_j)$, we have

$$\texttt{S}^i_j(\texttt{ax},\texttt{bx}) \overset{\text{def}}{=} \top * \big(\texttt{fork}^{i+1}_j(\texttt{x},\texttt{y}) \land @^i_{\texttt{ax}}\langle t\rangle(\Diamond\texttt{s} \land \Diamond\texttt{x}) \\ \land @^i_{\texttt{bx}}\langle t\rangle(\Diamond\texttt{s} \land \Diamond\texttt{y}) \land [\texttt{x} = \texttt{y}]^{i+1}_j \land @^{i+1}_{\texttt{x}}\neg\texttt{val} \land @^{i+1}_{\texttt{y}}\texttt{val}\big)$$

$$\texttt{L}^i_j(\texttt{ax},\texttt{bx}) \overset{\text{def}}{=} \neg\big(\top * \big(\texttt{fork}^{i+1}_j(\texttt{x},\texttt{y}) \land @^i_{\texttt{ax}}\langle t\rangle(\Diamond\texttt{l} \land \Diamond\texttt{x}) \land @^i_{\texttt{bx}}\langle t\rangle(\Diamond\texttt{l} \land \Diamond\texttt{y}) \\ \land [\texttt{x} = \texttt{y}]^{i+1}_j \land \neg(@^{i+1}_{\texttt{x}}\texttt{val} \Leftrightarrow @^{i+1}_{\texttt{y}}\texttt{val})\big)\big).$$

Both $\texttt{fork}^{i+1}_j(\texttt{x},\texttt{y})$ and $[\texttt{x} = \texttt{y}]^{i+1}_j$ used in these formulae are defined recursively. The formula $\texttt{S}^i_j(\texttt{ax},\texttt{bx})$ states that there is a submodel $\mathfrak{M}' \sqsubseteq \mathfrak{M}$ such that

I. $\mathfrak{M}', w \models \mathtt{fork}_j^{i+1}(\mathsf{x},\mathsf{y})$;
II. s_{ax} corresponds to the nominal x at depth $i+1$;
III. s_{bx} corresponds to the nominal y at depth $i+1$;
IV. $\mathbf{n}(s_{\mathsf{ax}}) = \mathbf{n}(s_{\mathsf{bx}})$,
V. $\mathfrak{M}, s_{\mathsf{ax}} \not\models \mathtt{val}$, and
VI. $\mathfrak{M}, s_{\mathsf{bx}} \models \mathtt{val}$.

The enumeration I-VI refers to the conjuncts in the formula.

$\mathsf{S}_j^i(\mathsf{ax},\mathsf{bx})$ correctly models the first condition of (LESS). Regarding $\mathsf{L}_j^i(\mathsf{ax},\mathsf{bx})$ and (LESS), a similar analysis can be performed. We define $\mathsf{LS}_j^i(\mathsf{ax},\mathsf{bx}) \stackrel{\text{def}}{=} \mathsf{L}_j^i(\mathsf{ax},\mathsf{bx}) \wedge \mathsf{S}_j^i(\mathsf{ax},\mathsf{bx})$.

Let us consider $[\mathsf{bx} = \mathsf{ax{+}1}]_j$. Under the hypothesis that $\mathfrak{M}, w \models \mathtt{fork}_j^i(\mathsf{ax},\mathsf{bx})$, this formula must express $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$ for the two (distinct) worlds $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^i(w)$. Then, as done for defining $[\mathsf{ax} < \mathsf{bx}]_j^i$, we take advantage of arithmetical properties on binary numbers and we search for two partitions $P_{\mathsf{ax}} = \{L_{\mathsf{ax}}, S_{\mathsf{ax}}, R_{\mathsf{ax}}\}$ and $P_{\mathsf{bx}} = \{L_{\mathsf{bx}}, S_{\mathsf{bx}}, R_{\mathsf{bx}}\}$ of the t-children of w_{ax} and w_{bx}, respectively, such that P_{ax} and P_{bx} satisfy (LSR) as well as the condition below:

(PLUS): P_{ax} and P_{bx} have the arithmetical properties of the successor relation:

- P_{ax} and P_{bx} satisfy (LESS);
- for every $r_{\mathsf{ax}} \in R_{\mathsf{ax}}$, we have $\mathfrak{M}, r_{\mathsf{ax}} \models \mathtt{val}$;
- for every $r_{\mathsf{bx}} \in R_{\mathsf{bx}}$, we have $\mathfrak{M}, r_{\mathsf{ax}} \not\models \mathtt{val}$,

where $S_{\mathsf{ax}} = \{s_{\mathsf{ax}}\}$ and $S_{\mathsf{bx}} = \{s_{\mathsf{bx}}\}$, as required by (LSR).

The definition of $[\mathsf{bx} = \mathsf{ax{+}1}]_j$ is similar to $[\mathsf{ax} < \mathsf{bx}]_j^i$:

$$[\mathsf{bx} = \mathsf{ax{+}1}]_j \stackrel{\text{def}}{=} \top{*}\big(\mathsf{nom}_1(\mathsf{ax}{\neq}\mathsf{bx}) \wedge [t]\mathtt{lsr}(j-1) \wedge \mathsf{LS}_j^1(\mathsf{ax},\mathsf{bx}) \wedge \mathsf{R}(\mathsf{ax},\mathsf{bx})\big),$$

where $\mathsf{R}(\mathsf{ax},\mathsf{bx}) \stackrel{\text{def}}{=} @_{\mathsf{ax}}^1[t](\Diamond \mathsf{r} \Rightarrow \mathtt{val}) \wedge @_{\mathsf{bx}}^1[t](\Diamond \mathsf{r} \Rightarrow \neg\mathtt{val})$ captures the last two conditions of (PLUS). We prove a technical lemma that will help us with the proof of correctness of $[\mathsf{ax} < \mathsf{bx}]_j^i$ and $[\mathsf{bx} = \mathsf{ax{+}1}]_j$ stated in Lemma 4.17 and Lemma 4.18 below.

Lemma 4.16. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $1 \leq i < j$. *Suppose that* $(\mathfrak{M}, w)$ *is such that* $R^i(w) = \{w_{\mathsf{ax}}, w_{\mathsf{bx}}\}$ *for some t-nodes* w_{ax} *and* w_{bx} *in* W, *and these two worlds satisfy the conditions of* $\mathtt{lsr}(j-i)$, *that is, for every* $b \in \{\mathsf{ax},\mathsf{bx}\}$,

(A) $\mathfrak{M}, w_b \models \mathtt{type}(j-i)$;
(B) every t-node in $R(w_b)$ *has exactly one* Aux*-child satisfying an atomic proposition from* $\{\mathsf{l},\mathsf{s},\mathsf{r}\}$;
(C) exactly one t-node in $R(w_b)$ *(say* $w_{b,\mathsf{s}}$*) has an* Aux*-child satisfying* s;
(D) given $w' \in R(w_b)$, w' *has an* Aux*-child satisfying* l *if and only if* $\mathbf{n}(w') > \mathbf{n}(w_{b,\mathsf{s}})$;
(E) given $w' \in R(w_b)$, w' *has an* Aux*-child satisfying* r *if and only if* $\mathbf{n}(w') < \mathbf{n}(w_{b,\mathsf{s}})$.

Then,

I. $\mathfrak{M}, w \models \mathsf{S}_j^i(\mathsf{ax},\mathsf{bx})$ *if and only if* $\mathbf{n}(w_{\mathsf{ax,s}}) = \mathbf{n}(w_{\mathsf{bx,s}})$, $\mathfrak{M}, w_{\mathsf{ax,s}} \models \neg\mathtt{val}$ *and* $\mathfrak{M}, w_{\mathsf{bx,s}} \models \mathtt{val}$;
II. $\mathfrak{M}, w \models \mathsf{L}_j^i(\mathsf{ax},\mathsf{bx})$ *if and only if (* $\mathfrak{M}, w_{\mathsf{ax,l}} \models \mathtt{val}$ *iff* $\mathfrak{M}, w_{\mathsf{bx,l}} \models \mathtt{val}$*), for all* $w_{\mathsf{ax,l}} \in R(w_{\mathsf{ax}})$ *and* $w_{\mathsf{bx,l}} \in R(w_{\mathsf{bx}})$ *s.t.* $\mathbf{n}(w_{\mathsf{ax,l}}) > \mathbf{n}(w_{\mathsf{ax,s}})$, $\mathbf{n}(w_{\mathsf{bx,l}}) > \mathbf{n}(w_{\mathsf{bx,s}})$ *and* $\mathbf{n}(w_{\mathsf{ax,l}}) = \mathbf{n}(w_{\mathsf{bx,l}})$.
III. If $i = 1$ *then,* $\mathfrak{M}, w \models \mathsf{R}(\mathsf{ax},\mathsf{bx})$ *if and only if*
- *for every world* $w_{\mathsf{ax,r}} \in R(w_{\mathsf{ax}})$, *if* $\mathbf{n}(w_{\mathsf{ax,r}}) < \mathbf{n}(w_{\mathsf{ax,s}})$ *then* $\mathfrak{M}, w_{\mathsf{ax,r}} \models \mathtt{val}$;
- *for every world* $w_{\mathsf{bx,r}} \in R(w_{\mathsf{bx}})$, *if* $\mathbf{n}(w_{\mathsf{bx,r}}) < \mathbf{n}(w_{\mathsf{bx,s}})$ *then* $\mathfrak{M}, w_{\mathsf{bx,r}} \models \neg\mathtt{val}$.

See the proof in Appendix E.

Lemma 4.17. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $1 \leq i < j$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j) \wedge \mathtt{fork}_j^i(\mathsf{ax},\mathsf{bx})$. *Then,* $\mathfrak{M}, w \models [\mathsf{ax} < \mathsf{bx}]_j^i$ *if and only if there are two distinct t-nodes* $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R^i(w)$ *such that* w_{ax} *corresponds to the nominal* ax, w_{bx} *corresponds to the nominal* bx *and* $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$.

See the proof in Appendix F.

Lemma 4.18. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $1 \leq i < j$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathtt{fork}^1_j(\mathsf{ax}, \mathsf{bx})$. *Then,* $\mathfrak{M}, w \models [\mathsf{bx} = \mathsf{ax}{+}1]_j$ *if and only if there are two distinct t-nodes* $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$ *such that* w_{ax} *corresponds to the nominal* ax, w_{bx} *corresponds to the nominal* bx *and* $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$.

Proof. We recall the definition of $[\mathsf{bx} = \mathsf{ax}{+}1]_j$:

$$[\mathsf{bx} = \mathsf{ax}{+}1]_j \overset{\text{def}}{=} \top{*}\big(\mathsf{nom}_1(\mathsf{ax}{\neq}\mathsf{bx}) \land [t]\mathtt{lsr}(j-1) \land \mathsf{S}^1_j(\mathsf{ax}, \mathsf{bx}) \land \mathsf{L}^1_j(\mathsf{ax}, \mathsf{bx}) \land \mathsf{R}(\mathsf{ax}, \mathsf{bx})\big).$$

As in Lemma 4.8, the proof uses standard properties of numbers encoded in binary. Again, let x, y be two natural numbers that can be represented in binary by using n bits. Let us denote with x_i (resp. y_i) the i-th bit of the binary representation of x (resp. y). We have that $y = x + 1$ if and only if

(A) there is a position $i \in [1, n]$ such that $x_i = 0$ and $y_i = 1$;
(B) for every position $j > i$, $x_j = 0 \Leftrightarrow y_j = 0$;
(C) for every position $j < i$, $x_j = 1$ and $y_j = 0$.

The formula $[\mathsf{bx} = \mathsf{ax}{+}1]_j$ uses this characterisation to state that $\mathbf{n}(w_{\mathsf{bx}}) = \mathbf{n}(w_{\mathsf{ax}}) + 1$.

One can see that the formula $[\mathsf{bx} = \mathsf{ax}{+}1]_j$ can be obtained (syntactically) from the formula $[\mathsf{ax} < \mathsf{bx}]^1_j \overset{\text{def}}{=} \top * (\mathsf{nom}_1(\mathsf{ax} \neq \mathsf{bx}) \land [t]^i\mathtt{lsr}(j-1) \land \mathsf{S}^1_j(\mathsf{ax}, \mathsf{bx}) \land \mathsf{L}^1_j(\mathsf{ax}, \mathsf{bx}))$ by simply adding the conjunct $\mathsf{R}(\mathsf{ax}, \mathsf{bx})$ to the right of $\mathsf{L}^1_j(\mathsf{ax}, \mathsf{bx})$. Then, it is easy to see that the proof of this lemma follows very closely the structure of the proof of Lemma 4.17. Indeed, to prove (A) and (B) we essentially rely on Lemma 4.16 (I and II), whereas (C) is shown using the third point of Lemma 4.16. □

To define $\mathtt{uniq}(j)$ and $\mathtt{compl}(j)$, we rely on $\mathtt{fork}^i_j(\mathsf{ax}, \mathsf{bx})$, $[\mathsf{ax} < \mathsf{bx}]^i_j$ and $[\mathsf{bx} = \mathsf{ax}{+}1]_j$.

$$\mathtt{uniq}(j) \overset{\text{def}}{=} \neg\big(\top * (\mathtt{fork}^1_j(\mathsf{x}, \mathsf{y}) \land [\mathsf{x} = \mathsf{y}]^1_j)\big)$$

$$\mathtt{compl}(j) \overset{\text{def}}{=} \neg\Big(\Box\bot * \Big([t](\mathtt{type}_{\mathtt{lsr}}(j-1) \land \Diamond\mathsf{y}) \land \mathsf{nom}_1(\mathsf{x}) \land @^1_{\mathsf{x}}\neg 1_j \land \neg\big(\top * (\mathtt{fork}^1_j(\mathsf{x}, \mathsf{y}) \land [\mathsf{y} = \mathsf{x}{+}1]_j)\big)\Big)\Big),$$

where $1_j \overset{\text{def}}{=} [t]\mathtt{val}$ reflects the encoding of $\mathfrak{t}(j, n){-}1$ for $j > 1$. The main difference between $\mathtt{compl}(1)$ and $\mathtt{compl}(j)$ ($j > 1$) is that the conjunct $[t]\Diamond\mathsf{y}$ of $\mathtt{compl}(1)$ is replaced by $[t](\mathtt{type}_{\mathtt{lsr}}(j{-}1) \land \Diamond\mathsf{y})$ in $\mathtt{compl}(j)$, as needed to correctly evaluate $\mathtt{fork}^1_j(\mathsf{x}, \mathsf{y})$. Indeed, the difference between $\mathtt{fork}^1_1(\mathsf{x}, \mathsf{y})$ and $\mathtt{fork}^1_j(\mathsf{x}, \mathsf{y})$ is precisely that the latter requires $[t]\mathtt{type}_{\mathtt{lsr}}(j-1)$. The definition of $\mathtt{type}(j)$ is now complete.

Lemma 4.19. *Let* $j \geq 2$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathsf{aux}$. *Then,* $\mathfrak{M}, w \models \mathtt{uniq}(j)$ *if and only if* $(\mathfrak{M}, w)$ *satisfies (uniq$_j$), i.e. distinct t-nodes in* $R(w)$ *encode different numbers.*

Proof. As in Lemma 4.9, but using Lemma 4.17 on the inductive formula $[\mathsf{x} = \mathsf{y}]^1_j$. □

Lemma 4.20. *Let* $j \geq 2$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathsf{aux}$. *Then,* $\mathfrak{M}, w \models \mathtt{compl}(j)$ *if and only if* $(\mathfrak{M}, w)$ *satisfies (compl$_j$), i.e. for every t-node* $w_1 \in R(w)$, *if* $\mathbf{n}(w_1) < \mathfrak{t}(j, n){-}1$ *then* $\mathbf{n}(w_2) = \mathbf{n}(w_1){+}1$ *for some t-node* $w_2 \in R(w)$.

Proof. As in Lemma 4.10, but using Lemma 4.18 and the formula $\mathtt{type}_{\mathtt{lsr}}(j-1)$ in order to properly evaluate $\mathtt{fork}^1_j(\mathsf{x}, \mathsf{y})$. □

Finally, we can state the correctness of the definition of $\mathtt{type}(j)$.

Lemma 4.21. *Let* $\mathfrak{M}, w \models \mathtt{init}(j)$. *We have* $\mathfrak{M}, w \models \mathtt{type}(j)$ *if and only if* $(\mathfrak{M}, w)$ *satisfies (sub$_j$), (zero$_j$), (uniq$_j$), (compl$_j$) and (aux).*

Proof. It follows directly from Lemmata 4.5, 4.19 and 4.20. □

The size of $\mathtt{type}(j)$ is exponential in $j > 1$ and polynomial in $n \geq 1$. As its size is elementary, we can use this formula as a starting point to reduce $\mathtt{Tile}_k$.

We finish this section by showing that the formulae $\mathtt{init}(j)$ and $\mathtt{type}(j)$ are (simultaneously) satisfiable, i.e., there exists a pointed forest $\mathfrak{M}, w$ such that $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathtt{type}(j)$. This result is useful in the next section, as we will need to show that a model encoding a grid actually exists.

Lemma 4.22. *Let $j \geq 2$. $\mathtt{init}(j) \land \mathtt{type}(j)$ is satisfiable.*

Proof. Let $j \geq 2$. By induction on j, we suppose that $\mathtt{init}(j-1) \land \mathtt{type}(j-1)$ is satisfiable (we already treated the base case for $j = 1$ in Lemma 4.12). Let us consider $w_0, \dots, w_{\mathfrak{t}(j,n)-1}$ distinct worlds. By the induction hypothesis, we can construct $\mathfrak{t}(j,n)$ models $\mathfrak{M}_i = (W_i, R_i, V_i)$ ($i \in [0, \mathfrak{t}(j,n)-1]$), so that $w_i \in W_i$ and $\mathfrak{M}_i, w_i \models \mathtt{init}(j-1) \land \mathtt{type}(j-1)$. W.l.o.g. we can assume, for each two distinct $i, i' \in [0, \mathfrak{t}(j,n)-1]$, $W_i \cap W_{i'} = \emptyset$. Similarly, we can assume that each $\mathfrak{M}_i$ is minimal, i.e. for every $\mathfrak{M}' \sqsubseteq \mathfrak{M}_i$ different from $\mathfrak{M}'$, $\mathfrak{M}', w_i \not\models \mathtt{init}(j-1) \land \mathtt{type}(j-1)$. This implies that w_i does not have any $\mathtt{Aux}$-children, and every t-node in $R_i(w_i)$ does not have $\{\mathtt{l}, \mathtt{s}, \mathtt{r}\}$-children (as these two properties are not guaranteed by (aux)).

Let w be a fresh world not appearing in the aforementioned models. Similarly, for every $i \in [0, \mathfrak{t}(j,n)-1]$, let $w_i^{\mathtt{x}}$ and $w_i^{\mathtt{y}}$ be fresh worlds. Lastly, we also introduce, for every world $\overline{w} \in R_i(w_i)$, three (distinct) new worlds $w_{\overline{w}}^{\mathtt{l}}$, $w_{\overline{w}}^{\mathtt{s}}$ and $w_{\overline{w}}^{\mathtt{r}}$.

Then, let us consider the model $\mathfrak{M} = (W, R, V)$ defined as follows:

(1) $W \stackrel{\text{def}}{=} \{w\} \cup W_i \cup \{w_i^{\mathtt{x}}, w_i^{\mathtt{y}} \mid i \in [0, \mathfrak{t}(j,n)-1]\} \cup \{w_{\overline{w}^{\mathtt{l}}}, w_{\overline{w}^{\mathtt{s}}}, w_{\overline{w}^{\mathtt{r}}}, \mid i \in [0, \mathfrak{t}(j,n)-1], \overline{w} \in R_i(w_i)\}$

(2) $R \stackrel{\text{def}}{=} \{(w, w_0), \dots, (w, w_{\mathfrak{t}(j,n)-1})\} \cup \bigcup_{i \in [0, \mathfrak{t}(j,n)-1]} R_i \cup \{(w_i, w_i^{\mathtt{x}}), (w_i, w_i^{\mathtt{y}}) \mid i \in [0, \mathfrak{t}(j,n)-1]\}$
$\cup \{(\overline{w}, w_{\overline{w}^{\mathtt{l}}}), (\overline{w}, w_{\overline{w}^{\mathtt{s}}}), (\overline{w}, w_{\overline{w}^{\mathtt{r}}}), \mid i \in [0, \mathfrak{t}(j,n)-1], \overline{w} \in R_i(w_i)\}$

(3) V is such that
- for every $i \in [0, \mathfrak{t}(j,n)-1]$, $p \in \mathrm{AP}$ and every $w' \in R_i^2(w_i)$, $w' \in V(p)$ if and only if $w' \in V_i(p)$. Hence, w.r.t. $(\mathfrak{M}, w)$, the evaluations w.r.t. worlds in $R_i^3(w) \cap W_i$ is unchanged compared to the one in $(\mathfrak{M}_i, w_i)$.
- For every $i \in [0, \mathfrak{t}(j,n)-1]$ and every $w' \in R_i(w_i)$, $w' \in V(\mathtt{val})$ if and only if w.r.t. $(\mathfrak{M}_i, w_i)$, the $\mathbf{n}(w')$-bit in the binary representation of i is 1. Notice that this will lead to $\mathbf{n}(w_i) = i$.
- For every $i \in [0, \mathfrak{t}(j,n)-1]$ and $\mathtt{ax} \in \mathtt{Aux}$, $w_i^{\mathtt{x}} \in V(\mathtt{ax})$ if and only if $\mathtt{ax} = \mathtt{x}$. Similarly, $w_i^{\mathtt{y}} \in V(\mathtt{ax})$ if and only if $\mathtt{ax} = \mathtt{y}$. Thus, every $w_i^{\mathtt{x}}$ is a $\mathtt{x}$-node, whereas every $w_i^{\mathtt{y}}$ is a $\mathtt{y}$-node.
- For every $\mathtt{ax} \in \mathtt{Aux}$, $w \notin V(\mathtt{ax})$ and for every $i \in [0, \mathfrak{t}(j,n)-1]$, $w_i \notin V(\mathtt{ax})$. Moreover, for every $\overline{w} \in R_i(w_i)$, $\overline{w} \notin V(\mathtt{ax})$ (notice that, by minimality, $\overline{w}$ is a t-node also in $\mathfrak{M}_i$). Thus, w, w_i and $\overline{w}$ (as above) are all t-nodes.
- For every $\mathtt{ax} \in \mathtt{Aux}$, $w \notin V(\mathtt{ax})$ and for every $i \in [0, \mathfrak{t}(j,n)-1]$ and $\overline{w} \in R_i(w_i)$, (1) $w_{\overline{w}}^{\mathtt{l}} \in V(\mathtt{ax})$ iff $\mathtt{ax} = \mathtt{l}$, (2) $w_{\overline{w}}^{\mathtt{s}} \in V(\mathtt{ax})$ iff $\mathtt{ax} = \mathtt{s}$, (3) $w_{\overline{w}}^{\mathtt{r}} \in V(\mathtt{ax})$ iff $\mathtt{ax} = \mathtt{r}$. Hence, every $w_{\overline{w}}^{\mathtt{l}}$, $w_{\overline{w}}^{\mathtt{s}}$ and $w_{\overline{w}}^{\mathtt{r}}$ (as above) is a $\mathtt{l}$-node, $\mathtt{s}$-node and $\mathtt{r}$-node, respectively.

We can check that $(\mathfrak{M}, w)$ satisfies $\mathtt{init}(j)$ as well as (sub_j), (zero_j), (uniq_j), (compl_j) and (aux). Thus, by Lemma 4.21, $\mathfrak{M}, w \models \mathtt{init}(j) \land \mathtt{type}(j)$. □

4.6 Tiling a grid $[0, \mathfrak{t}(k,n)-1] \times [0, \mathfrak{t}(k,n)-1]$

In this section we explain how to use previous developments to define a uniform reduction from $\mathtt{Tile}_k$, for every $k \geq 2$. Several adaptations are needed to encode smoothly the grid, but the hardest part was the design of the formula $\mathtt{type}(j)$, which we already achieved in the previous section.

As usual, in the following let $\mathfrak{M} = (W, R, V)$ be a finite forest and consider $w \in W$.

Let $k \geq 2$ and let $(\mathscr{T}, \mathtt{c})$ be an instance of $\mathtt{Tile}_k$, where $\mathscr{T} = (\mathcal{T}, \mathcal{H}, \mathcal{V})$ and $\mathtt{c} \in \mathcal{T}$ (see Section 4.1 for a formal definition). Recall that a solution for $(\mathscr{T}, \mathtt{c})$ w.r.t. $\mathtt{Tile}_k$ is a map τ :

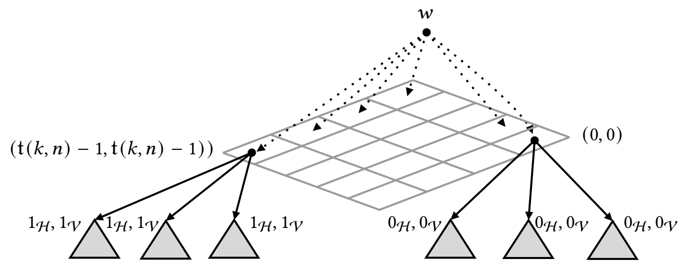


Fig. 4. Schema of a model satisfying $\mathtt{grid}_{\mathscr{T}}(k)$ (for $k \geq 2$).

$[0, \mathfrak{t}(k, n) - 1] \times [0, \mathfrak{t}(k, n) - 1] \to \mathcal{T}$ satisfying (first) and (hor&vert). W.l.o.g. we assume $\mathcal{T}$ is also understood as a set of atomic propositions, disjoint from $\{p_1, \ldots, p_n, \mathtt{val}\} \cup \mathtt{Aux}$ used in the definition of $\mathtt{type}(j)$. We construct a formula $\mathtt{tiling}_{\mathscr{T},\mathtt{c}}(k)$ that is satisfiable iff $(\mathscr{T}, \mathtt{c})$ as a solution.

Let us first describe how to represent a grid $[0, \mathfrak{t}(k, n) - 1]^2$ in the pointed forest $(\mathfrak{M}, w)$. We use the same ideas needed in order to define $\mathtt{type}(k)$, but with some minor modifications. As previously stated, if $\mathfrak{M}, w \models \mathtt{type}(k)$ then given a t-node $w' \in R(w)$, the number $\mathbf{n}(w') \in [0, \mathfrak{t}(k, n) - 1]$ is encoded using the t-children of w', where the numbers encoded by these children represent positions in the binary encoding of $\mathbf{n}(w')$. Instead of being a single number, a position in the grid is a pair of numbers $(h, v) \in [0, \mathfrak{t}(k, n) - 1]^2$. Hence, in a model $(\mathfrak{M}, w)$ satisfying $\mathtt{tiling}_{\mathscr{T},\mathtt{c}}(k)$ we require that $w' \in R(w)$ encodes two numbers $\mathbf{n}_{\mathcal{H}}(w')$ and $\mathbf{n}_{\mathcal{V}}(w')$, and say that w' encodes the position (h, v) if and only if $\mathbf{n}_{\mathcal{H}}(w') = h$ and $\mathbf{n}_{\mathcal{V}}(w') = v$. Since both numbers are from $[0, \mathfrak{t}(k, n) - 1]$, the same amount of t-children as in $\mathtt{type}(k)$ can be used in order to encode both $\mathbf{n}_{\mathcal{H}}(w')$ and $\mathbf{n}_{\mathcal{V}}(w')$. Thus, we rely on the formula $\mathtt{type}(k - 1)$ to force w' to have the correct amount of t-children, by requiring it to hold in $(\mathfrak{M}, w')$. Similarly to what is done previously for $\mathtt{type}(j)$ $(j \geq 2)$, we encode the numbers $\mathbf{n}_{\mathcal{H}}(w')$ and $\mathbf{n}_{\mathcal{V}}(w')$ by using the truth value, on the t-children of w', of two new atomic propositions $\mathtt{val}_{\mathcal{H}}$ and $\mathtt{val}_{\mathcal{V}}$, respectively. Then, we use similar formulae to $\mathtt{zero}(k)$, $\mathtt{uniq}(k)$ and $\mathtt{compl}(k)$ to state that w witnesses exactly one child for each position in the grid. Once the grid is encoded, the tiling conditions are enforced rather easily.

Figure 4 schematises a pointed forest satisfying a formula $\mathtt{grid}_{\mathscr{T}}(k)$ that properly encodes the $[0, \mathfrak{t}(k, n) - 1]^2$ grid. The actual grid is drawn in the picture to illustrate the intended meaning of the worlds in $R(w)$. As mentioned earlier, each world $w' \in R(w)$ encodes two numbers, corresponding to the respective horizontal and vertical coordinates of the grid. So, dotted arrows connect w with exactly one world for each position of the grid (for simplicity, we only draw some of these arrows). Thus, w has $\mathfrak{t}(k, n)^2$ children. These children must satisfy $\mathtt{type}(k-1)$, therefore they have $\mathfrak{t}(k-1, n)$ children that represent pairs of numbers via $\mathtt{val}_{\mathcal{H}}$ and $\mathtt{val}_{\mathcal{V}}$, as described before. In the picture the values $1_{\mathcal{H}}$ and $0_{\mathcal{H}}$ stand for $\mathtt{val}_{\mathcal{H}}$ being true and false, respectively (similarly for $1_{\mathcal{V}}$ and $0_{\mathcal{V}}$ w.r.t. $\mathtt{val}_{\mathcal{V}}$). For instance, in the rightmost child of w all "bits" are set to 0, both for horizontal and for vertical position, so it corresponds to the initial position $(0, 0)$ of the grid. Similarly, in the leftmost child, by setting all "bits" to 1 we encode the position $(\mathfrak{t}(k, n) - 1, \mathfrak{t}(k, n) - 1)$ of the grid.

Now we introduce the formula $\mathtt{grid}_{\mathscr{T}}(k)$ that characterises the set of models encoding the $[0, \mathfrak{t}(k, n) - 1]^2$ grid. A model $(\mathfrak{M} = (W, R, V), w)$ satisfying $\mathtt{grid}_{\mathscr{T}}(k)$ is such that:

(zero$_{\mathscr{T},k}$) there is a t-node $\tilde{w}$ in $R(w)$ that encodes the position $(\mathbf{n}_{\mathcal{H}}(\tilde{w}), \mathbf{n}_{\mathcal{V}}(\tilde{w})) = (0, 0)$;
(uniq$_{\mathscr{T},k}$) for all two distinct t-nodes $w_1, w_2 \in R(w)$, $\mathbf{n}_{\mathcal{H}}(w_1) \neq \mathbf{n}_{\mathcal{H}}(w_2)$ or $\mathbf{n}_{\mathcal{V}}(w_1) \neq \mathbf{n}_{\mathcal{V}}(w_2)$;
(compl$_{\mathscr{T},k}$) for every t-node $w_1 \in R(w)$,

- if $\mathbf{n}_{\mathcal{H}}(w_1) < \mathfrak{t}(k, n) - 1$ then there is a t-node $w_2 \in R(w)$ such that $\mathbf{n}_{\mathcal{H}}(w_2) = \mathbf{n}_{\mathcal{H}}(w_1) + 1$ and $\mathbf{n}_{\mathcal{V}}(w_2) = \mathbf{n}_{\mathcal{V}}(w_1)$;

- if $\mathbf{n}_{\mathcal{V}}(w_1) < \mathfrak{t}(k,n) - 1$ then there is a t-node $w_2 \in R(w)$ such that $\mathbf{n}_{\mathcal{V}}(w_2) = \mathbf{n}_{\mathcal{V}}(w_1) + 1$ and $\mathbf{n}_{\mathcal{H}}(w_2) = \mathbf{n}_{\mathcal{H}}(w_1)$;

(init/sub/aux) $(\mathfrak{M}, w)$ satisfies $\mathtt{init}(k)$, $\mathtt{sub}(k)$ and $\mathtt{aux}$.

It is easy to see that, with these conditions, $(\mathfrak{M}, w)$ correctly encodes the grid. The definition of $\mathtt{grid}_{\mathscr{T}}(k)$ follows rather closely the definition of $\mathtt{type}(j)$. It is defined as

$$\mathtt{grid}_{\mathscr{T}}(k) \stackrel{\text{def}}{=} \mathtt{zero}_{\mathscr{T}}(k) \land \mathtt{uniq}_{\mathscr{T}}(k) \land \mathtt{compl}_{\mathscr{T}}(k) \land \mathtt{init}(k) \land \mathtt{sub}(k) \land \mathtt{aux},$$

where each conjunct expresses the homonymous property above. To define the first three conjuncts of $\mathtt{grid}_{\mathscr{T}}(k)$ (hence completing its definition) we start by defining the formulae $[\mathsf{ax} \stackrel{D}{=} \mathsf{bx}]_k$ and $[\mathsf{bx} \stackrel{D}{=} \mathsf{ax{+}1}]_k$, where $D \in \{\mathcal{H}, \mathcal{V}\}$. These formulae will be defined similarly to $[\mathsf{ax} = \mathsf{bx}]_k^1$ and $[\mathsf{bx} = \mathsf{ax{+}1}]_k$. Given a pointed model $(\mathfrak{M}, w)$ (with $\mathfrak{M} = (W, R, V)$) satisfying $\mathtt{fork}_k^1(\mathsf{ax}, \mathsf{bx})$, and the two t-nodes $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$ corresponding to the nominals ax and bx, respectively,

$[\mathsf{ax} \stackrel{D}{=} \mathsf{bx}]_k$ states that $\mathbf{n}_D(w_{\mathsf{ax}}) = \mathbf{n}_D(w_{\mathsf{bx}})$; $[\mathsf{bx} \stackrel{D}{=} \mathsf{ax{+}1}]_k$ states that $\mathbf{n}_D(w_{\mathsf{bx}}) = \mathbf{n}_D(w_{\mathsf{ax}}) + 1$.

To encode $[\mathsf{ax} \stackrel{D}{=} \mathsf{bx}]_k$ we simply require that for all two t-children $w_{\mathsf{ax}} \in R(w_{\mathsf{ax}})$ and $w_{\mathsf{bx}} \in R(w_{\mathsf{bx}})$, if $\mathbf{n}_D(w_{\mathsf{ax}}) = \mathbf{n}_D(w_{\mathsf{bx}})$ then w_{ax} and w_{bx} agree on the satisfaction of $\mathtt{val}_D$. The following formula expresses this property (whose correctness is proved immediately after its definition):

$$[\mathsf{ax} \stackrel{D}{=} \mathsf{bx}]_k \stackrel{\text{def}}{=} \neg\big(\top * (\mathtt{fork}_k^2(\mathsf{x}, \mathsf{y}) \land @_{\mathsf{ax}}^1 \langle t \rangle \Diamond \mathsf{x} \land @_{\mathsf{bx}}^1 \langle t \rangle \Diamond \mathsf{y} \land [\mathsf{x} = \mathsf{y}]_k^2 \land \neg(@_{\mathsf{x}}^2 \mathtt{val}_D \Leftrightarrow @_{\mathsf{y}}^2 \mathtt{val}_D))\big).$$

Lemma 4.23. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $k \geq 2$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(k) \land \mathtt{fork}_k^1(\mathsf{ax}, \mathsf{bx})$. *Then,* $\mathfrak{M}, w \models [\mathsf{ax} \stackrel{D}{=} \mathsf{bx}]_k$ *if and only if there are two distinct t-nodes* $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$ *such that* w_{ax} *corresponds to the nominal* ax, w_{bx} *corresponds to the nominal* bx *and* $\mathbf{n}_D(w_{\mathsf{ax}}) = \mathbf{n}_D(w_{\mathsf{bx}})$.

Proof. This proof is similar to the one of Lemma 4.16 (II). Since $\mathfrak{M}, w \models \mathtt{init}(k) \land \mathtt{fork}_k^1(\mathsf{ax}, \mathsf{bx})$, by Lemma 4.14 there are two worlds w_{ax} and w_{bx} in $R(w)$ corresponding to the nominals (for the depth 1) ax and bx, respectively.

($\Rightarrow$): Suppose $\mathfrak{M}, w \models [\mathsf{ax} \stackrel{D}{=} \mathsf{bx}]_k$. Then, for every $\mathfrak{M}' = (W, R_1, V)$, if $\mathfrak{M}' \sqsubseteq \mathfrak{M}$ and $\mathfrak{M}', w \models \mathtt{fork}_k^2(\mathsf{x}, \mathsf{y}) \land @_{\mathsf{ax}}^1 \langle t \rangle \Diamond \mathsf{x} \land @_{\mathsf{bx}}^1 \langle t \rangle \Diamond \mathsf{y} \land [\mathsf{x} = \mathsf{y}]_k^2$ then $\mathfrak{M}', w \models @_{\mathsf{x}}^2 \mathtt{val}_D \Leftrightarrow @_{\mathsf{y}}^2 \mathtt{val}_D$. Now, from $\mathfrak{M}, w \models \mathtt{fork}_k^1(\mathsf{ax}, \mathsf{bx})$ we have $\mathfrak{M}, w_{\mathsf{ax}} \models \mathtt{type}(k-1)$ and $\mathfrak{M}, w_{\mathsf{bx}} \models \mathtt{type}(k-1)$ (notice that then, all the worlds in $R(w_{\mathsf{ax}}) \cup R(w_{\mathsf{bx}})$ satisfy $\mathtt{type}(k-2)$). Thus, let us consider two arbitrary worlds w_{x} and w_{y} such that

- $w_{\mathsf{x}} \in R(w_{\mathsf{ax}})$ and $w_{\mathsf{y}} \in R(w_{\mathsf{bx}})$;
- $\mathbf{n}_{k-1}(w_{\mathsf{x}}) = \mathbf{n}_{k-1}(w_{\mathsf{y}})$.

We show that $\mathfrak{M}, w_{\mathsf{x}} \models \mathtt{val}_D$ if and only if $\mathfrak{M}, w_{\mathsf{y}} \models \mathtt{val}_D$, thus concluding that $\mathbf{n}_D(w_{\mathsf{ax}}) = \mathbf{n}_D(w_{\mathsf{bx}})$. Let us consider the finite forest $\mathfrak{M}' = (W, R_1, V)$ where R_1 is obtained from R by removing every edge $(w_b, w') \in R$ where $b \in \{\mathsf{ax}, \mathsf{bx}\}$, and w' is a t-node different from w_{x} and w_{y}. We also remove the edge $(w_{\mathsf{x}}, w') \in R$ where w' is the only y-child of w_{x}, as well as (w_{y}, w'') where w'' is the only x-child of w_{y}. The existence of these nodes is guaranteed by $\mathfrak{M}, w_{\mathsf{ax}} \models \mathtt{type}(k-1)$ and $\mathfrak{M}, w_{\mathsf{bx}} \models \mathtt{type}(k-1)$. By Lemma 4.14, we have $\mathfrak{M}', w \models \mathtt{fork}_k^2(\mathsf{x}, \mathsf{y})$, where w_{x} corresponds to the nominal (at depth 2) x, whereas w_{y} corresponds to the nominal (at depth 2) y. Moreover, Lemma 4.14 ensures that $\mathfrak{M}, w_{\mathsf{x}} \models \mathtt{type}(k-2)$ and $\mathfrak{M}, w_{\mathsf{y}} \models \mathtt{type}(k-2)$, hence by Lemma 4.13 we conclude that w_{x} (resp. w_{y}) encodes the same number w.r.t. $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w)$. Again from the definition of R_1 it is easy to see that $\mathfrak{M}', w \models @_{\mathsf{ax}}^1 \langle t \rangle \Diamond \mathsf{x} \land @_{\mathsf{bx}}^1 \langle t \rangle \Diamond \mathsf{y}$. Lastly, by hypothesis on w_{x} and w_{y}, together with Lemma 4.17 and that $[\mathsf{x} = \mathsf{y}]_k^2$ is equal to $\neg([\mathsf{x} < \mathsf{y}]_k^2 \lor [\mathsf{y} < \mathsf{x}]_k^2)$ by definition, we conclude that $\mathfrak{M}', w \models [\mathsf{x} = \mathsf{y}]_k^2$. Thus, by hypothesis, $\mathfrak{M}', w \models @_{\mathsf{x}}^2 \mathtt{val}_D \Leftrightarrow @_{\mathsf{y}}^2 \mathtt{val}_D$, concluding the proof.

($\Leftarrow$): This direction is proved analogously by mainly relying on Lemma 4.17 and Lemma 4.13. □

The formula $[\mathsf{bx} \overset{D}{=} \mathsf{ax{+}1}]_k$ can be defined by slightly modifying the formula $[\mathsf{bx} = \mathsf{ax{+}1}]_k$. We start by defining the formulae $\mathsf{L}[D]_k(\mathsf{ax},\mathsf{bx})$, $\mathsf{S}[D]_k(\mathsf{ax},\mathsf{bx})$ and $\mathsf{R}[D](\mathsf{ax},\mathsf{bx})$ with semantics similar to $\mathsf{L}^1_k(\mathsf{ax},\mathsf{bx})$, $\mathsf{S}^1_k(\mathsf{ax},\mathsf{bx})$ and $\mathsf{R}(\mathsf{ax},\mathsf{bx})$, respectively, but where, for a given t-node in $R^2(w)$, we are interested in the satisfaction of $\mathtt{val}_D$ instead of $\mathtt{val}$. For example, the formula $\mathsf{S}[D]_k(\mathsf{ax},\mathsf{bx})$ is defined as

$$\mathsf{S}[D]_k(\mathsf{ax},\mathsf{bx}) \overset{\text{def}}{=} \top * \big(\mathtt{fork}^2_k(\mathsf{x},\mathsf{y}) \land @^1_{\mathsf{ax}}\langle t\rangle(\Diamond\mathsf{s} \land \Diamond\mathsf{x}) \land \\ @^1_{\mathsf{bx}}\langle t\rangle(\Diamond\mathsf{s} \land \Diamond\mathsf{y}) \land [\mathsf{x}=\mathsf{y}]^2_k \land @^2_{\mathsf{x}}\neg\mathtt{val}_D \land @^2_{\mathsf{y}}\mathtt{val}_D\big),$$

i.e., by replacing the two last conjuncts of $\mathsf{S}^1_k(\mathsf{ax},\mathsf{bx})$, $@^2_{\mathsf{x}}\neg\mathtt{val}$ and $@^2_{\mathsf{y}}\mathtt{val}$ with $@^2_{\mathsf{x}}\neg\mathtt{val}_D$ and $@^2_{\mathsf{y}}\mathtt{val}_D$, respectively. Similarly, $\mathsf{L}[D]_k(\mathsf{ax},\mathsf{bx})$ is defined from $\mathsf{L}^1_k(\mathsf{ax},\mathsf{bx})$ by replacing the last conjunct of this formula, i.e. $\neg(@^2_{\mathsf{x}}\mathtt{val} \Leftrightarrow @^2_{\mathsf{y}}\mathtt{val})$, by $\neg(@^2_{\mathsf{x}}\mathtt{val}_D \Leftrightarrow @^2_{\mathsf{y}}\mathtt{val}_D)$. Lastly, $\mathsf{R}[D](\mathsf{ax},\mathsf{bx})$ is defined from $\mathsf{R}(\mathsf{ax},\mathsf{bx})$ by replacing every occurrence of $\mathtt{val}$ by $\mathtt{val}_D$. The formula $[\mathsf{bx} \overset{D}{=} \mathsf{ax{+}1}]_k$ is then defined as follows:

$$[\mathsf{bx} \overset{D}{=} \mathsf{ax{+}1}]_k \overset{\text{def}}{=} \top * \big(\mathtt{nom}_1(\mathsf{ax}{\neq}\mathsf{bx}) \land [t]\mathtt{lsr}(k-1) \land \mathsf{L}[D]_k(\mathsf{ax},\mathsf{bx}) \land \mathsf{S}[D]_k(\mathsf{ax},\mathsf{bx}) \land \mathsf{R}[D](\mathsf{ax},\mathsf{bx})\big).$$

Lemma 4.24. *Let* $\mathsf{ax} \neq \mathsf{bx} \in \mathsf{Aux}$ *and* $k \geq 2$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(k) \land \mathtt{fork}^1_k(\mathsf{ax},\mathsf{bx})$. *Then,* $\mathfrak{M}, w \models [\mathsf{bx} \overset{D}{=} \mathsf{ax{+}1}]_k$ *if and only if there are two distinct t-nodes* $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$ *such that* w_{ax} *corresponds to the nominal* ax, w_{bx} *corresponds to the nominal* bx *and* $\mathbf{n}_D(w_{\mathsf{bx}}) = \mathbf{n}_D(w_{\mathsf{ax}}) + 1$.

Proof. The proof unfolds as the proofs of Lemmata 4.8 and 4.18. □

We are now ready to define the formulae $\mathtt{zero}_{\mathcal{T}}(k)$, $\mathtt{uniq}_{\mathcal{T}}(k)$ and $\mathtt{compl}_{\mathcal{T}}(k)$, achieving the conditions ($\mathrm{zero}_{\mathcal{T},k}$), ($\mathrm{uniq}_{\mathcal{T},k}$) and ($\mathrm{compl}_{\mathcal{T},k}$), respectively. All these formulae follow closely the definitions of $\mathtt{zero}(k)$, $\mathtt{uniq}(k)$ and $\mathtt{compl}(k)$ of the previous sections, hence we refer to these latter formulae for an informal description on how they work. The formula $\mathtt{zero}_{\mathcal{T}}(k)$ is defined as:

$$\mathtt{zero}_{\mathcal{T}}(k) \overset{\text{def}}{=} \langle t\rangle([t](\neg\mathtt{val}_{\mathcal{H}} \land \neg\mathtt{val}_{\mathcal{V}})).$$

Lemma 4.25. $\mathfrak{M}, w \models \mathtt{zero}_{\mathcal{T}}(k)$ *if and only if* $(\mathfrak{M}, w)$ *satisfies* ($\mathrm{zero}_{\mathcal{T},k}$).

Proof. The proof is direct, by definition of $\mathtt{zero}_{\mathcal{T}}(k)$ and how $(0,0)$ is encoded in the grid. □

The formula $\mathtt{uniq}_{\mathcal{T}}(k)$ is defined from $\mathtt{uniq}(k)$ by replacing $[\mathsf{x}=\mathsf{y}]^1_k$ with $[\mathsf{x} \overset{\mathcal{H}}{=} \mathsf{y}]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k$:

$$\mathtt{uniq}_{\mathcal{T}}(k) = \neg\big(\top * (\mathtt{fork}^1_k(\mathsf{x},\mathsf{y}) \land [\mathsf{x} \overset{\mathcal{H}}{=} \mathsf{y}]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k)\big).$$

Lemma 4.26. *Let* $k \geq 2$. *Suppose* $\mathfrak{M}, w \models \mathtt{init}(k) \land \mathsf{aux}$. *Then,* $\mathfrak{M}, w \models \mathtt{uniq}(k)$ *if and only if* $(\mathfrak{M}, w)$ *satisfies* ($\mathrm{uniq}_{\mathcal{T},k}$)*, i.e. distinct t-nodes in* $R(w)$ *encode different pairs of numbers.*

Proof. This lemma is proven as Lemma 4.9 and Lemma 4.19, by relying on Lemma 4.23 in order to show that, given two distinct worlds w_{x} and w_{y} corresponding to nominals (for the depth 1) x and y, respectively, $[\mathsf{x} \overset{\mathcal{H}}{=} \mathsf{y}]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k$ holds if and only if $\mathbf{n}_{\mathcal{H}}(w_{\mathsf{x}}) = \mathbf{n}_{\mathcal{H}}(w_{\mathsf{y}})$ and $\mathbf{n}_{\mathcal{V}}(w_{\mathsf{x}}) = \mathbf{n}_{\mathcal{V}}(w_{\mathsf{y}})$. □

Lastly, $\mathtt{compl}_{\mathcal{T}}(k) \overset{\text{def}}{=} \mathtt{compl}[\mathcal{H}]_{\mathcal{T}}(k) \land \mathtt{compl}[\mathcal{V}]_{\mathcal{T}}(k)$ where

$$\mathtt{compl}[\mathcal{H}]_{\mathcal{T}}(k) \overset{\text{def}}{=} \neg\Big(\Box\bot * \Big([t](\mathtt{type}_{\mathtt{lsr}}(k{-}1) \land \Diamond\mathsf{y}) \land \mathtt{nom}_1(\mathsf{x}) \land \\ @^1_{\mathsf{x}}\neg 1^{\mathcal{H}}_k \land \neg\big(\top * (\mathtt{fork}^1_j(\mathsf{x},\mathsf{y}) \land [\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x{+}1}]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k)\big)\Big)\Big),$$

and $\mathtt{compl}[\mathcal{V}]_{\mathcal{T}}(k)$ is defined from $\mathtt{compl}[\mathcal{H}]_{\mathcal{T}}(k)$ by replacing $1^{\mathcal{H}}_k$, $[\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x{+}1}]_k$ and $[\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k$ with $1^{\mathcal{V}}_k$, $[\mathsf{y} \overset{\mathcal{V}}{=} \mathsf{x{+}1}]_k$ and $[\mathsf{x} \overset{\mathcal{H}}{=} \mathsf{y}]_k$, respectively. Here, 1^D_k ($D \in \{\mathcal{H},\mathcal{V}\}$) is defined as $[t]\mathtt{val}_D$, and hence it is satisfied by the t-nodes $w' \in R(w)$ such that $\mathbf{n}_D(w') = \mathfrak{t}(k,n) - 1$.

Lemma 4.27. *Let $k \geq 2$. Suppose $\mathfrak{M}, w \models \mathtt{init}(k) \wedge \mathtt{aux}$. $\mathfrak{M}, w \models \mathtt{compl}_{\mathscr{T}}(k)$ if and only if $(\mathfrak{M}, w)$ satisfies (compl$_{\mathscr{T},k}$). More precisely,*

(1) $\mathfrak{M}, w \models \mathtt{compl}[\mathcal{H}]_{\mathscr{T}}(k)$ if and only if for every t-node $w_1 \in R(w)$, if $\mathbf{n}_{\mathcal{H}}(w_1) < \mathfrak{t}(k,n) - 1$ then there is a t-node $w_2 \in R(w)$ such that $\mathbf{n}_{\mathcal{H}}(w_2) = \mathbf{n}_{\mathcal{H}}(w_1) + 1$ and $\mathbf{n}_{\mathcal{V}}(w_2) = \mathbf{n}_{\mathcal{V}}(w_1)$;

(2) $\mathfrak{M}, w \models \mathtt{compl}[\mathcal{V}]_{\mathscr{T}}(k)$ if and only if for every t-node $w_1 \in R(w)$, if $\mathbf{n}_{\mathcal{V}}(w_1) < \mathfrak{t}(k,n) - 1$ then there is a t-node $w_2 \in R(w)$ such that $\mathbf{n}_{\mathcal{H}}(w_2) = \mathbf{n}_{\mathcal{H}}(w_1)$ and $\mathbf{n}_{\mathcal{V}}(w_2) = \mathbf{n}_{\mathcal{V}}(w_1) + 1$.

Proof. Both (1) and (2) are proved as Lemma 4.10 and Lemma 4.20, with the sole difference that we rely on Lemma 4.23 and Lemma 4.24 in order to show that, given two distinct worlds w_{x} and w_{y} corresponding to nominals (for the depth 1) x and y, respectively, $[\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x}{+}1]_k \wedge [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k$ holds if and only if $\mathbf{n}_{\mathcal{H}}(w_{\mathsf{x}}) = \mathbf{n}_{\mathcal{H}}(w_{\mathsf{y}}) + 1$ and $\mathbf{n}_{\mathcal{V}}(w_{\mathsf{x}}) = \mathbf{n}_{\mathcal{V}}(w_{\mathsf{y}})$ (in the proof of 1). Similarly, (in the proof of 2) $[\mathsf{y} \overset{\mathcal{V}}{=} \mathsf{x}{+}1]_k \wedge [\mathsf{x} \overset{\mathcal{H}}{=} \mathsf{y}]_k$ holds if and only if $\mathbf{n}_{\mathcal{H}}(w_{\mathsf{x}}) = \mathbf{n}_{\mathcal{H}}(w_{\mathsf{y}})$ and $\mathbf{n}_{\mathcal{V}}(w_{\mathsf{x}}) = \mathbf{n}_{\mathcal{V}}(w_{\mathsf{y}}) + 1$. □

This concludes the definition of $\mathtt{grid}_{\mathscr{T}}(k)$. It is proved correct in the following lemma.

Lemma 4.28. *$\mathfrak{M}, w \models \mathtt{grid}_{\mathscr{T}}(k)$ if and only if $(\mathfrak{M}, w)$ satisfies (zero$_{\mathscr{T},k}$), (uniq$_{\mathscr{T},k}$), (compl$_{\mathscr{T},k}$) and (init/sub/aux).*

Proof. Directly from Lemmata 4.1, 4.5 and 4.25 to 4.27. □

Corollary 4.29. *The formula $\mathtt{grid}_{\mathscr{T}}(k)$ is satisfiable.*

Proof. (sketch) The satisfiability of $\mathtt{grid}_{\mathscr{T}}(k)$ can be established by Lemma 4.28 as (zero$_{\mathscr{T},k}$), (uniq$_{\mathscr{T},k}$), (compl$_{\mathscr{T},k}$) and (init/sub/aux) can be simultaneously satisfied. A model satisfying these constraints can be defined similarly to what is done in Lemma 4.22. The main difference is that now the root shall have $\mathfrak{t}(k,n)^2$ children (one for each position of the grid) satisfying $\mathtt{type}(k-1)$. □

We can now proceed to the encoding of the tiling conditions (first) and (hor&vert). Given a model $(\mathfrak{M} = (W, R, V), w)$ satisfying $\mathtt{grid}_{\mathscr{T}}(k)$, the existence of a solution for $(\mathscr{T}, \mathsf{c})$, w.r.t. $\mathtt{Tile}_k$, can be expressed with the following conditions:

(one$_{\mathscr{T}}$) every t-node in $R(w)$ satisfies exactly one tile in $\mathcal{T}$;
(first$_{\mathscr{T},\mathsf{c}}$) for all $\tilde{w} \in R(w)$, if $\mathbf{n}_{\mathcal{H}}(\tilde{w}) = \mathbf{n}_{\mathcal{V}}(\tilde{w}) = 0$ then $\tilde{w} \in V(\mathsf{c})$;
(hor$_{\mathscr{T}}$) for all $w_1, w_2 \in R(w)$, if $\mathbf{n}_{\mathcal{H}}(w_2) = \mathbf{n}_{\mathcal{H}}(w_1) + 1$ and $\mathbf{n}_{\mathcal{V}}(w_2) = \mathbf{n}_{\mathcal{V}}(w_1)$ then there is $(\mathsf{c}_1, \mathsf{c}_2) \in \mathcal{H}$ such that $w_1 \in V(\mathsf{c}_1)$ and $w_2 \in V(\mathsf{c}_2)$;
(vert$_{\mathscr{T}}$) for all $w_1, w_2 \in R(w)$, if $\mathbf{n}_{\mathcal{V}}(w_2) = \mathbf{n}_{\mathcal{V}}(w_1) + 1$ and $\mathbf{n}_{\mathcal{H}}(w_2) = \mathbf{n}_{\mathcal{H}}(w_1)$ then there is $(\mathsf{c}_1, \mathsf{c}_2) \in \mathcal{V}$ such that $w_1 \in V(\mathsf{c}_1)$ and $w_2 \in V(\mathsf{c}_2)$.

Then, the formula $\mathtt{tiling}_{\mathscr{T},\mathsf{c}}(k)$ can be defined as

$$\mathtt{tiling}_{\mathscr{T},\mathsf{c}}(k) \overset{\text{def}}{=} \mathtt{grid}_{\mathscr{T}}(k) \wedge \mathtt{one}_{\mathscr{T}} \wedge \mathtt{first}_{\mathscr{T},\mathsf{c}}(k) \wedge \mathtt{hor}_{\mathscr{T}}(k) \wedge \mathtt{vert}_{\mathscr{T}}(k),$$

where the last four conjuncts express the homonymous property above. Given the toolkit of formulae introduced up to now, these four formulae are easy to define. The formula $\mathtt{one}_{\mathscr{T}}$ is simply defined as $[t] \bigvee_{\mathsf{c}_1 \in \mathcal{T}} (\mathsf{c}_1 \wedge \bigwedge_{\mathsf{c}_2 \in \mathcal{T}} \neg \mathsf{c}_2)$. Similarly, $\mathtt{first}_{\mathscr{T},\mathsf{c}}(k)$ is also straightforward to define:

$$\mathtt{first}_{\mathscr{T},c}(k) \overset{\text{def}}{=} [t]\big([t](\neg\mathtt{val}_{\mathcal{H}} \wedge \neg\mathtt{val}_{\mathcal{V}}) \Rightarrow \mathsf{c}\big).$$

Notice that, in this formula, we use the fact that the t-node $w' \in R(w)$ encoding $(0,0)$ is the only one, among the t-children of w, satisfying $[t](\neg\mathtt{val}_{\mathcal{H}} \wedge \neg\mathtt{val}_{\mathcal{V}})$.

Lemma 4.30. *Let $k \geq 2$ and suppose $\mathfrak{M}, w \models \mathtt{grid}_{\mathscr{T}}(k)$. Then,*

I. $\mathfrak{M}, w \models \mathtt{one}_{\mathscr{T}}$ if and only if $(\mathfrak{M}, w)$ satisfies (one$_{\mathscr{T}}$);
II. $\mathfrak{M}, w \models \mathtt{first}_{\mathscr{T},\mathsf{c}}(k)$ if and only if $(\mathfrak{M}, w)$ satisfies (first$_{\mathscr{T},\mathsf{c}}$).

Proof. Both I and II are easily proven directly from the definition of $\mathtt{one}_{\mathscr{T}}$ and $\mathtt{first}_{\mathscr{T},\mathsf{c}}(k)$. □

For the formula $\mathtt{hor}_{\mathscr{T}}(k)$, we essentially state that there cannot be two t-nodes $w_1, w_2 \in R(w)$ such that w_2 encodes the position $(\mathbf{n}_{\mathcal{H}}(w_1)+1, \mathbf{n}_{\mathcal{V}}(w_1))$ and $w_1 \in V(\mathtt{c}_1)$, $w_2 \in V(\mathtt{c}_2)$ does not hold for any $(\mathtt{c}_1, \mathtt{c}_2) \in \mathcal{H}$. In formula:

$$\mathtt{hor}_{\mathscr{T}}(k) \stackrel{\text{def}}{=} \neg\big(\top * \big(\mathtt{fork}^1_k(\mathtt{x},\mathtt{y}) \wedge [\mathtt{y} \stackrel{\mathcal{H}}{=} \mathtt{x{+}1}]_k \wedge [\mathtt{x} \stackrel{\mathcal{V}}{=} \mathtt{y}]_k \wedge \neg \textstyle\bigvee_{(\mathtt{c}_1,\mathtt{c}_2)\in\mathcal{H}}(@^1_{\mathtt{x}}\mathtt{c}_1 \wedge @^1_{\mathtt{y}}\mathtt{c}_2)\big)\big).$$

Lastly, $\mathtt{vert}_{\mathscr{T}}(k)$ is defined as $\mathtt{hor}_{\mathscr{T}}(k)$, but replacing $\mathcal{H}$ by $\mathcal{V}$ and vice-versa:

$$\mathtt{vert}_{\mathscr{T}}(k) \stackrel{\text{def}}{=} \neg\big(\top * \big(\mathtt{fork}^1_k(\mathtt{x},\mathtt{y}) \wedge [\mathtt{y} \stackrel{\mathcal{V}}{=} \mathtt{x{+}1}]_k \wedge [\mathtt{x} \stackrel{\mathcal{H}}{=} \mathtt{y}]_k \wedge \neg \textstyle\bigvee_{(\mathtt{c}_1,\mathtt{c}_2)\in\mathcal{V}}(@^1_{\mathtt{x}}\mathtt{c}_1 \wedge @^1_{\mathtt{y}}\mathtt{c}_2)\big)\big).$$

Lemma 4.31. *Let $k \geq 2$ and suppose $\mathfrak{M}, w \models \mathtt{grid}_{\mathscr{T}}(k)$. Then,*

I. $\mathfrak{M}, w \models \mathtt{hor}_{\mathscr{T}}(k)$ if and only if $(\mathfrak{M}, w)$ satisfies (hor$_{\mathscr{T}}$);
II. $\mathfrak{M}, w \models \mathtt{vert}_{\mathscr{T}}(k)$ if and only if $(\mathfrak{M}, w)$ satisfies (vert$_{\mathscr{T}}$).

See the proof in Appendix G. This concludes the definition of $\mathtt{tiling}_{\mathscr{T},\mathtt{c}}(k)$.

Lemma 4.32. *$\mathfrak{M}, w \models \mathtt{tiling}_{\mathscr{T},\mathtt{c}}(k)$ if and only if $(\mathfrak{M}, w)$ satisfies (zero$_{\mathscr{T},k}$), (uniq$_{\mathscr{T},k}$), (compl$_{\mathscr{T},k}$), (init/sub/aux), (one$_{\mathscr{T}}$), (first$_{\mathscr{T},\mathtt{c}}$), (hor$_{\mathscr{T}}$) and (vert$_{\mathscr{T}}$).*

Proof. Directly from Lemmata 4.28, 4.30 and 4.31. □

We can now prove Lemma 4.33 (shown below), leading directly to Theorem 4.34.

Lemma 4.33. *Let $k \geq 2$ and let $(\mathscr{T}, \mathtt{c})$ be an instance of $\mathtt{Tile}_k$, where $\mathscr{T} = (\mathcal{T}, \mathcal{H}, \mathcal{V})$ and $\mathtt{c} \in \mathcal{T}$. Then, $(\mathscr{T}, \mathtt{c})$ is a solution for $\mathtt{Tile}_k$ iff the formula $\mathtt{tiling}_{\mathscr{T},\mathtt{c}}(k)$ is satisfiable.*

The proof can be found in Appendix H. It should be noticed that the reduction from tiling to Sat(ML($*$)) we provided is (only) exponential in k. Therefore, with this last lemma at hand, we can finally conclude with the intended result in this section.

Theorem 4.34. Sat(ML($*$)) *is* Tower*-complete.*

Summing up, unlike ML($|$) whose complexity is AExp$_{\text{Pol}}$-complete (so, below ExpSpace), the satisfiability problem for ML($*$) is Tower-complete, which does not correspond to an elementary class. However, as we will see in the next section, ML($*$) is surprisingly strictly less expressive than ML($|$). Note also that related Tower-hard logics can be found in [39].

5 ML($*$) STRICTLY LESS EXPRESSIVE THAN GML

Below, we study the expressivity of ML($*$). We establish the inclusion ML($*$) $\preceq$ GML (Section 5.1) and then prove its strictness (Section 5.2). The former result takes advantage of the notion of g-bisimulation, i.e. the underlying structural indistinguishability relation of GML, studied in [22]. This notion is instrumental in the proofs but for the sake of conciseness, the statements in the body of the paper are stated in terms of modal equivalence. To show ML($*$) $\prec$ GML, we define an ad hoc notion of Ehrenfeucht-Fraïssé games for ML($*$), see e.g. [35] for classical definitions and [15, 20] for similar approaches, and design a GML formula that cannot be expressed in ML($*$).

5.1 ML($*$) is at most as expressive as GML

To establish that ML($*$) $\preceq$ GML, we proceed as in Section 3.2. In fact, by Lemma 2.2, given φ_1, φ_2 in GML, the formula $\varphi_1 * \varphi_2$ is equivalent to $\blacklozenge(\varphi_1 | \varphi_2)$. Moreover, we know that given φ_1, φ_2 in GML, $\varphi_1 | \varphi_2$ is equivalent to some formula in GML, as shown in Section 3. So, to prove that ML($*$) $\preceq$ GML by applying the proof schema of Theorem 3.7, it is sufficient to show that given φ in GML, there is ψ in GML such that $\blacklozenge\varphi \equiv \psi$. To do so, we rely on the indistinguishability relation of GML, called g-bisimulation [22].

Formal definitions about g-bisimulation are recalled in Appendix I but are not required in this section. Nevertheless, let us recall that a g-bisimulation is a refinement of the classical back-and-forth conditions of a bisimulation (see e.g. [10]), tailored towards capturing graded modalities. It relates models with similar structural properties, but up to parameters $m, k \in \mathbb{N}$ responsible for the modal degree and the graded rank, respectively. The following invariance result holds: g-bisimilar models are modally equivalent in GML (up to formulae of modal degree m and graded rank at most k). For simplicity, we present the construction of the above-mentioned formula ψ by directly using the notion of modal equivalence, without going explicitly through g-bisimulations. The notion of g-bisimulation is used explicitely in the proofs developed in the appendices.

Given $m, k \in \mathbb{N}$ and $\mathrm{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$, we write $\mathsf{GML}[m, k, \mathrm{P}]$ to denote the set of GML formulae ψ having $\mathrm{md}(\psi) \leq m$, $\mathrm{gr}(\psi) \leq k$ and propositional variables from P. It is known that $\mathsf{GML}[m, k, \mathrm{P}]$ is finite up to logical equivalence [22]. Given pointed forests $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$, we write $(\mathfrak{M}, w) \equiv^{\mathrm{P}}_{m,k} (\mathfrak{M}', w')$ whenever $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ are $\mathsf{GML}[m, k, \mathrm{P}]$-*indistinguishable*, i.e. for every ψ in $\mathsf{GML}[m, k, \mathrm{P}]$, $\mathfrak{M}, w \models \psi$ iff $\mathfrak{M}', w' \models \psi$. We write $\mathcal{T}^{\mathrm{P}}(m, k)$ to denote the quotient set induced by the equivalence relation $\equiv^{\mathrm{P}}_{m,k}$. As $\mathsf{GML}[m, k, \mathrm{P}]$ is finite up to logical equivalence, we get that $\mathcal{T}^{\mathrm{P}}(m, k)$ is a finite set.

To establish that GML is closed under $\blacklozenge$, we show that there is a function $\mathfrak{f} : \mathbb{N}^2 \to \mathbb{N}$ such that for all $m, k \in \mathbb{N}$ and $\mathrm{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$, if two models are in the same equivalence class of $\equiv^{\mathrm{P}}_{m,\mathfrak{f}(m,k)}$, then they satisfy the same formulae of the form $\blacklozenge\varphi$, where φ is in $\mathsf{GML}[m, k, \mathrm{P}]$. Then, we can conclude that $\blacklozenge\varphi$ is equivalent to a formula in $\mathsf{GML}[m, \mathfrak{f}(m, k), \mathrm{P}]$, see the proof of Lemma 5.2. Similar ideas are followed in [24, 26, 38]. As we are not interested in the size of the equivalent formula, we can simply use the cardinality of $\mathcal{T}^{\mathrm{P}}(m, k)$ in order to inductively define a suitable function:

$$\mathfrak{f}(0, k) \stackrel{\text{def}}{=} k, \qquad \mathfrak{f}(m + 1, k) \stackrel{\text{def}}{=} k \cdot (|\mathcal{T}^{\mathrm{P}}(m, \mathfrak{f}(m, k))| + 1).$$

In conformity with the results in Section 4, the map $\mathfrak{f}$ can be shown to be a non-elementary function. To prove that $\mathfrak{f}$ satisfies the required properties, we start by showing a technical lemma which essentially formalises a simulation argument on the relation $\equiv^{\mathrm{P}}_{m,\mathfrak{f}(m,k)}$ with respect to the submodel relation. By taking submodels as with the $\blacklozenge$ operator, equivalence in GML is preserved.

Lemma 5.1. *Consider* $(\mathfrak{M}, w) \equiv^{\mathrm{P}}_{m,\mathfrak{f}(m,k)} (\mathfrak{M}', w')$ *where* $m, k \in \mathbb{N}$, $\mathrm{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$, $\mathfrak{M} = (W, R, V)$ *and* $\mathfrak{M}' = (W', R', V')$. *Let* $R_1 \subseteq R$. *There is* $R'_1 \subseteq R'$ *such that* $((W, R_1, V), w) \equiv^{\mathrm{P}}_{m,k} ((W', R'_1, V'), w')$ *and if* $R_1(w) = R(w)$, *then* $R'_1(w') = R'(w')$.

Intuitively, Lemma 5.1 states that given two models satisfying the same formulae up to the parameters m and $\mathfrak{f}(m, k)$, we can extract submodels satisfying the same formulae up to m and k (reduced graded rank). This allows us to conclude that if φ is in GML, there is some GML formula equivalent to $\blacklozenge\varphi$ (Lemma 5.2). In other words, the operator $\blacklozenge$ can be eliminated to obtain a GML formula. The last condition about $R_1(w) = R(w)$ will serve in the proof of Lemma 5.2, as it allows us to capture the semantics of $\blacklozenge$, by preserving the children of the world w'.

The proof of Lemma 5.1 is in Appendix J and goes by induction on m. It relies on the properties of g-bisimulations [22] to define a binary relation $\leftrightarrow$ between the worlds of $R(w)$ and $R'(w')$. Every $w_1 \leftrightarrow w'_1$ is such that $(\mathfrak{M}, w_1) \equiv^{\mathrm{P}}_{m-1,\mathfrak{f}(m-1,k)} (\mathfrak{M}', w'_1)$. The operator $\blacklozenge$ does not necessarily preserve the children of w_1 and w'_1, so that the induction hypothesis, naturally defined from the statement of Lemma 5.1, is applied on models where the condition $R_1(w_1) = R(w_1)$ may not hold. We show that for all $R_1 \subseteq R$, it is possible to construct $R'_1 \subseteq R'$ such that, for all $w_1 \leftrightarrow w'_1$, $((W, R_1, V), w_1) \equiv^{\mathrm{P}}_{m-1,k} ((W', R'_1, V'), w'_1)$. The result is then lifted to $((W, R_1, V), w) \equiv^{\mathrm{P}}_{m,k} ((W', R'_1, V'), w')$ in Lemma 5.2, again thanks to the properties of the g-bisimulation. The proof of this lemma is in Appendix K.

Lemma 5.2. *For every* $\varphi \in \mathsf{GML}[m, k, \mathrm{P}]$ *there is* $\psi \in \mathsf{GML}[m, \mathfrak{f}(m, k), \mathrm{P}]$ *such that* $\blacklozenge\varphi \equiv \psi$.

Game on $[(\mathfrak{M}_1{=}(W_1, R_1, V_1), w_1), (\mathfrak{M}_2{=}(W_2, R_2, V_2), w_2), (m, s, \mathrm{P})]$.

if there is $p \in \mathrm{P}$ such that $w_1 \in V_1(p)$ iff $w_2 \notin V_2(p)$ then the spoiler wins.
else the spoiler chooses $i \in \{1, 2\}$ and plays on $\mathfrak{M}_i$. The duplicator replies on $\mathfrak{M}_j$ where $j \neq i$. The spoiler must choose one of the following moves, otherwise the duplicator wins:
modal move: if $m \geq 1$ and $R_i(w_i) \neq \emptyset$ then the spoiler **can** choose to play a modal move by selecting an element $w_i' \in R_i(w_i)$. Then,

- the duplicator must reply with a $w_j' \in R_j(w_j)$ (else, the spoiler wins);
- the game continues on $[(\mathfrak{M}_1, w_1'), (\mathfrak{M}_2, w_2'), (m-1, s, \mathrm{P})]$.

spatial move: if $s \geq 1$ then the spoiler **can** choose to play a spatial move by selecting two finite forests $\mathfrak{M}_i^1$ and $\mathfrak{M}_i^2$ such that $\mathfrak{M}_i^1 + \mathfrak{M}_i^2 = \mathfrak{M}_i$. Then,

- the duplicator replies with two finite forests $\mathfrak{M}_j^1$ and $\mathfrak{M}_j^2$ such that $\mathfrak{M}_j^1 + \mathfrak{M}_j^2 = \mathfrak{M}_j$;
- The game continues on $[(\mathfrak{M}_1^k, w_1), (\mathfrak{M}_2^k, w_2), (m, s-1, \mathrm{P})]$, where $k \in \{1, 2\}$ is chosen by the spoiler.

Fig. 5. Ehrenfeucht-Fraïssé games for $\mathsf{ML}(*)$

Hence, Lemma 5.2 together with Lemma 2.2 and Theorem 3.7 entail $\mathsf{ML}(*) \preceq \mathsf{GML}$.

Lemma 5.3. $\mathsf{ML}(*) \preceq \mathsf{GML}$.

Proof. Let φ be in $\mathsf{ML}(*)$. As $\Diamond\psi \equiv \Diamond_{\geq 1}\,\psi$, we can replace every occurrence of the modality $\Diamond$ appearing in φ with the modality $\Diamond_{\geq 1}$. Moreover, by Lemma 2.2, we can replace every subformula of the form $\psi * \chi$ with the formula $\blacklozenge(\psi \,|\, \chi)$. In this way, we obtain a formula φ' that is equivalent to φ and where all the modalities are of the form $\Diamond_{\geq 1}$, $|$ and $\blacklozenge$. If φ' has no occurrence of $|$ or $\blacklozenge$, we are done. Otherwise, let ψ be a subformula of φ' of the form $\blacklozenge(\varphi_1 \,|\, \varphi_2)$ where φ_1 and φ_2 are in GML.

- By Theorem 3.7, there is a formula ψ' in GML such that $\psi' \equiv \varphi_1 \,|\, \varphi_2$.
- By Lemma 5.2, there is a formula ψ'' in GML such that $\psi'' \equiv \blacklozenge\psi'$.

We have $\varphi' \equiv \varphi'[\psi \leftarrow \psi'']$, where $\varphi'[\psi \leftarrow \psi'']$ is obtained from φ' by replacing every occurrence of ψ by ψ''. Note that the number of occurrences of $\blacklozenge$ and $|$ in $\varphi'[\psi \leftarrow \psi'']$ is strictly less than the number of occurrences of $\blacklozenge$ and $|$ in φ'. By repeating such a type of replacement, we eventually obtain a formula φ'' in GML such that $\varphi' \equiv \varphi''$. Indeed, all the occurrences of $\blacklozenge$ and $|$ only appear as instances of the pattern $\blacklozenge(\psi \,|\, \chi)$. Hence, we get a formula in GML logically equivalent to φ. ◻

5.2 Showing $\mathsf{ML}(*) \prec \mathsf{GML}$ with EF games for $\mathsf{ML}(*)$

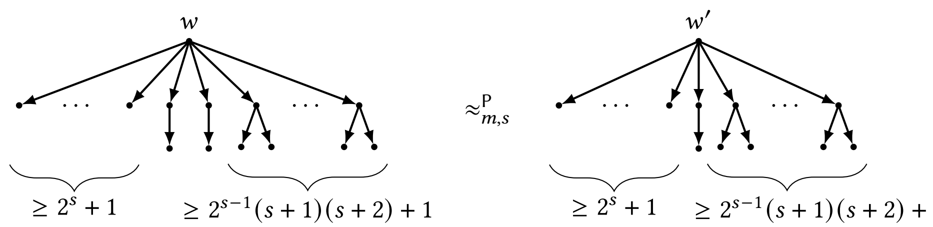
We tackle the problem of showing that $\mathsf{ML}(*)$ is strictly less expressive than GML. To do so, we adapt the notion of Ehrenfeucht-Fraïssé games (EF games, in short) [35] to $\mathsf{ML}(*)$, which gives us the corresponding structural equivalence between models that are logically indistinguishable. With this definition at hand, we design a GML formula that is not expressible in $\mathsf{ML}(*)$: we will find two models that are indistinguishable for $\mathsf{ML}(*)$ but distinguishable for GML . We write $\mathsf{ML}(*)[m, s, \mathrm{P}]$ for the set of formulae φ of $\mathsf{ML}(*)$ having $\mathrm{md}(\varphi) \leq m$, at most s nested $*$, and atomic propositions from $\mathrm{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$. It is easy to see that $\mathsf{ML}(*)[m, s, \mathrm{P}]$ is finite up to logical equivalence.

We introduce the EF games for $\mathsf{ML}(*)$. A game is played between two players: the *spoiler* and the *duplicator*. A game state is a triple made of two pointed forests $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ and a rank (m, s, P), where $m, s \in \mathbb{N}$ and $\mathrm{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$. The goal of the spoiler is to show that the two models are different. The goal of the duplicator is to counter the spoiler and to show that the two models are similar. Two models are different whenever there is $\varphi \in \mathsf{ML}(*)[m, s, \mathrm{P}]$ that is satisfied by only one of the two models. The EF games for $\mathsf{ML}(*)$ are formally defined in Figure 5. The exact correspondence between the game and the logic is formalised in Lemma 5.4.

Using the standard definitions in [35], the duplicator has a *winning strategy* for the game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathrm{P}))$ if she can play in a way that guarantees her to win regardless of how the spoiler plays. When this is the case, we write $(\mathfrak{M}, w) \approx_{m,s}^{\mathrm{P}} (\mathfrak{M}', w')$. Similarly, the spoiler has a *winning strategy*, written $(\mathfrak{M}, w) \not\approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$, if he can play in a way that guarantees him to win, regardless of how the duplicator plays. Lemma 5.4 guarantees that the games are well-defined.

Lemma 5.4. $(\mathfrak{M}, w) \not\approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$ *iff there is* $\varphi \in \mathsf{ML}(*)[m, s, \mathsf{P}]$ *s.t.* $\mathfrak{M}, w \models \varphi$ *and* $\mathfrak{M}', w' \not\models \varphi$.

Lemma 5.4 is proven with standard arguments from [35] (see the details in [9, Page 46]). For instance the left-to-right direction, i.e. the *completeness of the game*, is by induction on the rank (m, s, P). Thanks to the EF games, we characterise a notion of model equivalence for $\mathsf{ML}(*)$. Then, by designing a formula φ that distinguishes two $\mathsf{ML}(*)$ equivalent models, we are able to find a GML formula that is not expressible in $\mathsf{ML}(*)$. By Lemma 2.1 and as $\mathsf{ML}(\,|\,) \approx \mathsf{GML}$, such a formula is necessary of modal degree at least 2. Happily, $\varphi = \Diamond_{=2} \Diamond_{=1} \top$ does the job and cannot be expressed in $\mathsf{ML}(*)$. For the proof, we show that for every rank (m, s, P), there are two structures $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ such that $(\mathfrak{M}, w) \approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$, $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}', w' \not\models \varphi$. The inexpressibility of φ then stems from Lemma 5.4. The two structures are represented below ($(\mathfrak{M}, w)$ on the left).



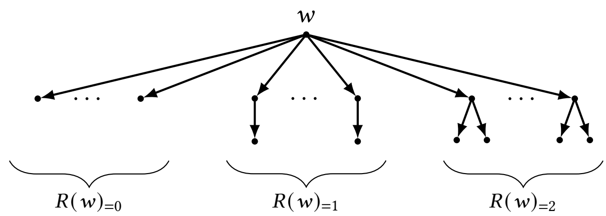
In the following, we say that a world has *type* i if it has i children. As one can see in the figure above, children of the current worlds w and w' are of three types: 0, 1 or 2. When the spoiler performs a spatial move in the game, a world of type i can take, in the submodels, a type between 0 and i. That is, the number of children of a world weakly monotonically decreases when taking submodels. This monotonicity, together with the finiteness of the game, lead to bounds on the number of children of each type, over which the duplicator is guaranteed to win. For instance, the bound for worlds of type 2 is given by the value $2^s(s+1)(s+2)$, where s is the number of spatial moves in the game. In the two presented pointed forests, one child of type 0 and one of type 2 are added with respect to these bounds, so that the duplicator can make up for the different numbers of children of type 1.

Lemma 5.5. $\mathsf{ML}(*)$ *cannot characterise the class of pointed models satisfying* $\Diamond_{=2} \Diamond_{=1} \top$.

Proof. (sketch) As usual, the non-expressivity of $\Diamond_{=2} \Diamond_{=1} \top$ is shown by proving that for every rank (m, s, P) there are two structures $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ such that $(\mathfrak{M}, w) \approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$, and $\mathfrak{M}, w \models \Diamond_{=2} \Diamond_{=1} \top$ whereas $\mathfrak{M}', w' \not\models \Diamond_{=2} \Diamond_{=1} \top$. The proof follows by establishing two properties of $\approx^{\mathsf{P}}_{m,s}$, named below (A) and (B). We start with some preliminary definitions. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$. We denote with $R(w)_{=i}$ the set of worlds in $R(w)$ having type i, i.e. $\{w_1 \in R(w) \mid |R(w_1)| = i\}$. During the proof, we only use pointed forests $(\mathfrak{M}, w)$ satisfying the following properties:

I $V(p) = \emptyset$ for every $p \in \mathrm{AP}$;
II $R(w)_{=0}$, $R(w)_{=1}$ and $R(w)_{=2}$ form a partition of $R(w)$;
III $R^3(w) = \emptyset$, i.e. the set of worlds reachable from w in at least three steps is empty.

Below, we represent schematically the models satisfying the properties I, II and III.



The first property of $\approx^{\mathsf{P}}_{m,s}$ is presented below (see its proof in Appendix L).

Property (A). *Consider a rank* (m, s, P) *and let* $(\mathfrak{M} = (W, R, V), w)$ *and* $(\mathfrak{M}' = (W', R', V'), w')$ *be two pointed forests satisfying I, II and III and such that*

- $\min(|R(w)_{=0}|, 2^s) = \min(|R'(w')_{=0}|, 2^s)$;
- $\min(|R(w)_{=1}|, 2^s(s+1)) = \min(|R'(w')_{=1}|, 2^s(s+1))$; *and*
- $\min(|R(w)_{=2}|, 2^{s-1}(s+1)(s+2)) = \min(|R'(w')_{=2}|, 2^{s-1}(s+1)(s+2))$.

Then, $(\mathfrak{M}, w) \approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$.

As worlds in our models do not satisfy any propositional symbol, the spoiler cannot win because of distinct propositional valuations. The proof is by cases on m and on the moves done by the spoiler, and by induction on s. The only significant case to be dealt with corresponds to the case $s \geq 1$ and the spoiler decides to perform a spatial move.

By relying on (A), the second property (B) can be established (see its proof in Appendix M).

Property (B). *Consider a rank* (m, s, P) *and let* $(\mathfrak{M} = (W, R, V), w)$ *and* $(\mathfrak{M}' = (W', R', V'), w')$ *be two pointed forests satisfying I, II and III and such that*

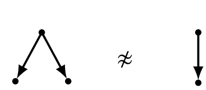
- $|R(w)_{=0}| \geq 2^s + 1$ *and* $|R'(w')_{=0}| \geq 2^s + 1$;
- $|R(w)_{=1}| = 2$ *and* $|R'(w')_{=1}| = 1$; *and*
- $|R(w)_{=2}| \geq 2^{s-1}(s+1)(s+2) + 1$ *and* $|R'(w')_{=2}| \geq 2^{s-1}(s+1)(s+2) + 1$.

Then, $(\mathfrak{M}, w) \approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$.

Obviously, (A) and (B) are quite close. The first condition of (B) satisfies the first condition of (A). Similarly, the third condition of (B) satisfies the third condition of (A). However, the second condition of (B) does not satisfy the second condition of (A) and this is the crucial difference.

It is also worth noticing that (B) implies the statement of the lemma, as $\mathfrak{M}, w \models \Diamond_{=2} \Diamond_{=1} \top$ whereas $\mathfrak{M}', w' \not\models \Diamond_{=2} \Diamond_{=1} \top$. Indeed, ad absurdum suppose that such an $\mathsf{ML}(*)$ formula φ exists. Let m be its modal degree, s be its maximal number of imbricated $*$ and P be the set of propositional variables occurring in φ. Let us consider two pointed forests $(\mathfrak{M}_1, w_1)$ and $(\mathfrak{M}_2, w_2)$ such that $\mathfrak{M}_1, w_1 \models \Diamond_{=2} \Diamond_{=1} \top$, $\mathfrak{M}_2, w_2 \not\models \Diamond_{=2} \Diamond_{=1} \top$ and satisfying the conditions in (B). This would lead to a contradiction, as $(\mathfrak{M}_1, w_1)$ and $(\mathfrak{M}_2, w_2)$ are supposed to satisfy φ (or not) equivalently. □

We conclude by noticing that $\mathsf{ML}(*)$ is more expressive than ML. Indeed, the formula $\Diamond\top * \Diamond\top$ distinguishes the following two models, which are bisimilar (as the valuations at every world are empty) and hence indistinguishable in ML [53]:



Theorem 5.6. $\mathsf{ML} \prec \mathsf{ML}(*) \prec \mathsf{GML} \approx \mathsf{ML}(|)$.

Proof. By $\mathsf{ML}(*) \preceq \mathsf{GML}$, Lemma 5.5 and Theorem 3.7. □

Trees

$$T := \mathtt{0} \mid \mathsf{n}[T] \mid T \,|\, T$$

Semantics

$T \models \top$		always holds
$T \models \mathtt{0}$	iff	$T \equiv \mathtt{0}$
$T \models \mathsf{n}[\varphi]$	iff	$\exists T'$ s.t. $T \equiv \mathsf{n}[T']$ and $T' \models \varphi$
$T \models \varphi \,|\, \psi$	iff	$\exists T_1, T_2$ s.t. $T \equiv T_1 \,|\, T_2, T_1 \models \varphi$ and $T_2 \models \psi$

Structural equivalence

- $T \,|\, \mathtt{0} \equiv T$
- $T_1 \equiv T_2 \Rightarrow T_2 \equiv T_1$
- $T_1 \equiv T_2, T_2 \equiv T_3 \;\Rightarrow\; T_1 \equiv T_3$
- $T_1 \,|\, T_2 \equiv T_2 \,|\, T_1$
- $(T_1 \,|\, T_2) \,|\, T_3 \equiv T_1 \,|\, (T_2 \,|\, T_3)$
- $T_1 \equiv T_2 \;\Rightarrow\; T_1 \,|\, T \equiv T_2 \,|\, T$
- $T_1 \equiv T_2 \;\Rightarrow\; \mathsf{n}[T_1] \equiv \mathsf{n}[T_2]$

Fig. 6. Interpretation and semantics of SAL($|$).

6 ML($|$) AND STATIC AMBIENT LOGIC

Static ambient logic (SAL) is a formalism proposed to reason about spatial properties of concurrent processes specified in the ambient calculus [17]. In [14], the satisfiability and validity problems for a very expressive fragment of SAL are shown to be decidable and conjectured to be in PSpace (see [14, Section 6]). We invalidate this conjecture (under standard complexity-theoretic assumptions) by showing that the intensional fragment of SAL (see [36]), herein denoted SAL($|$), is already AExp$_{\mathrm{Pol}}$-complete. More precisely, we design semantically faithful reductions between Sat(ML($|$)) and Sat(SAL($|$)) (in both directions), leading to the above-mentioned result by Theorem 3.12. In [8], these results are shown with respect to Kripke-like structures that can be shown isomorphic to the syntactical trees historically used in ambient calculus. Here, we provide the reductions directly on these syntactical trees. Let us start by introducing SAL($|$). This correspondence between SAL($|$) and ML($|$) is rather intuitive but a presentation of the complete formal developments could be too long to be included herein due to space restrictions. However, the proofs can be found in the preliminary report [9] (the complete version of [8] with its proofs) and in Mansutti's PhD thesis [40].

Let Σ be a countably infinite set of *ambient names*. The formulae of SAL($|$) are built from:

$$\varphi := \top \mid \mathtt{0} \mid \mathsf{n}[\varphi] \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi \,|\, \varphi,$$

where $\mathsf{n} \in \Sigma$. SAL($|$) is interpreted on edge-labelled finite trees: syntactical objects equipped with a structural equivalence relation $\equiv$. We denote with $\mathbb{T}_{\mathsf{SAL}}$ the set of these finite trees. The grammar used to construct these structures, their structural equivalence as well as the satisfaction relation $\models$ for SAL($|$) are provided in Figure 6 (the cases for $\wedge$ and $\neg$ being omitted). We will also use $\sum_{i \in I} T_i$, for a given set of indices $I = \{i_1, \ldots, i_m\}$, as an abbreviation of $T_{i_1} \,|\, T_{i_2} \,|\, \ldots \,|\, T_{i_m}$.

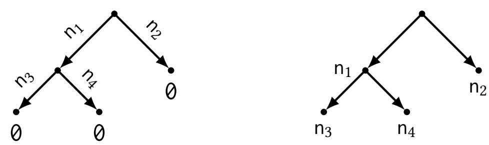
Obviously SAL($|$) and ML($|$) are strongly related, but how close? For example, $\mathsf{n}[\varphi] \,|\, \top$ can be seen as a relativised version of $\Diamond$ of the form $\Diamond(\mathsf{n} \wedge \varphi)$. To formalise this intuition, we borrow the syntax from Hennesy-Milner logic (HML) [31] and define the formula $\langle \mathsf{n} \rangle \varphi \stackrel{\text{def}}{=} \mathsf{n}[\varphi] \,|\, \top$ and its dual $[\mathsf{n}]\varphi \stackrel{\text{def}}{=} \neg\langle \mathsf{n} \rangle \neg\varphi$. Below, w.l.o.g. we assume $\Sigma = \mathrm{AP}$ (for the sake of clarity).

6.1 From Sat(SAL($|$)) to Sat(ML($|$)).

The reduction from Sat(SAL($|$)) to Sat(ML($|$)) is quite simple as SAL($|$) is essentially interpreted on finite trees where each world satisfies a single propositional variable (its ambient name). Let $T \in \mathbb{T}_{\mathsf{SAL}}$ be a tree built with ambient names from $\mathrm{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$, $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$. We say that $(\mathfrak{M}, w)$ *encodes* T if and only if:

(1) every $w' \in R^*(w)$ satisfies at most one symbol in P;
(2) there is $\mathfrak{f} : W \rightarrow \mathbb{T}_{\mathsf{SAL}}$ such that $\mathfrak{f}(w) \equiv T$ and for all $w' \in R^*(w)$, we have $\mathfrak{f}(w') \equiv \sum_{i=1}^{K} \mathsf{n}_i[\mathfrak{f}(w_i)]$ where $\{w_1, \ldots, w_K\} = R(w')$ and $w_i \in V(\mathsf{n}_i)$ for all $1 \leq i \leq K$.

It is easy to verify that every tree in $\mathbb{T}_{\mathsf{SAL}}$ has an encoding. The figure just below depicts a tree T (on the left) and one of its possible encodings as a finite forest (on the right).



Given a formula φ of $\mathsf{SAL}(|)$, we define its translation $\tau(\varphi)$ in $\mathsf{ML}(|)$. The translation τ is homomorphic for Boolean connectives and $\top$, and otherwise it is inductively defined as follows:

$\tau(\mathtt{0}) \stackrel{\text{def}}{=} \Box\bot;$ $\quad\tau(\varphi|\psi) \stackrel{\text{def}}{=} \tau(\varphi)|\tau(\psi);$ $\quad\tau(\mathsf{n}[\varphi]) \stackrel{\text{def}}{=} \Diamond(\mathsf{n} \land \tau(\varphi)) \land \neg(\Diamond\top|\Diamond\top).$

The following lemma states that the translation is correct.

Lemma 6.1. *If $(\mathfrak{M}, w)$ encodes $T \in \mathbb{T}_{\mathsf{SAL}}$ then for every φ in $\mathsf{SAL}(|)$ we have $T \models \varphi$ iff $\mathfrak{M}, w \models \tau(\varphi)$.*

The proof can be achieved with an easy structural induction and therefore we omit it herein. So, we can complete the reduction.

Theorem 6.2. *Let φ be in $\mathsf{SAL}(|)$ built over $\mathsf{P} \subseteq_{\text{fin}} \mathrm{AP}$ and $p \notin \mathsf{P}$. φ is satisfiable if and only if $\tau(\varphi) \land \bigwedge_{i\in[1,\mathsf{size}(\varphi)]} \Box^i \bigvee_{\mathsf{n}\in\mathsf{P}\cup\{p\}} \big(\mathsf{n} \land \bigwedge_{\mathsf{m}\in(\mathsf{P}\cup\{p\})\setminus\{\mathsf{n}\}} \neg\mathsf{m}\big)$ is satisfiable.*

Proof. Suppose φ satisfiable. Then, there is T such that $T \models \varphi$. In general, it could be that T contains ambient names that do not appear in φ. However, we can assume that there is only one name in T that does not appear in φ and that name is p (as in the statement of this theorem). Indeed, this assumption relies on the following property of static ambient logic.

Lemma 6.3 ([14], Lemma 8). *Let p and q be two ambient names not appearing in φ. Then, $T \models \varphi$ iff $T[p \leftarrow q] \models \varphi$, where $T[p \leftarrow q]$ is the tree obtained from T by replacing every occurrence of p with q.*

Let $(\mathfrak{M}, w)$ be a pointed forest, where $\mathfrak{M} = (W, R, V)$, encoding of T (it always exists). From Lemma 6.1, $\mathfrak{M}, w \models \tau(\varphi)$. Let us recall the properties of the encoding of T by a model $(\mathfrak{M}, w)$:

(1) every world in W satisfies at most one propositional symbol in P;
(2) there is a function $\mathfrak{f}$ from W to $\mathbb{T}_{\mathsf{SAL}}$ such that $\mathfrak{f}(w) \equiv T$ and for every $w' \in R^*(w)$, we have $\mathfrak{f}(w') \equiv \sum_{i\in[1,K]} \mathsf{n}_i[\mathfrak{f}(w_i)]$ where $\{w_1, \dots, w_K\} = R(w')$ and for all $i \in [1, K]$, $w_i \in V(\mathsf{n}_i)$.

The first property together with the last part of the second property imply that every world reachable in at least one step from w satisfies exactly one propositional symbol of P. Then,

$$\mathfrak{M}, w \models \bigwedge_{i=1}^{\mathsf{size}(\varphi)} \Box^i \bigvee_{\mathsf{n}\in\mathsf{P}\cup\{p\}} \big(\mathsf{n} \land \bigwedge_{\mathsf{m}\in(\mathsf{P}\cup\{p\})\setminus\{\mathsf{n}\}} \neg\mathsf{m}\big).$$

Conversely, suppose $\psi = \tau(\varphi) \land \bigwedge_{i=1}^{\mathsf{size}(\varphi)} \Box^i \bigvee_{\mathsf{n}\in\mathsf{P}\cup\{p\}} \big(\mathsf{n} \land \bigwedge_{\mathsf{m}\in(\mathsf{P}\cup\{p\})\setminus\{\mathsf{n}\}} \neg\mathsf{m}\big)$ satisfiable. To prove the result it is sufficient to show that there is a pair $(\mathfrak{M}, w)$ encoding a tree T that satisfies ψ. Indeed, if this is the case then by $\mathfrak{M}, w \models \tau(\varphi)$ we obtain $T \models \varphi$ by Lemma 6.1. As ψ is satisfiable, we know that there is a forest $\mathfrak{M} = (W, R, V)$ and a world $w \in W$ such that $\mathfrak{M}, w \models \psi$. It is important to notice that, as in Theorem 6.5, we can get rid of all the parts beyond $\mathsf{md}(\varphi)$, so we can ensure that as $\mathfrak{M}, w \models \psi$, then it is an encoding of some T, and therefore, $T \models \varphi$. ☐

6.2 From Sat(ML(|)) to Sat(SAL(|)).

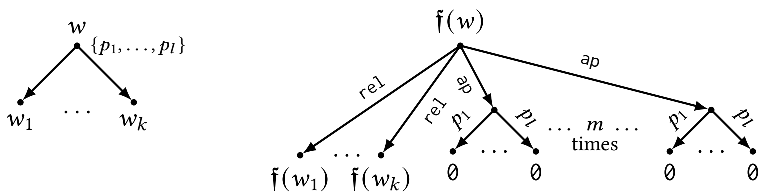
One of the main challenges in order to obtain a polynomial-time reduction from Sat(ML(|)) to Sat(SAL(|)), is to understand how to encode a finite set of propositional symbols. This problem arises since Kripke-style finite forests can satisfy multiple atomic propositions at each world, whereas each ambient of an information tree only satisfies exactly one atomic proposition: its ambient name. To solve this, it is crucial to deal with two issues: we need to avoid an exponential blow up in the representation, and we have to maintain information about the children of a node.

We solve both issues by representing a propositional symbol p as a particular ambient, and copying enough times the ambient encoding p. Let $\mathsf{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$ and $n \in \mathbb{N}^{>0}$, where $\mathbb{N}^{>0}$ denotes the set of positive natural numbers. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$. Let $\mathtt{rel}$ and $\mathtt{ap}$ be two ambient names not in P. The ambient name $\mathtt{rel}$ encodes the relation R whereas $\mathtt{ap}$ can be seen as a *container* for propositional variables holding on the current world. We say that $T \in \mathbb{T}_{\mathsf{SAL}}$ is an *encoding* of $(\mathfrak{M}, w)$ with respect to P and n if and only if

(1) every ambient name in T is from $\mathsf{P} \cup \{\mathtt{rel}, \mathtt{ap}\}$;
(2) there is a function $\mathfrak{f}$ from W to $\mathbb{T}_{\mathsf{SAL}}$ s.t. $\mathfrak{f}(w) \equiv T$ and for each $w' \in R^*(w)$ there is $m \geq n$ s.t.

$$\mathfrak{f}(w') \equiv \Big(\sum_{i=1}^{m} \mathtt{ap}[\sum_{\substack{p \in \mathsf{P} \\ w' \in V(p)}} p[\mathtt{0}]] \Big) \mid \sum_{w'' \in R(w')} \mathtt{rel}[\mathfrak{f}(w'')].$$

The figure below shows on the right a possible encoding of the model on the left.



It is easy to verify that $(\mathfrak{M}, w)$ always admits such an encoding. We define the translation of φ, written $\tau(\varphi)$, into $\mathsf{SAL}(\mid)$. It is homomorphic for Boolean connectives and $\top$, $\tau(p) \stackrel{\text{def}}{=} \langle \mathtt{ap} \rangle \langle \mathrm{p} \rangle \top$ and otherwise it is inductively defined (using the notation from HML):

$$\tau(\Diamond\varphi) \stackrel{\text{def}}{=} \langle \mathtt{rel} \rangle \tau(\varphi); \qquad \tau(\varphi \mid \psi) \stackrel{\text{def}}{=} \big(\tau(\varphi) \wedge \langle \mathtt{ap} \rangle_{\geq \mathrm{size}(\varphi)} \top\big) \mid \big(\tau(\psi) \wedge \langle \mathtt{ap} \rangle_{\geq \mathrm{size}(\psi)} \top\big),$$

where $\langle \mathrm{n} \rangle_{\geq k} \varphi$ is the graded modality defined as $\top$ for $k = 0$, otherwise $(\langle \mathrm{n} \rangle \varphi) \mid \langle \mathrm{n} \rangle_{\geq k-1} \varphi$. In the translation of $\mid$, the model of $\mathsf{SAL}(\mid)$ has to be split in such a way that both subtrees contain enough $\mathtt{ap}$ ambients to correctly answer to the formula $\langle \mathtt{ap} \rangle \langle \mathrm{p} \rangle \top$. It is easy to see that the size of $\tau(\varphi)$ is quadratic in $\mathrm{size}(\varphi)$.

Lemma 6.4. *Let $\mathfrak{M}$ be a finite forest and w be one of its worlds. Let $\mathsf{P} \subseteq_{\mathrm{fin}} \mathrm{AP}$ and $n \in \mathbb{N}^{>0}$. Let T be an encoding of $(\mathfrak{M}, w)$ w.r.t P and n. For every formula φ built over P with $\mathrm{size}(\varphi) \leq n$, we have $\mathfrak{M}, w \models \varphi$ if and only if $T \models \tau(\varphi)$.*

The proof is by structural induction on φ and it is quite straightforward. Then, with this result at hand, we can state the intended result.

Theorem 6.5. *Let φ be in $\mathsf{ML}(\mid)$ built over P. Then φ is satisfiable iff ψ below is satisfiable:*

$$\psi \stackrel{\text{def}}{=} \tau(\varphi) \wedge \bigwedge_{i=0}^{\mathrm{size}(\varphi)} [\mathtt{rel}]^i \Big(\langle \mathtt{ap} \rangle_{\geq \mathrm{size}(\varphi)} \top \wedge \bigwedge_{p \in \mathsf{P}} \big(\langle \mathtt{ap} \rangle \langle \mathrm{p} \rangle \top \Rightarrow [\mathtt{ap}] \langle \mathrm{p} \rangle \top \big) \wedge [\mathtt{ap}] \sum_{p \in \mathsf{P}} (p[\mathtt{0}] \vee \mathtt{0}) \Big).$$

As a corollary of the reductions we provided in this section, and appealing to Theorem 3.12, we can establish the following complexity results.

Corollary 6.6. $\mathrm{Sat}(\mathsf{SAL}(\mid))$ *is AExp$_{\mathrm{POL}}$-complete.* $\mathrm{Sat}(\mathsf{SAL})$ *with* SAL *from [14] is AExp$_{\mathrm{POL}}$-hard.*

7 ML($*$) AND MODAL SEPARATION LOGIC

The family of *modal separation logics* (MSL), combining separating and modal connectives, has been recently introduced in [23]. Its models, inspired from the memory states used in separation logic (see also [19]), are Kripke-style structures $\mathfrak{M} = (W, R, V)$, where $W = \mathbb{N}$ and $R \subseteq W \times W$ is finite and functional. Hence, unlike finite forests, $\mathfrak{M}$ may have loops.

Among the fragments studied in [23], the modal separation logic $\mathsf{MSL}(*, \Diamond^{-1})$ was left with a huge complexity gap: between PSpace-hardness and a Tower upper bound. We fill this gap, by showing that the logic is Tower-hard, by reducing Sat($\mathsf{ML}(*)$) to Sat($\mathsf{MSL}(*, \Diamond^{-1})$). Full details of the reduction can be found in [40, Section 9.4.2].

Formulae of $\mathsf{MSL}(*, \Diamond^{-1})$ are defined from

$$\varphi := p \ \mid \ \Diamond^{-1}\varphi \ \mid \ \varphi \wedge \varphi \ \mid \ \neg\varphi \ \mid \ \varphi * \varphi \,.$$

The satisfaction relation is as in $\mathsf{ML}(*)$ for $p \in \mathrm{AP}$, Boolean connectives and $\varphi_1 * \varphi_2$, otherwise

$$\mathfrak{M}, w \models \Diamond^{-1}\varphi \Leftrightarrow \exists w' \text{ s.t. } (w', w) \in R \text{ and } \mathfrak{M}, w' \models \varphi.$$

Since $\mathsf{MSL}(*, \Diamond^{-1})$ is interpreted over a finite and functional relation, $\Diamond^{-1}$ effectively works as the $\Diamond$ modality of $\mathsf{ML}(*)$. Then, assume we want to check the satisfiability of φ in $\mathsf{ML}(*)$ by relying on an algorithm for Sat($\mathsf{MSL}(*, \Diamond^{-1})$). We simply need to consider the formula $\varphi[\Diamond \leftarrow \Diamond^{-1}]$ obtained from φ by replacing every occurrence of $\Diamond$ by $\Diamond^{-1}$, and check if it can be satisfied by a *locally acyclic* model $(\mathfrak{M}, w)$ of MSL, i.e. one where w does not belong to a loop of length $\leq \mathsf{md}(\varphi)$. Notice that given a finite forest (W, R, V), the structure (W, R^{-1}, V) is locally acyclic. The next lemma establishes the correspondence between the satisfaction of a formula in a model, in the two logics.

Lemma 7.1. *Let φ in $\mathsf{ML}(*)$. Let (W, R, V) be a finite forest and $w \in W$. Then, $(W, R, V), w \models \varphi$ in $\mathsf{ML}(*)$ if and only if $(W, R^{-1}, V), w \models \varphi[\Diamond \leftarrow \Diamond^{-1}]$ in $\mathsf{MSL}(*, \Diamond^{-1})$.*

Proof. The result is proven with a rather straightforward structural induction on φ. □

In order to provide a complete reduction from Sat($\mathsf{ML}(*)$) to Sat($\mathsf{MSL}(*, \Diamond^{-1})$), we need to make sure that the formulae are being checked against the appropriate class of models. Notice that in $\mathsf{ML}(*)$, only the worlds that are reachable from the current one in at most $\mathsf{md}(\varphi)$ steps are relevant for the satisfiability of φ (see Lemma A.1 in Appendix A). Thus, for a given formula φ, we can restrict ourselves to the class of MSL models in which the current point of evaluation is not reachable by any world in more than $\mathsf{md}(\varphi) + 1$ steps. The formula doing the job is $(\Box^{-1})^{\mathsf{md}(\varphi)}\bot$, where $\Box^{-1}\varphi \stackrel{\text{def}}{=} \neg\Diamond^{-1}\neg\varphi$, and $(\Box^{-1})^n\varphi$ with $n \in \mathbb{N}$ is defined as expected. Then, we can conclude:

Lemma 7.2. *Let φ in $\mathsf{ML}(*)$, φ is satisfiable in $\mathsf{ML}(*)$ if and only if $\varphi[\Diamond \leftarrow \Diamond^{-1}] \wedge (\Box^{-1})^{\mathsf{md}(\varphi)}\bot$ is satisfiable in $\mathsf{MSL}(*, \Diamond^{-1})$.*

Proof. The proof is rather straightforward, relying on Lemma 7.1. □

Hence, the results in Section 4 allow us to close the complexity gap from [23].

Corollary 7.3. Sat($\mathsf{MSL}(*, \Diamond^{-1})$) *is Tower-complete.*

8 CONCLUSION

We have studied and compared the logics $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$, two modal logics interpreted on finite forests and featuring composition operators. We have not only characterised the expressive power and the complexity for both logics, but also identified remarkable differences and export our results to other logics. $\mathsf{ML}(|)$ is shown as expressive as GML, and its satisfiability problem is found to be $\mathrm{AExp}_{\mathrm{Pol}}$-complete. Besides the obvious similarities between $\mathsf{ML}(|)$ and $\mathsf{ML}(*)$, these results are counter-intuitive: though the logic $\mathsf{ML}(*)$ is strictly less expressive than GML (and consequently, than $\mathsf{ML}(|)$), Sat($\mathsf{ML}(*)$) is Tower-complete. Our proof techniques go beyond what is known in the literature. For instance, to design the Tower-hardness proof we needed substantial modifications from the proof introduced in [7] for QK^t. On the other hand, to show the expressivity inclusion of $\mathsf{ML}(*)$ within GML, we provided a novel definition of Ehrenfeucht-Fraïssé games for $\mathsf{ML}(*)$.

Lastly, our framework led to the characterisation of the satisfiability problems for two sister logics . We proved that the satisfiability problem for the modal separation logic $\mathsf{MSL}(*, \Diamond^{-1})$ is Tower-complete [23]. Moreover, the satisfiability problem for the static ambient logic $\mathsf{SAL}(\,|\,)$ is AExp_{Pol}-complete, solving open problems from [14, 23] and paving the way to study the complexity of the full SAL.

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ELECTRONIC APPENDIX (ON COMPOSING FINITE FORESTS WITH MODAL LOGICS)

Bartosz Bednarczyk, TU Dresden, Germany & University of Wrocław, Poland
Stéphane Demri, Université Paris-Saclay, ENS Paris-Saclay, CNRS, LMF, 91190, Gif-sur-Yvette, France
Raul Fervari, FAMAF, Universidad Nacional de Córdoba & CONICET, Argentina, & GTIIT, China
Alessio Mansutti, University of Oxford, UK & IMDEA Software Institute, Spain

A PROOF OF LEMMA 2.1

Proof. We start the proof by stating a classical property of ML and GML which carries over to $\mathsf{ML}(*)$ and $\mathsf{ML}(\mathbin{|})$. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$. We introduce the notation

$$R|_w^{\leq n} \stackrel{\text{def}}{=} \{(w', w'') \in R \mid w' \in R^i(w) \text{ for some } i \in [0, n-1]\}.$$

Informally, $R|_w^{\leq n}$ is the maximal subset of R encoding exactly a subtree rooted at w having only paths of length at most n. We denote with $R|_w$ the set $\{(w', w'') \in R \mid w' \subseteq R^*(w)\}$, i.e. the maximal subset of R encoding exactly a subtree rooted at w. Alternatively, $R|_w = \bigcup_{n \in \mathbb{N}} R|_w^{\leq n}$.

Lemma A.1. *Let $n \in \mathbb{N}$ and φ be a formula of $\mathsf{ML}(\mathbin{|})$ or $\mathsf{ML}(*)$ such that $\mathsf{md}(\varphi) \leq n$. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$. $\mathfrak{M}, w \models \varphi$ if and only if $(W, R|_w^{\leq n}, V), w \models \varphi$.*

The proof is by structural induction on φ. Details are omitted as this poses no difficulty.

Now, let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$. Notice that if $\mathsf{md}(\varphi)$ is at most 1, by Lemma A.1 the satisfaction of φ only depends on the set of worlds $\{w\} \cup R(w)$. More precisely, $\mathfrak{M}, w \models \varphi$ iff $(W, R|_w^{\leq 1}, V), w \models \varphi$. The same holds for formulae in $\mathsf{ML}(*)$. Similarly, $\psi \stackrel{\text{def}}{=} \varphi[\mathbin{|} \leftarrow *]$ (as in the statement) has modal degree at most 1 and again by Lemma A.1 we have $\mathfrak{M}, w \models \psi$ iff $(W, R|_w^{\leq 1}, V), w \models \psi$. To conclude the proof it is sufficient then to prove the following:

$$(W, R|_w^{\leq 1}, V), w \models \varphi \quad \text{if and only if} \quad (W, R|_w^{\leq 1}, V), w \models \psi.$$

Notice that this result already trivially holds for $\mathsf{md}(\varphi) = 0$. Indeed, in this case the satisfaction of φ and ψ only depends on the satisfaction of propositional variables on the current world w and therefore not at all on the accessibility relation. Instead, the proof for $\mathsf{md}(\varphi) = 1$ boils down to the proof of the equivalence

$$(W, R|_w^{\leq 1}, V), w \models \varphi_1 \mathbin{|} \varphi_2 \quad \text{if and only if} \quad (W, R|_w^{\leq 1}, V), w \models \varphi_1 * \varphi_2.$$

depicted as follows. The statements below are equivalent.

- $(W, R|_w^{\leq 1}, V), w \models \varphi_1 \mathbin{|} \varphi_2$
- there are $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ s.t. $\mathfrak{M}_1 +_w \mathfrak{M}_2 = (W, R|_w^{\leq 1}, V)$, $\mathfrak{M}_1, w \models \varphi_1$ and $\mathfrak{M}_2, w \models \varphi_2$ (by definition of $\models$)
- there are disjoint R_1 and R_2 such that $R_1 \cup R_2 = R|_w^{\leq 1}$, $(W, R_1, V), w \models \varphi_1$ and $(W, R_2, V), w \models \varphi_2$ (by definition of $+_w$, as $R|_w^{\leq 1} = \{w\} \times R(w)$)
- there are $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $\mathfrak{M}_1 + \mathfrak{M}_2 = (W, R|_w^{\leq 1}, V)$, $\mathfrak{M}_1, w \models \varphi_1$ and $\mathfrak{M}_2, w \models \varphi_2$ (by definition of $+$)
- $(W, R|_w^{\leq 1}, V), w \models \varphi_1 * \varphi_2$ (by definition of $\models$). □

B PROOF OF LEMMA 2.2

Proof. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and $w \in W$.

For the left to right direction, suppose $\mathfrak{M}, w \models \varphi * \psi$. Then, by definition of $\models$, there are $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{M}$, $\mathfrak{M}_1, w \models \varphi$ and $\mathfrak{M}_2, w \models \psi$. By Lemma A.1 we can easily conclude that $(W, R_1|_w, V), w \models \varphi$ and $(W, R_2|_w, V), w \models \psi$, where

$R|_w \stackrel{\text{def}}{=} \{(w', w'') \in R \mid w' \in R^*(w)\}$. Indeed, this holds as by definition, for every $n \in \mathbb{N}$, $(R|_w)|_w^{\leq n} = R|_w^{\leq n}$. Now, consider the model $\widehat{\mathfrak{M}} = (W, R_1|_w \cup R_2|_w, V)$. It is easy to see that $(W, R_1|_w, V)$ and $(W, R_2|_w, V)$ are such that $(W, R_1|_w, V) +_w (W, R_2|_w, V) = \widehat{\mathfrak{M}}$. Hence $\widehat{\mathfrak{M}}, w \models \varphi \,|\, \psi$. Moreover by definition $R_1|_w \cup R_2|_w \subseteq R$ and $(R_1|_w \cup R_2|_w)(w) = R(w)$. We conclude that $\mathfrak{M}, w \models \blacklozenge(\varphi \,|\, \psi)$.

For the right to left direction, suppose $\mathfrak{M}, w \models \blacklozenge(\varphi \,|\, \psi)$. Then by definition of $\models$ there is a model $\widehat{\mathfrak{M}} = (W, \widehat{R}, V)$ such that $\widehat{R} \subseteq R$, $\widehat{R}(w) = R(w)$ and $\widehat{\mathfrak{M}}, w \models \varphi \,|\, \psi$. Again by definition of $\models$, there are $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $\mathfrak{M}_1 +_w \mathfrak{M}_2 = \widehat{\mathfrak{M}}$ and $\mathfrak{M}_1, w \models \varphi$ and $\mathfrak{M}_2, w \models \psi$. Consider now the set $\overline{R} = R \setminus \widehat{R}$. We define:

$$R_1' \stackrel{\text{def}}{=} R_1 \cup \{(w', w'') \in \overline{R} \mid w' \notin R_1^*(w)\}$$
$$R_2' \stackrel{\text{def}}{=} R_2 \cup (\overline{R} \setminus R_1')$$

By definition, it is easy to see that $R_1'|_w = R_1|_w$ and $R_2'|_w = R_2|_w$. Moreover, $R_1' \cap R_2' = \emptyset$ and $R_1' \cup R_2' = R$. Hence, again by using Lemma A.1 we can easily conclude that $(W, R_1', V), w \models \varphi$ and $(W, R_2', V), w \models \psi$. From the properties of R_1' and R_2' expressed above, we obtain $\mathfrak{M}, w \models \varphi * \psi$. □

C PROOF OF LEMMA 3.11

Proof. The proof of Lemma 3.11 essentially consists in proving the lemmas C.1 and C.2 below.

Given $\mathsf{P} = \{p_1, \ldots, p_m\}$ and a finite forest $\mathfrak{M} = (W, R, V)$, for all $w', w'' \in W$, we write $w' \approx_{\mathsf{P}} w''$ iff for all $i \in [1, m]$, we have $\mathfrak{M}, w' \models p_i$ iff $\mathfrak{M}, w'' \models p_i$, i.e. w' and w'' agree on the truth values of all the propositional variables in P. As done in Section 3.3, we recall that $\mathsf{Q} = \{q_1, \ldots, q_{n+1}\}$.

Lemma C.1. *Let $\emptyset \neq X \subseteq [1, n+1]$ and $(\mathfrak{M}, w)$ be a pointed forest such that $\mathfrak{M}, w \models \mathtt{uni}(\mathsf{Q})$. We have $\mathfrak{M}, w \models \mathtt{cp}(X)$ iff for all $w' \in R(w) \cap (\bigcup_{k \in X} V(q_k))$, $X \subseteq \{k \in [1, n+1] \mid$ there is $w'' \in R(w)$ such that $w' \approx_{\mathsf{P}} w''$ and $\mathfrak{M}, w'' \models q_k\}$.*

The second condition can be restated as follows: whenever a child of w satisfies a valuation with respect to P and belongs to $(\bigcup_{k \in X} V(q_k))$, then the valuation is satisfied in a child of w satisfying q_k for all $k \in X$. We recall that $\mathtt{cp}(X)$ is defined as follows.

$$\bigwedge_{k \neq k' \in X} \neg(\Box q_k \,|\, (\Diamond_{=1}\, q_k \wedge \neg(\top \,|\, \Diamond_{=1}\, q_k \wedge \Diamond_{=1}\, q_{k'} \wedge \bigwedge_{j \in [1,m]} \Diamond p_j \Rightarrow \Box p_j))).$$

Proof. In order to show the main equivalence of the statement, we proceed by showing intermediate properties for subformulae of $\mathtt{cp}(X)$. Actually, we shall state the properties, assuming that their proof are by an easy verification. In what follows, we always assume that $(\mathfrak{M}, w)$ be a pointed forest such that $\mathfrak{M}, w \models \mathtt{uni}(\mathsf{Q})$.

(unicity) The first property is related to the formula $\mathtt{uni}(\mathsf{Q}) \stackrel{\text{def}}{=} \Box(\bigwedge_{i \neq i' \in [1,n+1]} \neg(q_i \wedge q_{i'}) \wedge \bigvee_{i \in [1,n+1]} q_i)$, which allows us to state a unicity property. We have $\mathfrak{M}, w \models \mathtt{uni}(\mathsf{Q})$ iff for all $w' \in R(w)$, there is a unique $i \in [1, n+1]$ such that $\mathfrak{M}, w' \models q_i$.

(uniformity) The second property is related to the subformula $\bigwedge_{j \in [1,m]} \Diamond p_j \Rightarrow \Box p_j$ that states a uniformity condition. We have $\mathfrak{M}, w \models \bigwedge_{j \in [1,m]} \Diamond p_j \Rightarrow \Box p_j$ if and only if for all $w', w'' \in R(w)$, we have $w' \approx_{\mathsf{P}} w''$.

(two-witnesses) Let $k \neq k' \in X$ and $\psi_{k,k'} \stackrel{\text{def}}{=} (\top \,|\, \Diamond_{=1}\, q_k \wedge \Diamond_{=1}\, q_{k'} \wedge \bigwedge_{j \in [1,m]} \Diamond p_j \Rightarrow \Box p_j)$. We have $\mathfrak{M}, w \models \psi_{k,k'}$ iff there are $w' \neq w'' \in R(w)$ s.t. $\mathfrak{M}, w' \models q_k$, $\mathfrak{M}, w'' \models q_{k'}$ and $w' \approx_{\mathsf{P}} w''$.

(no-witness-1) Again, let $k \neq k' \in X$. We have $\mathfrak{M}, w \models \Diamond_{=1}\, q_k \wedge \neg\psi_{k,k'}$ iff there is a unique $w' \in R(w)$ such that $\mathfrak{M}, w' \models q_k$ and there is no $w'' \in R(w)$ s.t. $\mathfrak{M}, w'' \models q_{k'}$ and $w' \approx_{\mathsf{P}} w''$.

(no-witness-2) Finally, we have $\mathfrak{M}, w \models \Box q_k \,|\, (\Diamond_{=1}\, q_k \wedge \neg\psi_{k,k'})$ there is $w' \in R(w)$ such that $\mathfrak{M}, w' \models q_k$ and there is no $w'' \in R(w)$ such that $\mathfrak{M}, w'' \models q_{k'}$ and $w' \approx_{\mathsf{P}} w''$.

Consequently, $\mathfrak{M}, w \models \mathtt{cp}(X)$ iff for all $k \neq k' \in X$, there is no $w' \in R(w)$ such that $\mathfrak{M}, w' \models q_k$ and for which there is no $w'' \in R(w)$ such that $\mathfrak{M}, w'' \models q_{k'}$ and $w' \approx_{\mathsf{P}} w''$. Otherwise said, for all $w' \in R(w)$ such that $\mathfrak{M}, w' \models q_k$, there is $w'' \in R(w)$ such that $\mathfrak{M}, w'' \models q_{k'}$ and $w' \approx_{\mathsf{P}} w''$ (P and Q are disjoint). □

Let $(\mathfrak{M}, w)$ be a pointed forest satisfying $\mathtt{uni}(\mathrm{Q})$, $\mathfrak{T}$ be a team built upon P and $\emptyset \neq X \subseteq [1, n+1]$. We write $(\mathfrak{M}, w) \equiv^X_{\mathsf{P}} \mathfrak{T}$ iff the conditions below are satisfied.

(1) For all valuations $\mathfrak{v} \in \mathfrak{T}$, for all $k \in X$, there is $w' \in R(w)$ such that for all $i \in [1, m]$, we have $\mathfrak{M}, w' \models p_i$ iff $\mathfrak{v}(p_i) = \top$ (written $\mathfrak{M}, w' \models \mathfrak{v}$) and $\mathfrak{M}, w' \models q_k$.
(2) For all valuations $\mathfrak{v}$ such that (for all $k \in X$, there is $w'_k \in R(w)$ such that $\mathfrak{M}, w'_k \models \mathfrak{v}$ and $\mathfrak{M}, w'_k \models q_k$), we have $\mathfrak{v} \in \mathfrak{T}$.

Hence, when $(\mathfrak{M}, w) \equiv^X_{\mathsf{P}} \mathfrak{T}$, the children of w encodes the team $\mathfrak{T}$ with the property that each encoding of $\mathfrak{v} \in \mathfrak{T}$ is witnessed by $|X|$ witness worlds.

Given an PL[~] formula φ, its $\dot\vee$-*weight*, written $\mathsf{w}_{\dot\vee}(\varphi)$, is defined as the number of occurrences of $\dot\vee$ in φ.

Lemma C.2. *Let $\emptyset \neq X \subseteq [1, n+1]$, $(\mathfrak{M}, w)$ be a pointed forest such that $\mathfrak{M}, w \models \mathtt{uni}(\mathrm{Q}) \wedge \mathtt{cp}(X)$ and $\mathfrak{T}$ be a team built over* P *such that $(\mathfrak{M}, w) \equiv^X_{\mathsf{P}} \mathfrak{T}$. For all PL[~] formula ψ built over* P *such that $\mathsf{w}_{\dot\vee}(\psi) \leq |X| - 1$, we have $\mathfrak{T} \models \psi$ iff $\mathfrak{M}, w \models \tau(\psi, X)$.*

Proof. The proof is by structural induction.

Base case with $\psi = p_i$, $i \in [1, m]$. First, assume that $\mathfrak{T} \models p_i$, which means that for all valuations $\mathfrak{v} \in \mathfrak{T}$, we have $\mathfrak{v}(p_i) = \top$. *Ad absurdum*, suppose that there is $w' \in R(w) \cap (\bigcup_{k \in X} V(q_k))$, such that $\mathfrak{M}, w' \not\models p_i$. Let $\mathfrak{v}$ be the valuation over P satisfied by w'. As $\mathfrak{M}, w \models \mathtt{cp}(X)$, by Lemma C.1, the valuation $\mathfrak{v}$ is satisfied in a child of w satisfying q_k for all $k \in X$. By (2.) in the definition of $\equiv^X_{\mathsf{P}}$, this implies that $\mathfrak{v} \in \mathfrak{T}$, which leads to a contradiction. Consequently, for all $w' \in R(w) \cap (\bigcup_{k \in X} V(q_k))$, we have $\mathfrak{M}, w' \models p_i$, which can be expressed precisely with $\mathfrak{M}, w \models \Box((\bigvee_{j \in X} q_j) \Rightarrow p_i)$. Hence, $\mathfrak{M}, w \models \tau(p_i, X)$ by definition of τ. For the proof of the other direction, we assume that $\mathfrak{M}, w \models \Box((\bigvee_{j \in X} q_j) \Rightarrow p_i)$ and one can show $\mathfrak{T} \models p_i$ by using this time (1.). Indeed, *ad absurdum*, suppose that $\mathfrak{T} \not\models p_i$. So, there is a valuation $\mathfrak{v}$ such that $\mathfrak{v}(p_i) = \perp$. By (1.), for all $k \in X$, there is $w'_k \in R(w)$ such that $\mathfrak{M}, w'_k \not\models p_i$ and $\mathfrak{M}, w'_k \models q_k$. Since $w'_k \in R(w)$, $\mathfrak{M}, w'_k \models q_k$ and $\mathfrak{M}, w \models \Box((\bigvee_{j \in X} q_j) \Rightarrow p_i)$, we get $\mathfrak{M}, w'_k \models p_i$, which leads to a contradiction.

Base case with $\psi = \dot\neg p_i$, $i \in [1, m]$. Similar to the case $\psi = p_i$.

Induction step. The cases in the induction step for which the outermost connective of ψ is either $\wedge$ or ~ are by an easy verification. Let us consider the case $\psi = \psi_1 \dot\vee \psi_2$. Observe that $\mathsf{w}_{\dot\vee}(\psi) = \mathsf{w}_{\dot\vee}(\psi_1) + \mathsf{w}_{\dot\vee}(\psi_2) + 1$ and recall that $\mathsf{w}_{\dot\vee}(\psi) \leq |X| - 1$. Consequently, $\mathsf{w}_{\dot\vee}(\psi_1) + \mathsf{w}_{\dot\vee}(\psi_2) + 2 \leq |X|$ and let $X_i = \mathfrak{c}_i(X, \mathsf{w}_{\dot\vee}(\psi_1) + 1, \mathsf{w}_{\dot\vee}(\psi_2) + 1)$ for $i \in \{1, 2\}$.

Assume $\mathfrak{T} \models \psi_1 \dot\vee \psi_2$. By definition of $\models$ for PL[~], there are $\mathfrak{T}_1$ and $\mathfrak{T}_2$ such that $\mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2$, $\mathfrak{T}_1 \models \psi_1$ and $\mathfrak{T}_2 \models \psi_2$. We define $\mathfrak{M}_1 = (W, R_1, V_1)$ and $\mathfrak{M}_2 = (W, R_2, V_2)$ s.t. $\mathfrak{M} = \mathfrak{M}_1 +_w \mathfrak{M}_2$ and satisfying the conditions below (only the relevant part is explicitly specified).

- Assume $\mathfrak{v} \in \mathfrak{T}_1 \cap \mathfrak{T}_2$. As $(\mathfrak{M}, w) \equiv^X_{\mathsf{P}} \mathfrak{T}$, for all $k \in X$, there is $w'_k \in R(w)$ such that $\mathfrak{M}, w'_k \models \mathfrak{v}$ and $\mathfrak{M}, w'_k \models q_k$. For all $i \in \{1, 2\}$ and $k \in X$, for all $w' \in R(w) \cap V(q_k)$ such that $\mathfrak{M}, w' \models \mathfrak{v}$, if $k \in X_i$, then $(w, w') \in R_i$ by definition, otherwise $(w, w') \in R_{3-i}$. For all $w' \in R(w)$ such that $w' \notin (\bigcup_{k \in X} V(q_k))$ and $\mathfrak{M}, w' \models \mathfrak{v}$, it is irrelevant whether (w, w') belongs to R_1 or to R_2.
- Assume that $\mathfrak{v} \in \mathfrak{T}_j \setminus \mathfrak{T}_{3-j}$ for some $j \in \{1, 2\}$. For all $w' \in R(w)$ such that $\mathfrak{M}, w' \models \mathfrak{v}$, $(w, w') \in R_j$ by definition.

One can check that $\mathfrak{M}_1, w \equiv_{\mathsf{P}}^{X_1} \mathfrak{T}_1$, $\mathfrak{M}_2, w \equiv_{\mathsf{P}}^{X_2} \mathfrak{T}_2$, $\mathsf{w}_{\dot\vee}(\psi_1) \leq |X_1| - 1$ and $\mathsf{w}_{\dot\vee}(\psi_2) \leq |X_2| - 1$. By the induction hypothesis, we have $\mathfrak{M}_1, w \models \tau(\psi_1, X_1)$ and $\mathfrak{M}_2, w \models \tau(\psi_2, X_2)$. Moreover, as $\mathfrak{M}, w \models \mathsf{cp}(X)$, it is also easy to check that $\mathfrak{M}_1, w \models \mathsf{cp}(X_1)$ and $\mathfrak{M}_2, w \models \mathsf{cp}(X_2)$. Hence, $\mathfrak{M}, w \models (\tau(\psi_1, X_1) \wedge \mathsf{cp}(X_1)) \,|\, (\tau(\psi_2, X_2) \wedge \mathsf{cp}(X_2))$, i.e. $\mathfrak{M}, w \models \tau(\psi, X)$ by definition of τ.

Assume $\mathfrak{M}, w \models \tau(\psi_1 \dot\vee \psi_2, X)$**.** There are $\mathfrak{M}_1$, $\mathfrak{M}_2$ such that $\mathfrak{M} = \mathfrak{M}_1 +_w \mathfrak{M}_2$, $\mathfrak{M}_1, w \models \mathsf{cp}(X_1) \wedge \tau(\psi_1, X_1)$ and $\mathfrak{M}_2, w \models \mathsf{cp}(X_2) \wedge \tau(\psi_2, X_2)$. Let us define $\mathfrak{T}_1$ and $\mathfrak{T}_2$ such that $\mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2$, $\mathfrak{M}_1, w \equiv_{\mathsf{P}}^{X_1} \mathfrak{T}_1$ and $\mathfrak{M}_2, w \equiv_{\mathsf{P}}^{X_2} \mathfrak{T}_2$. Let $\mathfrak{v} \in \mathfrak{T}$ and $j \in \{1, 2\}$. We have $\mathfrak{v} \in \mathfrak{T}_j \overset{\text{def}}{\Leftrightarrow}$ for all $k \in X_j$, there is $w'_k \in R_j(w)$ such that $\mathfrak{M}_j, w'_k \models \mathfrak{v}$ and $\mathfrak{M}_j, w'_k \models q_k$. As $\mathfrak{M}, w \models \mathsf{cp}(X)$ and $X = X_1 \uplus X_2$, one can verify that the definition of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ is well-designed and the teams $\mathfrak{T}_1$ and $\mathfrak{T}_2$ satisfy the expected properties. Using that $\mathsf{w}_{\dot\vee}(\psi_1) + 1 \leq |X_1|$ and $\mathsf{w}_{\dot\vee}(\psi_2) + 1 \leq |X_2|$, by the induction hypothesis, we have $\mathfrak{T}_1 \models \psi_1$ and $\mathfrak{T}_2 \models \psi_2$. Consequently, $\mathfrak{T} \models \psi$. □

The proof of Lemma 3.11 can be now easily completed. Let φ be an PL[~] formula built upon $\mathsf{P} = \{p_1, \ldots, p_m\}$ with $\mathsf{w}_{\dot\vee}(\varphi) = n$ and $\mathsf{Q} = \{q_1, \ldots, q_{n+1}\}$.

Suppose that φ is satisfiable, meaning that there is a team $\mathfrak{T} = \{\mathfrak{v}_1, \ldots, \mathfrak{v}_K\}$ satisfying φ. Let $\mathfrak{M} = (W, R, V)$ be the finite forest such that $W = \{0\} \cup [1, K] \times [1, n+1]$, $R = \{(0, (i, j)) \mid (i, j) \in [1, K] \times [1, n+1]\}$, and V is a valuation such that,

- $V(q_j) = [1, K] \times \{j\}$ for all $j \in [1, n+1]$,
- $V(p_s) = \{(i, j) \mid \mathfrak{v}_i(p_s) = \top\}$ for all $s \in [1, m]$.

One can show that $\mathfrak{M}, w \models \mathtt{uni}(\mathsf{Q}) \wedge \mathsf{cp}([1, n+1])$ and $\mathfrak{M}, w \equiv_{\mathsf{P}}^{[1,n+1]} \mathfrak{T}$. As $\mathsf{w}_{\dot\vee}(\varphi) = |[1, n+1]| - 1$ $(= n)$, by Lemma C.2, we have $\mathfrak{M}, w \models \tau(\varphi, [1, n+1])$.

Conversely, suppose that $\mathtt{uni}(\mathsf{Q}) \wedge \mathsf{cp}([1, n+1]) \wedge \tau(\varphi, [1, n+1])$ is satisfiable, meaning that there is a pointed forest $(\mathfrak{M}, w)$ satisfying it with $\mathfrak{M} = (W, R, V)$. We define the team $\mathfrak{T}$ such that for all valuations $\mathfrak{v}$ built over P, $\mathfrak{v}$ belongs to $\mathfrak{T}$ iff there is $w' \in R(w)$ such that $\mathfrak{M}, w' \models q_k$ for some $k \in [1, n+1]$ and $\mathfrak{M}, w' \models \mathfrak{v}$. Again, one can check that $\mathfrak{M}, w \equiv_{\mathsf{P}}^{[1,n+1]} \mathfrak{T}$ (here we use the fact the $\mathfrak{M}, w \models \mathtt{uni}(\mathsf{Q}) \wedge \mathsf{cp}([1, n+1]))$ and by Lemma C.2, we have $\mathfrak{T} \models \varphi$. □

D PROOF OF LEMMA 4.2

Proof. Recall that $\mathsf{nom}_i(\mathsf{ax})$ is defined as follows:

$$nom_i(ax) \overset{\text{def}}{=} \langle t\rangle^i \Diamond \mathsf{ax} \wedge \bigwedge_{k \in [0, i-1]} [t]^k \neg\big(\langle t\rangle^{i-k} \Diamond \mathsf{ax} * \langle t\rangle^{i-k} \Diamond \mathsf{ax}\big).$$

(⇒): Suppose $\mathfrak{M}, w \models \mathsf{nom}_i(\mathsf{ax})$. By definition of $\models$ and the relativised modality $\langle t\rangle$, there exists a path of t-worlds $w_1, w_2, \ldots, w_i$, such that $wRw_1Rw_2 \ldots Rw_i$, and there exists w' such that $(w_i, w') \in R$ and $\mathfrak{M}, w' \models \mathsf{ax}$. The second conjunct of $\mathsf{nom}_i(\mathsf{ax})$ guarantees that there is only one such paths, leading to w_i being a nominal for the depth i. Indeed, suppose *ad absurdum* that there is a second world $w'_i \in R^i(w)$, distinct from w_i, such that $\mathfrak{M}, w'_i \models \Diamond \mathsf{ax}$. Since $\mathfrak{M}, w \models \mathtt{init}(j)$, w'_i must be a t-node and there must be a path of t-worlds $w'_1, w'_2, \ldots, w'_i$ such that $wRw'_1Rw'_2 \ldots Rw'_i$. Then, there must be $k \in [0, i-1]$ such that for every $j \leq k$, $w_j = w'_j$, and for every $l \in [j+1, i]$, $w_l \neq w'_l$. By considering the pointed forest $(\mathfrak{M}, w_k)$, we can easily show that $\mathfrak{M}, w_k \models \langle t\rangle^{i-k} \Diamond \mathsf{ax} * \langle t\rangle^{i-k} \Diamond \mathsf{ax}$. This implies that $\mathfrak{M}, w \models \langle t\rangle^k\big(\langle t\rangle^{i-k} \Diamond \mathsf{ax} * \langle t\rangle^{i-k} \Diamond \mathsf{ax}\big)$, in contradiction with the second conjunct of $\mathsf{nom}_i(\mathsf{ax})$. Hence, w'_i cannot be distinct from w_i.

(⇐): This direction is analogous. Suppose that $\mathfrak{M}, w \models \mathtt{init}(j)$ and ax is a nominal for the depth i. By definition, there is a unique t-world w' in $R^i(w)$ having a child satisfying ax. Since $\mathfrak{M}, w \models \mathtt{init}(j)$, the path from w to w' must only witness t-nodes. Hence $\mathfrak{M}, w \models \langle t\rangle^i \Diamond \mathsf{ax}$. Moreover, by the uniqueness of this path we conclude that $\mathfrak{M}, w \models \bigwedge_{k \in [0, i-1]} [t]^k \neg\big(\langle t\rangle^{i-k} \Diamond \mathsf{ax} * \langle t\rangle^{i-k} \Diamond \mathsf{ax}\big)$ also holds. Thus, $\mathfrak{M}, w \models \mathsf{nom}_i(\mathsf{ax})$. □

E PROOF OF LEMMA 4.16

Proof. We prove each item.

(Proof of I) We recall that $\mathsf{S}^i_j(\mathsf{ax},\mathsf{bx})$ is defined as

$$\top * \big(\mathtt{fork}^{i+1}_j(\mathsf{x},\mathsf{y}) \land @^i_{\mathsf{ax}}\langle t\rangle(\Diamond \mathsf{s} \land \Diamond \mathsf{x}) \land @^i_{\mathsf{bx}}\langle t\rangle(\Diamond \mathsf{s} \land \Diamond \mathsf{y}) \land [\mathsf{x}=\mathsf{y}]^{i+1}_j \land @^{i+1}_{\mathsf{x}}\neg\mathtt{val} \land @^{i+1}_{\mathsf{y}}\mathtt{val}\big).$$

($\Rightarrow$): Suppose $\mathfrak{M}, w \models \mathsf{S}^i_j(\mathsf{ax},\mathsf{bx})$. By unfolding the definition above, there exists $\mathfrak{M}' = (W, R_1, V)$, such that $\mathfrak{M}' \sqsubseteq \mathfrak{M}$ and:

(a) w has exactly two t-children and exactly two paths of t-nodes, both of length $i+1$;

(b) one of these two paths ends on a world (say w_{x}) corresponding to the nominal x whereas the other ends on a world (say w_{y}) corresponding to the nominal y;

(c) there is a t-world $w_{\mathsf{ax}} \in R^i_1(w)$ corresponding to the nominal ax s.t. $\mathfrak{M}', w_{\mathsf{ax}} \models \langle t\rangle(\Diamond \mathsf{s} \land \Diamond \mathsf{x})$;

(d) there is a t-world $w_{\mathsf{bx}} \in R^i_1(w)$ corresponding to the nominal bx s.t. $\mathfrak{M}', w_{\mathsf{bx}} \models \langle t\rangle(\Diamond \mathsf{s} \land \Diamond \mathsf{y})$;

(e) $\mathfrak{M}', w \models [\mathsf{x}=\mathsf{y}]^{i+1}_j$;

(f) $\mathfrak{M}', w_{\mathsf{x}} \models \neg\mathtt{val}$ and $\mathfrak{M}', w_{\mathsf{y}} \models \mathtt{val}$.

Let $w_{\mathsf{ax},\mathsf{s}} \in R_1(w_{\mathsf{ax}})$ and $w_{\mathsf{bx},\mathsf{s}} \in R_1(w_{\mathsf{bx}})$ be such that they are the only t-children of w_{ax} and w_{bx} respectively, having a child satisfying s (notice they exist due to the hypothesis (C)). Notice by item (b) above, there exists $w' \in R_1(w_{\mathsf{ax}})$ such that $\mathfrak{M}', w' \models t$ and $\mathfrak{M}', w' \models \Diamond \mathsf{s} \land \Diamond \mathsf{x}$. Since $w_{\mathsf{ax},\mathsf{s}}$ is the only child of w_{ax} having an s-child, then $w_{\mathsf{ax},\mathsf{s}} = w'$, and as a consequence $\mathfrak{M}', w_{\mathsf{ax},\mathsf{s}} \models \Diamond \mathsf{x}$. The same argument can be applied by using item (c) above in order to get $\mathfrak{M}', w_{\mathsf{bx},\mathsf{s}} \models \Diamond \mathsf{y}$. By item (a) and (b) above, we have that w_{x} and w_{y} must be the unique t-worlds at distance $i+1$ of w having x and y children, respectively. Therefore, we have necessarily $w_{\mathsf{ax},\mathsf{s}} = w_{\mathsf{x}}$ and $w_{\mathsf{bx},\mathsf{s}} = w_{\mathsf{y}}$, so $\mathfrak{M}, w_{\mathsf{ax},\mathsf{s}} \models \neg\mathtt{val}$ and $\mathfrak{M}, w_{\mathsf{bx},\mathsf{s}} \models \mathtt{val}$ as wanted (by using item (f) above).

Finally, by applying the induction hypothesis on item (e), together with Lemma 4.13, we get $\mathbf{n}(w_{\mathsf{ax},\mathsf{s}}) = \mathbf{n}(w_{\mathsf{bx},\mathsf{s}})$, which concludes the proof of this direction.

($\Leftarrow$): For this direction, we can use a similar argument backwards.

(Proof of II) We recall that $\mathsf{L}^i_j(\mathsf{ax},\mathsf{bx})$ is defined as

$$\neg\big(\top * \big(\mathtt{fork}^{i+1}_j(\mathsf{x},\mathsf{y}) \land @^i_{\mathsf{ax}}\langle t\rangle(\Diamond \mathtt{l} \land \Diamond \mathsf{x}) \land @^i_{\mathsf{bx}}\langle t\rangle(\Diamond \mathtt{l} \land \Diamond \mathsf{y}) \\ \land\ [\mathsf{x}=\mathsf{y}]^{i+1}_j \land \neg(@^{i+1}_{\mathsf{x}}\mathtt{val} \Leftrightarrow @^{i+1}_{\mathsf{y}}\mathtt{val})\big)\big).$$

Notice also that by definition of the satisfaction relation $\models$, we have that $\mathfrak{M}, w \models \mathsf{L}^i_j(\mathsf{ax},\mathsf{bx})$ if and only if for all $\mathfrak{M}' = (W, R_1, V)$ such that $\mathfrak{M}' \sqsubseteq \mathfrak{M}$, we have

$$\mathfrak{M}', w \models (\mathtt{fork}^{i+1}_j(\mathsf{x},\mathsf{y}) \land @^i_{\mathsf{ax}}\langle t\rangle(\Diamond \mathtt{l} \land \Diamond \mathsf{x}) \land @^i_{\mathsf{bx}}\langle t\rangle(\Diamond \mathtt{l} \land \Diamond \mathsf{y}) \land [\mathsf{x}=\mathsf{y}]^{i+1}_j) \Rightarrow \\ (@^{i+1}_{\mathsf{x}}\mathtt{val} \Leftrightarrow @^{i+1}_{\mathsf{y}}\mathtt{val})$$

($\Rightarrow$): Suppose $\mathfrak{M}, w \models \mathsf{L}^i_j(\mathsf{ax},\mathsf{bx})$. Then, for all $\mathfrak{M}' = (W, R_1, V)$ such that $\mathfrak{M}' \sqsubseteq \mathfrak{M}$, if the following conditions hold

(a) w has exactly two t-children and exactly two paths of t-nodes, both of length $i+1$;

(b) one of these two paths ends on a world (say w_{x}) corresponding to the nominal x whereas the other ends on a world (say w_{y}) corresponding to the nominal y;

(c) there is a t-world $w_{\mathsf{ax}} \in R^i_1(w)$ corresponding to the nominal ax s.t. $\mathfrak{M}', w_{\mathsf{ax}} \models \langle t\rangle(\Diamond \mathtt{l} \land \Diamond \mathsf{x})$;

(d) there is a t-world $w_{\mathsf{bx}} \in R_1^i(w)$ corresponding to the nominal bx s.t. $\mathfrak{M}', w_{\mathsf{bx}} \models \langle t \rangle(\Diamond \mathtt{l} \wedge \Diamond \mathsf{y})$;

(e) $\mathfrak{M}', w \models [\mathsf{x} = \mathsf{y}]_j^{i+1}$;

then, it follows that

(f) $\mathfrak{M}', w_{\mathsf{x}} \models \mathtt{val}$ if and only if $\mathfrak{M}', w_{\mathsf{y}} \models \mathtt{val}$.

By hypothesis, there exist $w_{\mathsf{ax}}, w_{\mathsf{bx}}$ at distance i from w corresponding to nominals ax and bx, respectively. Let $w_{\mathsf{ax},1} \in R(w_{\mathsf{ax}})$ and $w_{\mathsf{bx},1} \in R(w_{\mathsf{bx}})$ be such that $\mathbf{n}(w_{\mathsf{ax},1}) > \mathbf{n}(w_{\mathsf{ax},\mathsf{s}})$ and $\mathbf{n}(w_{\mathsf{bx},1}) > \mathbf{n}(w_{\mathsf{bx},\mathsf{s}})$. If we are able to satisfy all the conditions a.–e. above, we can conclude what we want. Suppose $\mathbf{n}(w_{\mathsf{ax},1}) = \mathbf{n}(w_{\mathsf{bx},1})$. By the induction hypothesis, together with Lemma 4.13, we get $\mathfrak{M}, w \models [\mathsf{x} = \mathsf{y}]_j^{i+1}$. Also, since by hypothesis $\mathfrak{M}, w_b \models \mathtt{type}(j - i)$, for $w_b \in \{w_{\mathsf{ax}}, w_{\mathsf{bx}}\}$, then it is easy to check that the remaining conditions above are satisfied. Therefore we can conclude $\mathfrak{M}', w_{\mathsf{x}} \models \mathtt{val}$ iff $\mathfrak{M}', w_{\mathsf{y}} \models \mathtt{val}$.

($\Leftarrow$): The other direction uses similar steps backwards.

(Proof of III) We recall that $\mathsf{R}(\mathsf{ax}, \mathsf{bx}) \stackrel{\text{def}}{=} @^1_{\mathsf{ax}}[t](\Diamond \mathsf{r} \Rightarrow \mathtt{val}) \wedge @^1_{\mathsf{bx}}[t](\Diamond \mathsf{r} \Rightarrow \neg\mathtt{val})$.

($\Rightarrow$): Suppose $\mathfrak{M}, w \models \mathsf{R}(\mathsf{ax}, \mathsf{bx})$. By unfolding the definition above, there exist two distinct t-nodes $w_{\mathsf{ax}}, w_{\mathsf{bx}} \in R(w)$, corresponding to nominals ax and bx respectively, such that:

(a) $\mathfrak{M}, w_{\mathsf{ax}} \models [t](\Diamond \mathsf{r} \Rightarrow \mathtt{val})$, and

(b) $\mathfrak{M}, w_{\mathsf{bx}} \models [t](\Diamond \mathsf{r} \Rightarrow \neg\mathtt{val})$.

By item (C) in the hypothesis, we know that there is exactly one t-node in $R(w_{\mathsf{ax}})$ (say $w_{\mathsf{ax},\mathsf{s}}$) having an Aux-child satisfying s. Let $w_{\mathsf{ax},\mathsf{r}} \in R(w_{\mathsf{ax}})$ be such that $\mathbf{n}(w_{\mathsf{ax},\mathsf{r}}) < \mathbf{n}(w_{\mathsf{ax},\mathsf{s}})$. By item ($E$) in the hypothesis, there exists $w' \in R(w_{\mathsf{ax},\mathsf{r}})$ such that $\mathfrak{M}, w' \models \mathsf{r}$, so $\mathfrak{M}, w_{\mathsf{ax},\mathsf{r}} \models \Diamond \mathsf{r}$. As a consequence, by the item (a) above, we have $\mathfrak{M}, w_{\mathsf{ax},\mathsf{r}} \models \mathtt{val}$.

By applying the same reasoning with $w_{\mathsf{bx},\mathsf{r}} \in R(w_{\mathsf{bx}})$ such that $\mathbf{n}(w_{\mathsf{bx},\mathsf{r}}) < \mathbf{n}(w_{\mathsf{bx},\mathsf{s}})$, and the item (b) above, we get $\mathfrak{M}, w_{\mathsf{bx},\mathsf{r}} \models \neg\mathtt{val}$.

($\Leftarrow$): This direction uses similar arguments (backwards). □

F PROOF OF LEMMA 4.17

Proof. Recall that $[\mathsf{ax} < \mathsf{bx}]_j^i$ is defined as

$$\top * (\mathtt{nom}_i(\mathsf{ax} \neq \mathsf{bx}) \wedge [t]^i \mathtt{lsr}(j - i) \wedge \mathsf{S}_j^i(\mathsf{ax}, \mathsf{bx}) \wedge \mathsf{L}_j^i(\mathsf{ax}, \mathsf{bx})).$$

As in Lemma 4.7, the proof uses standard properties of numbers encoded in binary. Again, let x, y be two natural numbers that can be represented in binary by using n bits. Let us denote with x_i (resp. y_i) the i-th bit of the binary representation of x (resp. y). We have that $x < y$ if and only if

(A) there is a position $i \in [1, n]$ such that $x_i = 0$ and $y_i = 1$;

(B) for every position $j > i$, $x_j = 0 \Leftrightarrow y_j = 0$.

The formula $[\mathsf{ax} < \mathsf{bx}]_j^i$ uses exactly this characterisation in order to state that $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$.

Suppose $\mathfrak{M}, w \models \mathtt{init}(j) \wedge \mathtt{fork}_j^i(\mathsf{ax}, \mathsf{bx})$. From Lemma 4.14, in $(\mathfrak{M}, w)$ it holds that

(i) w has exactly two t-children and exactly two paths of t-nodes, both of length i;

(ii) one of these two paths ends on a world (say w_{ax}) corresponding to the nominal ax whereas the other ends on a world (say w_{bx}) corresponding to the nominal bx;

(iii) $(\mathfrak{M}, w_{\mathsf{ax}})$ and $(\mathfrak{M}, w_{\mathsf{bx}})$ satisfy $\mathtt{type}_{\mathtt{lsr}}(j - i) \stackrel{\text{def}}{=} \mathtt{type}(j - i) \wedge [t](\Diamond \mathtt{l} \wedge \Diamond \mathsf{s} \wedge \Diamond \mathsf{r})$.

To complete the proof, we prove each direction separately.

($\Rightarrow$): Suppose $\mathfrak{M}, w \models [\mathsf{ax} < \mathsf{bx}]_j^i$. Then, by definition of the satisfaction relation $\models$, there exists $\mathfrak{M}' = (W, R', V)$, such that $\mathfrak{M}' \sqsubseteq \mathfrak{M}$ and

$$\mathfrak{M}', w \models \mathtt{nom}_i(\mathsf{ax} \neq \mathsf{bx}) \wedge [t]^i \mathtt{lsr}(j - i) \wedge \mathsf{S}_j^i(\mathsf{ax}, \mathsf{bx}) \wedge \mathsf{L}_j^i(\mathsf{ax}, \mathsf{bx}).$$

Then, from (i)–(iii), we can conclude that in $(\mathfrak{M}', w)$, the two worlds w_{ax} and w_{bx} (corresponding to the nominals ax and bx in $(\mathfrak{M}, w)$) are exactly the ones responsible for the satisfaction of $\mathtt{nom}_i(\mathsf{ax} \neq \mathsf{bx})$. Moreover, from $\mathfrak{M}', w \models [t]^i \mathtt{lsr}(j-i)$ and Lemma 4.15, we have $\mathfrak{M}', w_{\mathsf{ax}} \models \mathtt{type}(j-i)$. Then, by Lemma 4.13 we conclude that w_{ax} encodes the same number w.r.t. $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w)$. The same property holds for w_{bx}, since again by $\mathfrak{M}', w \models [t]^i \mathtt{lsr}(j-i)$ and Lemma 4.15, we have $\mathfrak{M}', w_{\mathsf{bx}} \models \mathtt{type}(j-i)$. Lastly, again from Lemma 4.15,

(1) every t-node in $R'(w_{\mathsf{ax}})$ and $R'(w_{\mathsf{bx}})$ has exactly one Aux-child satisfying an atomic proposition from $\{\mathtt{l}, \mathtt{s}, \mathtt{r}\}$;
(2) exactly one t-node in $R'(w_{\mathsf{ax}})$ (say $w_{\mathsf{ax},\mathsf{s}}$) has an Aux-child satisfying $\mathtt{s}$. Similarly, exactly one t-node in $R'(w_{\mathsf{bx}})$ (say $w_{\mathsf{bx},\mathsf{s}}$) has an Aux-child satisfying $\mathtt{s}$;
(3) given $w_{\mathsf{ax},1} \in R'(w_{\mathsf{ax}})$ (resp. $w_{\mathsf{bx},1} \in R'(w_{\mathsf{bx}})$), it has an Aux-child satisfying $\mathtt{l}$ if and only if $\mathbf{n}(w_{\mathsf{ax},1}) > \mathbf{n}(w_{\mathsf{ax},\mathsf{s}})$ (resp. $\mathbf{n}(w_{\mathsf{bx},1}) > \mathbf{n}(w_{\mathsf{bx},\mathsf{s}})$).

Recall that the number $\mathbf{n}(w_{\mathsf{ax}})$ (resp. $\mathbf{n}(w_{\mathsf{bx}})$) is represented by the binary encoding of the truth values of $\mathtt{val}$ on the t-children of w_{ax} (resp. w_{bx}) which, since $(\mathfrak{M}', w_{\mathsf{ax}}) \models \mathtt{type}(j-i)$ (resp. $(\mathfrak{M}', w_{\mathsf{bx}}) \models \mathtt{type}(j-i)$), are $\mathfrak{t}(j-i, n)$ children implicitly ordered by the number they, in turn, encode. As $(\mathfrak{M}', w)$ satisfies the hypothesis of Lemma 4.16, from $\mathfrak{M}', w \models \mathsf{S}^i_j(\mathsf{ax}, \mathsf{bx}) \wedge \mathsf{L}^i_j(\mathsf{ax}, \mathsf{bx})$ we conclude that

- $\mathbf{n}(w_{\mathsf{ax},\mathsf{s}}) = \mathbf{n}(w_{\mathsf{bx},\mathsf{s}})$, $\mathfrak{M}, w_{\mathsf{ax},\mathsf{s}} \models \neg\mathtt{val}$ and $\mathfrak{M}, w_{\mathsf{bx},\mathsf{s}} \models \mathtt{val}$. Thus, in the binary representation of $\mathbf{n}(w_{\mathsf{ax}})$, the $\mathbf{n}(w_{\mathsf{ax},\mathsf{s}})$th-bit is 0, whereas in the binary representation of $\mathbf{n}(w_{\mathsf{bx}})$, it is 1. Hence, the property (A) of numbers encoded in binary holds for $\mathbf{n}(w_{\mathsf{ax}})$ and $\mathbf{n}(w_{\mathsf{bx}})$;
- for all worlds $w_{\mathsf{ax},1} \in R(w_{\mathsf{ax}})$ and $w_{\mathsf{bx},1} \in R(w_{\mathsf{bx}})$ such that $\mathbf{n}(w_{\mathsf{ax},1}) > \mathbf{n}(w_{\mathsf{ax},\mathsf{s}})$ and $\mathbf{n}(w_{\mathsf{bx},1}) > \mathbf{n}(w_{\mathsf{bx},\mathsf{s}})$, if $\mathbf{n}(w_{\mathsf{ax},1}) = \mathbf{n}(w_{\mathsf{bx},1})$ then
$$\mathfrak{M}, w_{\mathsf{ax},1} \models \mathtt{val} \text{ if and only if } \mathfrak{M}, w_{\mathsf{bx},1} \models \mathtt{val}.$$
Thus, the binary representation of $\mathbf{n}(w_{\mathsf{ax}})$ and $\mathbf{n}(w_{\mathsf{bx}})$, is the same when restricted to the bits that are more significant than $\mathbf{n}(w_{\mathsf{ax},\mathsf{s}})$ (which is equal to $\mathbf{n}(w_{\mathsf{bx},\mathsf{s}})$ by the previous case). Hence, the property (B) is also verified by $\mathbf{n}(w_{\mathsf{ax}})$ and $\mathbf{n}(w_{\mathsf{bx}})$.

Directly, we then conclude that $\mathbf{n}(w_{\mathsf{ax}}) < \mathbf{n}(w_{\mathsf{bx}})$.

($\Leftarrow$): This direction is proven analogously by essentially relying on Lemma 4.16 (I and II). □

G PROOF OF LEMMA 4.31

Proof. We show the proof for I, the one for II being analogous. Recall that (hor$_{\mathcal{T}}$) stands for:
$$\forall w_1, w_2 \in R(w), \text{ if } \mathbf{n}_{\mathcal{H}}(w_2) = \mathbf{n}_{\mathcal{H}}(w_1) + 1 \text{ and } \mathbf{n}_{\mathcal{V}}(w_2) = \mathbf{n}_{\mathcal{V}}(w_1) \text{ then there is } (\mathsf{c}_1, \mathsf{c}_2) \in \mathcal{H} \text{ s.t. } w_1 \in V(\mathsf{c}_1) \text{ and } w_2 \in V(\mathsf{c}_2).$$
Suppose $\mathfrak{M}, w \models \mathtt{grid}_{\mathcal{T}}(k)$. Then in particular every world $w' \in R(w)$ encodes a pair of numbers $(\mathbf{n}_{\mathcal{H}}(w), \mathbf{n}_{\mathcal{V}}(w)) \in [0, \mathfrak{t}(k,n)-1]^2$.

($\Rightarrow$): Suppose $\mathfrak{M}, w \models \mathtt{hor}_{\mathcal{T}}(k)$. Then, by definition, for every $\mathfrak{M}' \sqsubseteq \mathfrak{M}$, if $\mathfrak{M}', w \models \mathtt{fork}^1_k(\mathsf{x}, \mathsf{y}) \wedge [\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x}{+}1]_k \wedge [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k$ then $\mathfrak{M}', w \models \bigvee_{(\mathsf{c}_1,\mathsf{c}_2) \in \mathcal{H}} (@^1_{\mathsf{x}} \mathsf{c}_1 \wedge @^1_{\mathsf{y}} \mathsf{c}_2)$. Consider now two worlds $w_{\mathsf{x}}, w_{\mathsf{y}} \in R(w)$ such that $\mathbf{n}_{\mathcal{H}}(w_{\mathsf{y}}) = \mathbf{n}_{\mathcal{H}}(w_{\mathsf{x}}) + 1$ and $\mathbf{n}_{\mathcal{V}}(w_{\mathsf{y}}) = \mathbf{n}_{\mathcal{V}}(w_{\mathsf{x}})$. Notice that $\mathfrak{M}$ at w_{x} and $\mathfrak{M}$ at w_{y} satisfy $\mathtt{type}(k-1)$, by definition of $\mathtt{grid}_{\mathcal{T}}(k)$. Let $\mathfrak{M}' = (W, R_1, V)$ be the submodel of $\mathfrak{M}$ where R_1 is defined from R by removing the following pairs of worlds:

- $(w, w') \in R$ where w' is different from w_{x} and w_{y};
- $(w_{\mathsf{x}}, w'') \in R$ where w'' is the only Aux-child of w_{x} satisfying y (this world exists as $\mathfrak{M}, w_{\mathsf{x}} \models \mathtt{type}(k-1)$, then it satisfies $\mathtt{init}(k-1)$ and $\mathtt{aux}$);
- $(w_{\mathsf{y}}, w''') \in R$ where w''' is the only Aux-child of w_{y} satisfying x (again, this world exists as $\mathfrak{M}, w_{\mathsf{y}} \models \mathtt{type}(k-1)$, then it satisfies $\mathtt{init}(k-1)$ and $\mathtt{aux}$).

We can easily check that the pointed forest $(\mathfrak{M}', w)$ satisfies $\mathtt{fork}^1_k(\mathsf{x}, \mathsf{y})$, where w_x and w_y correspond to two nominals (for the depth 1) x and y, respectively. Thus, $\mathfrak{M}', w_\mathsf{x} \models \mathtt{type}(k-1)$ and $\mathfrak{M}', w_\mathsf{y} \models \mathtt{type}(k-1)$. Therefore, by Lemma 4.13 (which can be easily extended in order to consider pairs of numbers described with $\mathtt{val}_\mathcal{H}$ and $\mathtt{val}_\mathcal{V}$, instead of a single number described with $\mathtt{val}$), we conclude that w_x and w_y keep encoding the same two pairs of numbers when $\mathfrak{M}$ is modified to $\mathfrak{M}'$. Then, since by hypothesis $\mathbf{n}_\mathcal{H}(w_\mathsf{y}) = \mathbf{n}_\mathcal{H}(w_\mathsf{x}) + 1$ and $\mathbf{n}_\mathcal{V}(w_\mathsf{y}) = \mathbf{n}_\mathcal{V}(w_\mathsf{x})$, by Lemmata 4.23 and 4.24 we conclude that $\mathfrak{M}', w \models [\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x}{+}1]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k$. Then, by hypothesis $\mathfrak{M}, w \models \mathtt{hor}_\mathscr{T}(k)$, we conclude that $\mathfrak{M}', w \models \bigvee_{(\mathsf{c}_1,\mathsf{c}_2)\in\mathcal{H}}(@^1_\mathsf{x}\mathsf{c}_1 \land @^1_\mathsf{y}\mathsf{c}_2)$. Thus, there must be a pair $(\mathsf{c}_1, \mathsf{c}_2) \in \mathcal{H}$ such that $\mathfrak{M}', w \models @^1_\mathsf{x}\mathsf{c}_1 \land @^1_\mathsf{y}\mathsf{c}_2$. Since w_x (resp. w_y) corresponds to the nominal (for the depth 1) x (resp. y), we conclude that $\mathfrak{M}, w_\mathsf{x} \models \mathsf{c}_1$ and $\mathfrak{M}, w_\mathsf{y} \models \mathsf{c}_2$. By definition, this implies that $(\mathfrak{M}, w)$ satisfies (hor$_\mathscr{T}$).

($\Leftarrow$): This direction is rather straightforward and, analogously to the left-to-right direction, relies on Lemmata 4.13, 4.23 and 4.24. Briefly, suppose that $(\mathfrak{M}, w)$ satisfies (hor$_\mathscr{T}$) and, *ad absurdum*, assume that $\mathfrak{M}, w \not\models \mathtt{hor}_\mathscr{T}(k)$. Therefore,

$$\mathfrak{M}, w \models \top * \big(\mathtt{fork}^1_k(\mathsf{x}, \mathsf{y}) \land [\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x}{+}1]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k \land \neg \bigvee_{(\mathsf{c}_1,\mathsf{c}_2)\in\mathcal{H}}(@^1_\mathsf{x}\mathsf{c}_1 \land @^1_\mathsf{y}\mathsf{c}_2)\big).$$

Then, there is a submodel $\mathfrak{M}' = (W, R, V)$ of $\mathfrak{M}$ such that $\mathfrak{M}', w \models \mathtt{fork}^1_k(\mathsf{x}, \mathsf{y}) \land [\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x}{+}1]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k \land \neg \bigvee_{(\mathsf{c}_1,\mathsf{c}_2)\in\mathcal{H}}(@^1_\mathsf{x}\mathsf{c}_1 \land @^1_\mathsf{y}\mathsf{c}_2)$. By $\mathfrak{M}', w \models \mathtt{fork}^1_k(\mathsf{x}, \mathsf{y})$ we conclude that there are two worlds w_x and w_y corresponding to two nominals (depth 1) x and y, respectively. Moreover, by Lemma 4.13, these worlds encode the same two numbers w.r.t. $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w)$. From $\mathfrak{M}', w \models [\mathsf{y} \overset{\mathcal{H}}{=} \mathsf{x}{+}1]_k \land [\mathsf{x} \overset{\mathcal{V}}{=} \mathsf{y}]_k$ and the fact that $(\mathfrak{M}, w)$ satisfies (hor$_\mathscr{T}$), together with Lemmata 4.23 and 4.24 we conclude that there is a pair $(\mathsf{c}_1, \mathsf{c}_2) \in \mathcal{H}$ such that $w_\mathsf{x} \in V(\mathsf{c}_1)$ and $w_\mathsf{y} \in V(\mathsf{c}_2)$. However, this contradicts $\mathfrak{M}', w \models \neg \bigvee_{(\mathsf{c}_1,\mathsf{c}_2)\in\mathcal{H}}(@^1_\mathsf{x}\mathsf{c}_1 \land @^1_\mathsf{y}\mathsf{c}_2)$. Thus, $\mathfrak{M}, w \models \mathtt{hor}_\mathscr{T}(k)$. ◻

H PROOF OF LEMMA 4.33

Proof. ($\Rightarrow$): Suppose that $(\mathscr{T}, \mathsf{c})$ has a solution $\tau : [0, \mathfrak{t}(k, n) - 1]^2 \to \mathcal{T}$. Let $\mathfrak{M} = (W, R, V)$ and $w \in W$ be such that $\mathfrak{M}, w \models \mathtt{grid}_\mathscr{T}(k)$ (such a pointed forest exists by Corollary 4.29). We slightly modify V so that the resulting model still satisfies $\mathtt{grid}_\mathscr{T}(k)$, but also satisfies (one$_\mathscr{T}$), (first$_{\mathscr{T},\mathsf{c}}$), (hor$_\mathscr{T}$) and (vert$_\mathscr{T}$). This can be done rather straightforwardly. Indeed, since $\mathfrak{M}, w \models \mathtt{grid}_\mathscr{T}(k)$, by Lemma 4.28 every t-node $w' \in R(w)$ encodes a pair of numbers $(\mathbf{n}_\mathcal{H}(w'), \mathbf{n}_\mathcal{V}(w')) \in [0, \mathfrak{t}(k, n)-1]$. Then, let us consider the model $\mathfrak{M}' = (W, R, V')$ such that

(1) for every $p \in \mathrm{AP} \setminus \mathcal{T}$, $V'(p) = V(p)$. This property leads to $\mathfrak{M}', w \models \mathtt{grid}_\mathscr{T}(k)$, since $\mathtt{grid}_\mathscr{T}(k)$ is written with propositional symbols not appearing in $\mathcal{T}$.
(2) for every $\mathsf{c} \in \mathcal{T}$ and $w' \in R(w)$, $w' \in V(\mathsf{c})$ if and only if $\tau(\mathbf{n}_\mathcal{H}(w'), \mathbf{n}_\mathcal{V}(w')) = \mathsf{c}$.

The second condition allows us to conclude that $(\mathfrak{M}', w)$ satisfies (one$_\mathscr{T}$), (first$_{\mathscr{T},\mathsf{c}}$), (hor$_\mathscr{T}$) and (vert$_\mathscr{T}$). Indeed, (one$_\mathscr{T}$) holds as τ is functional; (first$_{\mathscr{T},\mathsf{c}}$) holds as τ satisfies (first); whereas (hor$_\mathscr{T}$) and (vert$_\mathscr{T}$) hold as τ satisfies (hor&vert). Thus, $(\mathfrak{M}', w) \models \mathtt{tiling}_{\mathscr{T},\mathsf{c}}(k)$ and therefore $\mathtt{tiling}_{\mathscr{T},\mathsf{c}}(k)$ is satisfiable.

($\Leftarrow$): Suppose $\mathtt{tiling}_{\mathscr{T},\mathsf{c}}(k)$ satisfiable and let $\mathfrak{M} = (W, R, V)$ and $w \in W$ such that $\mathfrak{M}, w \models \mathtt{tiling}_{\mathscr{T},\mathsf{c}}(k)$. Let us consider the relation $\tau \subseteq [0, \mathfrak{t}(k, n) - 1] \times [0, \mathfrak{t}(k, n) - 1] \times \mathcal{T}$ defined as

$$(i, j, \mathsf{c}') \in \tau \text{ if and only if there is } w' \in R(w) \text{ such that } \mathbf{n}_\mathcal{H}(w') = i, \mathbf{n}_\mathcal{V}(w') = j \text{ and } w' \in V(\mathsf{c}').$$

Directly by Lemma 4.32 we have that:

I. from (uniq$_{\mathscr{T},k}$) and (one$_\mathscr{T}$), τ is (possibly weakly) functional in its first two components, i.e. for every $(i, j) \in [0, \mathfrak{t}(k, n) - 1]^2$ there is at most one c' such that $(i, j, \mathsf{c}') \in \tau$;
II. from (zero$_{\mathscr{T},k}$) and (compl$_{\mathscr{T},k}$), τ is total (hence not weakly functional), i.e. cannot be that there is $(i, j) \in [0, \mathfrak{t}(k, n) - 1]^2$ such that for every $\mathsf{c}' \in \mathcal{T}$, $(i, j, \mathsf{c}') \notin \tau$. Together with I, this means that τ is a map;

III. from (first$_{\mathcal{T},\mathtt{c}}$), $(0, 0, \mathtt{c}) \in \tau$;
IV. from (hor$_{\mathcal{T}}$) and (vert$_{\mathcal{T}}$), for all $i \in [0, \mathfrak{t}(k, n) - 1]$ and $j \in [0, \mathfrak{t}(k, n) - 2]$, $(\tau(j, i), \tau(j + 1, i)) \in \mathcal{H}$ and $(\tau(i, j), \tau(i, j + 1)) \in \mathcal{V}$.

Therefore, we conclude that τ is a solution for $\mathtt{Tile}_k$. □

I REMINDER ABOUT G-BISIMULATION

Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two finite forests. Let $m \in \mathbb{N}$, $k \in \mathbb{N}^{>0}$ and $\mathsf{P} \subseteq_{\text{fin}} \mathsf{AP}$. A *g-bisimulation* up to (m, k, P) between $\mathfrak{M}$ and $\mathfrak{M}'$ is a sequence of $m + 1$ k-uples $\mathcal{Z}^0 = (\mathcal{Z}^0_1, \mathcal{Z}^0_2, \ldots, \mathcal{Z}^0_k), \ldots, \mathcal{Z}^m = (\mathcal{Z}^m_1, \mathcal{Z}^m_2, \ldots, \mathcal{Z}^m_k)$ satisfying:

init: $\mathcal{Z}^0_1$ is not empty and for every $i \in [1, k]$ and $j \in [0, m]$, $\mathcal{Z}^j_i \subseteq \mathcal{P}(W) \times \mathcal{P}(W')$;
refine: for every $i \in [1, k]$ and $j \in [1, m]$, $\mathcal{Z}^j_i \subseteq \mathcal{Z}^{j-1}_i$;
size: if $X \mathcal{Z}^j_i Y$ then $|X| = |Y| = i$;
atoms: if $\{w\} \mathcal{Z}^0_1 \{w'\}$ then for every $p \in \mathsf{P}$, $w \in V(p)$ if and only if $w' \in V'(p)$;
m-forth: if $\{w\} \mathcal{Z}^{j+1}_1 \{w'\}$ and $X \subseteq R(w)$ with $|X| \in [1, k]$, then there is $Y \subseteq R'(w')$ such that $X \mathcal{Z}^j_{|X|} Y$;
m-back: if $\{w\} \mathcal{Z}^{j+1}_1 \{w'\}$ and $Y \subseteq R'(w')$ with $|Y| \in [1, k]$, then there is $X \subseteq R(w)$ such that $X \mathcal{Z}^j_{|Y|} Y$;
g-forth: if $X \mathcal{Z}^j_i Y$ and $w \in X$, then there is $w' \in Y$ such that $\{w\} \mathcal{Z}^j_1 \{w'\}$;
g-back: if $X \mathcal{Z}^j_i Y$ and $w' \in Y$, then there is $w \in X$ such that $\{w\} \mathcal{Z}^j_1 \{w'\}$.

We write $\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m,k} \mathfrak{M}', w'$ and we say that the two models are *g-bisimilar* whenever there is a g-bisimulation up to (m, k, P) between $\mathfrak{M}$ and $\mathfrak{M}'$, say $\mathcal{Z}^0, \ldots, \mathcal{Z}^m$, such that $\{w\} \mathcal{Z}^m_1 \{w'\}$. We write $\Gamma(\mathfrak{M}, w)^{\mathsf{P}}_{m,k}$ to denote the set of formulae in GML of rank (m, k) and with propositional symbols from P that are satisfied in $\mathfrak{M}, w$, i.e. $\Gamma(\mathfrak{M}, w)^{\mathsf{P}}_{m,k} \stackrel{\text{def}}{=} \{\psi \in \mathsf{GML}[m, k, \mathsf{P}] \mid \mathfrak{M}, w \models \psi\}$. We write $\mathcal{T}^{\mathsf{P}}(m, k)$ to denote the quotient set induced by the equivalence relation $\leftrightarrows^{\mathsf{P}}_{m,k}$. Let us summarise the main results from [22].

Proposition I.1 ([22]). *(1) $\Gamma(\mathfrak{M}, w)^{\mathsf{P}}_{m,k}$ contains finitely many non-equivalent formulae.*
(2) $\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m,k} \mathfrak{M}', w'$ if and only if $\Gamma(\mathfrak{M}, w)^{\mathsf{P}}_{m,k} = \Gamma(\mathfrak{M}', w')^{\mathsf{P}}_{m,k}$.
(3) $\leftrightarrows^{\mathsf{P}}_{m,k}$ is a finite index equivalence relation. $\mathcal{T}^{\mathsf{P}}(m, k)$ is finite.

So, $\equiv^{\mathsf{P}}_{m,k}$ and $\leftrightarrows^{\mathsf{P}}_{m,k}$ are identical relations (see the definitions for $\equiv^{\mathsf{P}}_{m,k}$ and $\mathsf{GML}[m, k, \mathsf{P}]$ in Section 5.1) and there is a finite set $\{\chi_1, \ldots, \chi_Q\} \subseteq \mathsf{GML}[m, k, \mathsf{P}]$ such that

- $\chi_1 \vee \cdots \vee \chi_Q$ is valid, and each χ_i is satisfiable,
- for all $i \neq j \in [1, Q]$, $\chi_i \wedge \chi_j$ is unsatisfiable,
- $(\mathfrak{M}, w) \equiv^{\mathsf{P}}_{m,k} (\mathfrak{M}', w')$ iff there is i such that $(\mathfrak{M}, w) \models \chi_i$ and $(\mathfrak{M}', w') \models \chi_i$.

Hence, χ_i characterises one equivalence class of $\equiv^{\mathsf{P}}_{m,k}$ (or equivalently of $\leftrightarrows^{\mathsf{P}}_{m,k}$).

In what follows, recall that $R|_w \stackrel{\text{def}}{=} \{(w', w'') \in R \mid w' \subseteq R^*(w)\}$.

Lemma I.2. *Let $m \in \mathbb{N}$, $k \in \mathbb{N}^{>0}$ and $\mathsf{P} \subseteq_{\text{fin}} \mathsf{AP}$. Let $\mathfrak{M} = (W, R, V)$ be a finite forest and let $w \in W$. Then, $\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m,k} (W, R|_w, V), w$.*

Proof. As $\leftrightarrows^{\mathsf{P}}_{m,k}$ is an equivalence relation (Proposition I.1.3), it is reflexive and hence $\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m,k} \mathfrak{M}, w$. There is therefore a g-bisimulation up to (m, k, P) between $\mathfrak{M}$ and itself, say $\mathcal{Z}^0, \ldots, \mathcal{Z}^m$ where $\mathcal{Z}^i = (\mathcal{Z}^i_1, \ldots, \mathcal{Z}^i_k)$ for every $i \in [0, m]$, such that $\{w\} \mathcal{Z}^m_1 \{w\}$. Consider now the restriction of $\mathcal{Z}^i_j$, where $i \in [0, m]$ and $j \in [1, k]$, to those sets where every element is reachable from w. Formally, we define $\widehat{\mathcal{Z}^i_j} = \{(X, Y) \in \mathcal{Z}^i_j \mid X \cup Y \subseteq R^*(w)\}$. It is easy to show that $\widehat{\mathcal{Z}^0}, \ldots, \widehat{\mathcal{Z}^m}$, where $\widehat{\mathcal{Z}^i} = (\widehat{\mathcal{Z}^i_1}, \dots, \widehat{\mathcal{Z}^i_k})$ for every $i \in [0, m]$, is a g-bisimulation up to (m, k, P) between $\mathfrak{M}$ and $(W, R|_w, V)$. As $\{w\}\widehat{\mathcal{Z}^m_1}\{w\}$ by definition, we conclude that $\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m,k} (W, R|_w, V), w$. □

J PROOF OF LEMMA 5.1

In the following, we denote with $\mathscr{T}^{\mathsf{P}}(m,k)$ the set $\mathcal{T}^{\mathsf{P}}(m, \mathfrak{f}(m,k))$. Then, notice that $\mathscr{T}^{\mathsf{P}}(m,k) = \mathcal{T}^{\mathsf{P}}(0,k)$ for $m = 0$, and otherwise ($m \geq 1$) $\mathscr{T}^{\mathsf{P}}(m,k) = \mathcal{T}^{\mathsf{P}}(m, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)| + 1))$. Since $\mathcal{T}^{\mathsf{P}'}(m', k')$ is finite for all m', k' and finite P', $\mathscr{T}^{\mathsf{P}}(m,k)$ is well-defined and finite. Lemma 5.1 can be reformulated using $\mathscr{T}^{\mathsf{P}}(m,k)$ as follows.

Lemma *Let $m, k \in \mathbb{N}$ and $\mathsf{P} \subseteq_{\text{fin}} \mathrm{AP}$. Let $(\mathfrak{M}, w)$, $(\mathfrak{M}', w')$ be pointed forests such that $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$. If $\{(\mathfrak{M}, w), (\mathfrak{M}', w')\} \subseteq \mathsf{T}$ for some $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m,k)$, then for every $R_1 \subseteq R$ there is $R'_1 \subseteq R'$ such that $((W, R_1, V), w) \equiv^{\mathsf{P}}_{m,k} ((W', R'_1, V'), w')$, and if $R_1(w) = R(w)$ then $R'_1(w') = R'(w')$.*

Proof. In the case $k = 0$, any formula in $\mathsf{GML}[m, 0, \mathsf{P}]$ is equivalent to a formula in the propositional calculus built over propositional variables in P as $\Diamond_{\geq 0}\, \psi$ is logically equivalent to $\top$. Hence, the lemma trivially holds.

Otherwise ($k \geq 1$), we prove semantically the lemma as $\equiv^{\mathsf{P}}_{m,k}$ and $\leftrightarrows^{\mathsf{P}}_{m,k}$ are identical relations. The proof is by induction on the modal depth m. The induction step is articulated in three steps:

(I) definition and proof of various properties of the two models,

(II) definition of a strategy to reduce R' to R'_1 that closely follows the relationship between R and R_1 with respect to the children of w and,

(III) a proof that the relation R'_1 is such that $(W, R_1, V), w \leftrightarrows^{\mathsf{P}}_{m,k} (W', R'_1, V'), w'$. By construction, we also obtain that if $R_1(w) = R(w)$ then $R'_1(w') = R'(w')$.

Let us begin with the base case.

Base case: $m = 0$**.** The base case is straightforward from the following property of g-bisimulations. When $m = 0$, given $\widehat{\mathfrak{M}} = (\widehat{W}, \widehat{R}, \widehat{V})$, $\widehat{R}_1 \subseteq \widehat{R}$, $\widehat{w} \in \widehat{W}$ and $\widehat{k} \in \mathbb{N}$, we have $\widehat{\mathfrak{M}}, \widehat{w} \leftrightarrows^{\mathsf{P}}_{0,\widehat{k}} (\widehat{W}, \widehat{R}_1, \widehat{V}), \widehat{w}$. This statement holds as it can be easily shown that the set of relations $\mathcal{Z}^0 = (\mathcal{Z}^0_1, \dots, \mathcal{Z}^0_{\widehat{k}})$ where $\mathcal{Z}^0_1 = \{(w, w)\}$ and $\mathcal{Z}^0_j = \emptyset$ for $j \in [2, \widehat{k}]$ satisfies all the requirements for being a g-bisimulation.

Then, with respect to the statement of the lemma, by definition, we have $(W, R_1, V), w \leftrightarrows^{\mathsf{P}}_{0,k} \mathfrak{M}, w$. Now, by definition $\mathscr{T}^{\mathsf{P}}(0,k) = \mathcal{T}^{\mathsf{P}}(0,k)$ and by hypothesis there is $\mathsf{T} \in \mathcal{T}^{\mathsf{P}}(0,k)$ such that $\{(\mathfrak{M}, w), (\mathfrak{M}', w')\} \subseteq \mathsf{T}$. By definition of $\mathcal{T}^{\mathsf{P}}(0,k)$, we have

$$\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{0,k} \mathfrak{M}', w'.$$

As $\leftrightarrows^{\mathsf{P}}_{0,k}$ is an equivalence relation, we conclude $(W, R_1, V), w \leftrightarrows^{\mathsf{P}}_{0,k} \mathfrak{M}', w'$ and therefore it is sufficient to take $R'_1 \stackrel{\text{def}}{=} R'$ to end the proof. Note that in this case, $R'_1(w') = R'(w')$ holds too.

Induction case. In particular, we have $m > 1$ and $\mathscr{T}^{\mathsf{P}}(m,k) = \mathcal{T}^{\mathsf{P}}(m, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)| + 1))$. Moreover, by hypothesis there exists $\mathsf{T} \in \mathcal{T}^{\mathsf{P}}(m, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)| + 1))$ such that

$$\{(\mathfrak{M}, w), (\mathfrak{M}', w')\} \subseteq \mathsf{T}.$$

By definition, we have

$$\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)} \mathfrak{M}', w'.$$

Let us explain the main idea of the proof. Let us pick one child w_1 of w in $\mathfrak{M}$. Obviously, the pointed forest $(\mathfrak{M}, w_1)$ belongs to a specific equivalence class $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$. The effect of reducing R to R_1 is that w_1, together with the updated model, "jumps"[2] to an equivalence class $\mathsf{T}_1 \in \mathcal{T}^{\mathsf{P}}(m-1,k)$. Obviously, $(\mathfrak{M}, w_1)$ already belongs to a class in $\mathcal{T}^{\mathsf{P}}(m-1,k)$. However (from the statement of the lemma), we are only interested in $\mathcal{T}^{\mathsf{P}}(m-1,k)$ when considering R_1, whereas we focus on $\mathscr{T}^{\mathsf{P}}(m-1,k)$ when studying R. To prove the result, we have to show that there is a child w_1' of w' in $\mathfrak{M}'$ so that $(\mathfrak{M}', w_1')$ is in the same equivalence class T of $(\mathfrak{M}, w_1)$ and to show that it is possible to update R' to make w_1' (together with the updated model) "jump" to the equivalence class T_1. However, we need to do this for all the children of w and w', respecting the constraints of being a g-bisimulation. The key step is to show that the graded rank $k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)$ is all we need to find enough children in $R'(w')$ and to be able to construct a relation R_1' so that the resulting models are g-bisimilar up to (m,k,P). Let us now formalise the proof, which requires some intermediate steps that are below $\boxed{\text{highlighted}}$.

We start by considering a single equivalence class $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$ (in fact, our proof is done modularly on these classes). We introduce the two following sets:

- $R(w)|_{\mathsf{T}} \stackrel{\text{def}}{=} \{w_1 \in R(w) \mid (\mathfrak{M}, w_1) \in \mathsf{T}\}$.
- $R'(w')|_{\mathsf{T}} \stackrel{\text{def}}{=} \{w_1' \in R'(w') \mid (\mathfrak{M}', w_1') \in \mathsf{T}\}$.

It is fairly simple to see that the following property holds:

$$\boxed{(\star)\text{: } \min(|R(w)|_{\mathsf{T}}|, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)) = \min(|R'(w')|_{\mathsf{T}}|, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1))}$$

Indeed, ad absurdum, suppose that

$$(\dagger)\text{:} \qquad |R(w)|_{\mathsf{T}}| < k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1) \text{ and } |R(w)|_{\mathsf{T}}| < |R'(w')|_{\mathsf{T}}|$$

The other case $|R'(w')|_{\mathsf{T}}| < k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)$ and $|R'(w')|_{\mathsf{T}}| < |R(w)|_{\mathsf{T}}|$ is analogous and therefore its treatment is omitted below. Since it holds by hypothesis that

$$\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m,k\cdot(|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)} \mathfrak{M}', w',$$

there is a g-bisimulation up to $(m, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1), \mathsf{P})$ between $\mathfrak{M}$ and $\mathfrak{M}'$, say $\mathcal{Z}^0, \dots, \mathcal{Z}^m$, such that $\{w\}\mathcal{Z}^m_1\{w'\}$.

- From (m-back), by taking Y as a subset of $R'(w')|_{\mathsf{T}}$ such that
$$|Y| = \min(|R'(w')|_{\mathsf{T}}|, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)),$$
it must hold that there is a subset $X \subseteq R(w)$ such that $X\mathcal{Z}^{m-1}_{|Y|}Y$.
- From (size), $|X| = |Y|$. Hence, by (†) there must be a world $w_2 \in X$ such that $(\mathfrak{M}, w_2) \notin \mathsf{T}$.
- From (g-forth), there is $w_2' \in Y$ such that $\{w_2\}\mathcal{Z}^{m-1}_1\{w_2'\}$.
- As $\{w_2\}\mathcal{Z}^{m-1}_1\{w_2'\}$, from the definition of g-bisimulation it holds that
$$\mathfrak{M}, w_2 \leftrightarrows^{\mathsf{P}}_{m-1,k\cdot(|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)} \mathfrak{M}', w_2'.$$
- Again by definition of g-bisimulation, it is easy to see that if two models are in the same equivalence class w.r.t. $\leftrightarrows^{\mathsf{P}}_{m',k'}$ then they are in the same equivalence class w.r.t. $\leftrightarrows^{\mathsf{P}}_{m',k''}$ for every $k'' \leq k'$. Therefore $\mathfrak{M}, w_2 \leftrightarrows^{\mathsf{P}}_{m-1,k\cdot(|\mathscr{T}^{\mathsf{P}}(m-2,k)|+1)} \mathfrak{M}', w_2'$. Notice that the set of equivalence classes induced by $\leftrightarrows^{\mathsf{P}}_{m-1,k\cdot(|\mathscr{T}^{\mathsf{P}}(m-2,k)|+1)}$ is $\mathscr{T}^{\mathsf{P}}(m-1,k)$. We conclude that $(\mathfrak{M}, w_2)$ and $(\mathfrak{M}', w_2')$ belong to the same class in $\mathscr{T}^{\mathsf{P}}(m-1,k)$. However, this leads to a contradiction as we have $w_2 \notin \mathsf{T}$ and $w_2' \in \mathsf{T}$ (where $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$).

This concludes the proof of $(\star)$.

Given an equivalence class T' in $\mathcal{T}^{\mathsf{P}}(m-1,k)$, we define the set below

$$R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'} \stackrel{\text{def}}{=} R(w)|_{\mathsf{T}} \cap R_1(w)|_{\mathsf{T}'}.$$

[2]We always put the word "jump" in quotes as it is used in an informal way.

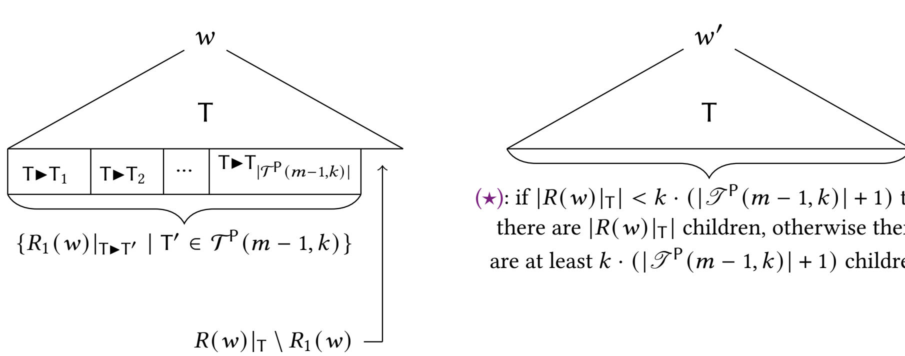
Following the proof idea presented above, a world $w_1 \in R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$ is a child of w such that $(\mathfrak{M}, w_1)$ is in the class T and "jumps" to the class T' when updating the accessibility relation from R to R_1. In what follows, we denote with $R|_{w_1}$ the restriction of R to those worlds reachable from w_1, i.e. the set $\{(w_2, w_3) \in R \mid \{w_2, w_3\} \subseteq R^*(w_1)\}$, as defined in the statement of Lemma I.2. We also consider similar restrictions for R' and R'_1. We are interested in the following key property:

$$\boxed{(\star\star): \quad \text{for all } w_1 \in R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'} \text{ and } w'_1 \in R'(w')|_{\mathsf{T}} \text{ there is } R'_{1,w'_1} \subseteq R'|_{w'_1} \text{ such that } (W, R_1|_{w_1}, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R'_{1,w'_1}, V'), w'_1}$$

Let us prove $(\star\star)$. By definition, we have $w_1 \in R(w)|_{\mathsf{T}}$ and $w'_1 \in R'(w')|_{\mathsf{T}}$. Therefore, $\{(\mathfrak{M}, w_1), (\mathfrak{M}', w'_1)\} \subseteq \mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$. By Lemma I.2, it follows that $(W, R|_{w_1}, V), w_1$ and $(W', R'|_{w'_1}, V'), w'_1$ are also in T. Moreover, by definition $R_1|_{w_1} \subseteq R|_{w_1}$. Then, we can use the induction hypothesis (notice that the modal degree is now $m-1$) to conclude that there is $R'_{1,w'_1} \subseteq R'|_{w'_1}$ such that $(W, R_1|_{w_1}, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R'_{1,w'_1}, V'), w'_1$, concluding the proof of $(\star\star)$. This intermediate result gives us an important information: every single "jump" (as informally expressed above) done while updating the accessibility relation of $\mathfrak{M}$ can be mimicked by updating $\mathfrak{M}'$. An important missing piece is proving that all jumps can be simultaneously mimicked. In order to prove this, we start by considering the following partition of $R(w)|_{\mathsf{T}}$:

$$R(w)^{\mathsf{T}}_{\blacktriangleright R_1} \stackrel{\text{def}}{=} \{R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'} \mid \mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)\} \cup \{R(w)|_{\mathsf{T}} \setminus R_1(w)\}.$$

Informally, $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ partitions the children of w in $R(w)|_{\mathsf{T}}$ in different sets depending on what is the set $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ they "jump" to. One additional set, i.e. $R(w)|_{\mathsf{T}} \setminus R_1(w)$, contains all the children of w in $R(w)|_{\mathsf{T}}$ that are lost when updating R to R_1. To be completely formal, let us first prove that $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ is a partition of $R(w)|_{\mathsf{T}}$. Indeed, $R(w)|_{\mathsf{T}}$ can be written as $(R(w)|_{\mathsf{T}} \cap R_1(w)) \cup (R(w)|_{\mathsf{T}} \setminus R_1(w))$. Moreover, by definition of $\mathcal{T}^{\mathsf{P}}(m-1,k)$ as the quotient set of $\leftrightarrows^{\mathsf{P}}_{m-1,k}$, we have $R_1(w) = \bigcup_{\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)} R_1(w)|_{\mathsf{T}'}$. Lastly, $R(w)|_{\mathsf{T}} \cap \bigcup_{\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)} R_1(w)|_{\mathsf{T}'}$ is equivalent to $\bigcup_{\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)} (R(w)|_{\mathsf{T}} \cap R_1(w)|_{\mathsf{T}'})$, which leads to the definition of the partition $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ from the definition of $R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$ together with the remaining component $R(w)|_{\mathsf{T}} \setminus R_1(w)$. The figure below presents schematically the results we have shown so far, only considering the children of w in $R(w)|_{\mathsf{T}}$ (on the left) and the children of w' in $R'(w')|_{\mathsf{T}}$ (on the right).To work towards the definition of R'_1 (as in the statement of the



lemma), we now deal with the children in $R'(w')|_{\mathsf{T}}$ and find suitable subsets of R'_1 in order to define a partition of $R'(w')|_{\mathsf{T}}$ that is similar to $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ (where "similar" here means that, later, we will be able to construct a g-bisimulation using this partition). More precisely, we show that:

($\star\star\star$): it is possible to construct a family of sets

$$R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'} \qquad \text{for every } \mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$$
$$\mathcal{G}_{\mathsf{T}}$$

satisfying the following properties.

(1) For every $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$, $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ is a set of pairs (R'_{1,w'_1}, w'_1) s.t. $w'_1 \in R'(w')|_{\mathsf{T}}$, $R'_{1,w'_1} \subseteq R'$, $((W', R'_{1,w'_1}, V'), w'_1) \in \mathsf{T}'$, and for all $(w'_2, w'_3) \in R'_{1,w'_1}$, $\{w'_2, w'_3\} \subseteq R'^*(w'_1)$.

(2) $\mathcal{G}_{\mathsf{T}} \subseteq R'(w')|_{\mathsf{T}}$.

(3) Every $w'_1 \in R'(w')|_{\mathsf{T}}$ appears in exactly one set among $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ (for all $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$) and $\mathcal{G}_{\mathsf{T}}$. Then, these sets underlie a partition of $R'(w')|_{\mathsf{T}}$.

(4) For every $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$, $\min(|R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}|, k) = \min(|R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}|, k)$.

(5) $\min(|R(w)|_{\mathsf{T}} \setminus R_1(w)|, k) = \min(|\mathcal{G}_{\mathsf{T}}|, k)$.

Let us informally explain these properties (apart from the second and third properties, which are self-explanatory). The first property basically requires us to modify R' so that the children of $R'(w')|_{\mathsf{T}}$ "jumps" to specific sets in $\mathcal{T}^{\mathsf{P}}(m-1,k)$, in line with the developments that lead to the proof of ($\star\star$). Instead, the set $\mathcal{G}_{\mathsf{T}}$ is dedicated to those worlds that should be made unaccessible from w'. The updates to R' cannot be arbitrary, and this is where the fourth and fifth properties come into play. These properties impose cardinality constraints on the sets we construct, in line with the graded rank k that is used in the equivalence relation $\leftrightarrows^{\mathsf{P}}_{m,k}$. For example, suppose that for a given set T' we have $|R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}| < k$. Then, we need to select exactly $|R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}|$ children in $R'(w')|_{\mathsf{T}}$ and modify R' so that all of them can be used to define the set $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$. If instead $|R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}| \geq k$, it is possible to select an arbitrary amount of children from $R'(w')|_{\mathsf{T}}$, as long as they are at least k. Again, after selecting these children we need to modify R' so that they define the set $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$. To comply with these two last properties we rely on ($\star$). The proof of ($\star\star\star$) distinguishes two cases (which are very similar in substance):

- $|R(w)|_{\mathsf{T}}| < k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)| + 1)$. By ($\star$) it follows that $|R'(w')|_{\mathsf{T}}| = |R(w)|_{\mathsf{T}}|$. This case is the easiest one. Consider a bijection $\mathfrak{f} : R(w)|_{\mathsf{T}} \to R'(w')|_{\mathsf{T}}$. Then define $\mathcal{G}_{\mathsf{T}}$ as the set $\{\mathfrak{f}(w_1) \mid w_1 \in R(w)|_{\mathsf{T}} \setminus R_1(w)\}$. By doing this, trivially the second and fifth properties required by ($\star\star\star$) are satisfied. In order to define the sets of the form $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$, we start by an initialisation to the empty set $\emptyset$ and then we populate them. Iteratively, for every $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ and every $w_1 \in R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$, consider $\mathfrak{f}(w_1)$. By ($\star\star$), there is $R'_{1,\mathfrak{f}(w_1)} \subseteq R'|_{\mathfrak{f}(w_1)}$ such that $(W, R_1|_{w_1}, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R'_{1,\mathfrak{f}(w_1)}, V'), \mathfrak{f}(w_1)$. By Lemma I.2, it follows that $(W, R_1, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R'_{1,\mathfrak{f}(w_1)}, V'), \mathfrak{f}(w_1)$ and therefore $((W', R'_{1,\mathfrak{f}(w_1)}, V'), \mathfrak{f}(w_1)) \in \mathsf{T}'$. Then, add to $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ the pair $(R'_{1,\mathfrak{f}(w_1)}, \mathfrak{f}(w_1))$. Notice that this pair satisfies the constraints required in the first property of ($\star\star\star$). After the iterations over all $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ and over all $w_1 \in R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$, the construction is completed. As we are guided by the bijection $\mathfrak{f}$, we obtain that every $w'_1 \in R'(w')|_{\mathsf{T}}$ appears in exactly one set among $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ for some $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ or in $\mathcal{G}_{\mathsf{T}}$ (condition 3 of ($\star\star\star$)). Moreover (again thanks to the bijection $\mathfrak{f}$) it holds that for every $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$, $|R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}| = |R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}|$, which implies condition 4 of ($\star\star\star$). Hence, ($\star\star\star$) is proved.
- $|R(w)|_{\mathsf{T}}| \geq k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)| + 1)$. By ($\star$), it follows that $|R'(w')|_{\mathsf{T}}| \geq k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)| + 1)$ too. For this case, it is easy to show that there is a set in the partition $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ of $R(w)|_{\mathsf{T}}$ that has cardinality at least k. Indeed, ad absurdum, suppose all the sets in $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ are

of cardinality less than k. As $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ partitions $R(w)|_{\mathsf{T}}$ and it contains $|\mathcal{T}^{\mathsf{P}}(m-1,)|+1$ sets (where the +1 refers to the set $R(w)|_{\mathsf{T}} \setminus R_1(w)$) this would imply that $|R(w)|_{\mathsf{T}}| \leq (k-1)\cdot(|\mathcal{T}^{\mathsf{P}}(m-1,k)|+1)$. This leads to a contradiction as by definition $|\mathcal{T}^{\mathsf{P}}(m-1,k)| \leq |\mathscr{T}^{\mathsf{P}}(m-1,k)|$ and we are in the case where $|R(w)|_{\mathsf{T}}| \geq k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)$. Hence, let Ω be a set in $R(w)^{\mathsf{T}}_{\blacktriangleright R_1}$ that has at least k elements.

For the construction, we initialise all the sets $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ and $\mathcal{G}_{\mathsf{T}}$ to the empty set $\emptyset$ and we show how to populate them. Moreover, we introduce an auxiliary set Δ which is initially equal to $R'(w')|_{\mathsf{T}}$ and keeps track of which elements of this latter set have not been already used in the construction (and are hence available). The set Δ can be understood as a copy of $R'(w')|_{\mathsf{T}}$ with unmarked elements and marked elements. Unmarked elements are the worlds yet to be handled by the algorithm. Iteratively,

(1) consider some $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ s.t. $R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'} \neq \Omega$ and that was not already treated;
(2) select $\beta = \min(|R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}|,k)$ worlds $w'_1,\ldots,w'_\beta$ from the pool of available worlds Δ.
(3) As in the previous case of the proof, by (★★) we have that for each $i \in [1,\beta]$ there is $R'_{1,w'_i} \subseteq R'|_{w'_i}$ such that for every $w_1 \in R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$ it holds that
$$(W,R_1|_{w_1},V),w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W',R'_{1,w'_i},V'),w'_i.$$
By Lemma I.2, it follows also that $(W,R_1,V),w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W',R'_{1,w'_i},V'),w'_i$ and therefore $((W',R'_{1,w'_i},V'),w'_i) \in \mathsf{T}'$. Then, define the set $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ as
$$\{(R'_{1,w'_i},w'_i) \mid i \in [1,\beta]\}.$$
Notice that by construction this set satisfies the first and fourth properties of (★ ★ ★).
(4) Remove $w'_1,\ldots,w'_\beta$ from Δ (they will not be used in the successive iterations).

After this iterative construction, only two sets still need to be handled: Ω and $R(w)|_{\mathsf{T}}\setminus R_1(w)$. In the case these two sets are different, we proceed as follows.

(1) We start by considering $R(w)|_{\mathsf{T}} \setminus R_1(w)$, and we select $\beta = \min(|R(w)|_{\mathsf{T}} \setminus R_1(w)|,k)$ worlds, say $w'_1,\ldots,w'_\beta$ from the pool of available worlds Δ.
(2) We define $\mathcal{G}_{\mathsf{T}}$ as $\{w'_1,\ldots,w'_\beta\}$ and remove these worlds from Δ. By construction, $\mathcal{G}_{\mathsf{T}}$ satisfies the second and fifth properties of (★ ★ ★).
(3) We consider Ω. A few things should be noted now.
    - There is $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ such that $\Omega = R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$, and by definition of Ω, we have $|R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}| \geq k$.
    - At this point of the construction, we dealt with $|\mathcal{T}^{\mathsf{P}}(m-1,k)|$ of the $|\mathcal{T}^{\mathsf{P}}(m-1,k)|+1$ sets needed for the construction. For each of these sets we used at most k new worlds of $R'(w')|_{\mathsf{T}}$. Hence, as $|R'(w')|_{\mathsf{T}}| \geq k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)$ and $|\mathscr{T}^{\mathsf{P}}(m-1,k)| \geq |\mathcal{T}^{\mathsf{P}}(m-1,k)|$, we conclude that Δ has at least k elements.
(4) Consider the set Δ. By (★★) we have that for each $w'_1 \in \Delta$ there is $R'_{1,w'_1} \subseteq R'|_{w'_1}$ such that for every $w_1 \in R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$ it holds that
$$(W,R_1|_{w_1},V),w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W',R'_{1,w'_1},V'),w'_1.$$
By Lemma I.2, it follows that $(W,R_1,V),w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W',R'_{1,w'_1},V'),w'_1$ and therefore $((W',R'_{1,w'_1},V'),w'_1) \in \mathsf{T}'$. Then, define the set $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ as $\{(R'_{1,w'_1},w'_1) \mid w'_1 \in \Delta\}$. By construction, this set satisfies the first and fourth properties of (★ ★ ★) (recall that both $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ and $R_1(w)|_{\mathsf{T}\blacktriangleright\mathsf{T}'}$ have at least k elements, see the previous point).
(5) Empty Δ as every remaining world in it is now used. We completed the construction in the case of $\Omega \neq R(w)|_{\mathsf{T}} \setminus R_1(w)$.

In the case $\Omega = R(w)|_{\mathsf{T}} \setminus R_1(w)$, the construction is trivially completed by adding to $\mathcal{G}_{\mathsf{T}}$ every world in Δ. Notice that for the same considerations done before (point 3 of the construction for $\Omega \neq R(w)|_{\mathsf{T}} \setminus R_1(w)$) it holds that Δ has at least k elements. Hence, $\mathcal{G}_{\mathsf{T}}$ satisfies both the second and the fifth properties of $(\star\star\star)$. Again, as a last step, we empty Δ as every remaining world is now used.

During the definition of the construction, we already detailed why the first, second, fourth and fifth properties of $(\star\star\star)$ are satisfied. The same holds true for the third one, as we relied on the set Δ to never use twice the same world, and at the end of the construction Δ was always empty.

Therefore $(\star\star\star)$ holds. A last note about this construction: from the first and third properties of $(\star\star\star)$, in particular that "for all $(w_2', w_3') \in R'_{1,w_1'}$, $\{w_2', w_3'\} \subseteq R'^*(w_1')\}$", it is easy to see that for all $(R'_{1,w_1'}, w_1') \in R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}_1}$ and $(R'_{1,w_2'}, w_2') \in R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}_2}$ with $w_1' \neq w_2'$, we have $R'_{1,w_1'} \cap R'_{1,w_2'} = \emptyset$. Keeping this in mind, we are now ready to construct R_1'.

We consider every $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$ and apply $(\star\star\star)$ to construct the sets $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ (for every $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$) and $\mathcal{G}_{\mathsf{T}}$. We then define R_1' as

$$R_1' \stackrel{\text{def}}{=} \bigcup_{\substack{\mathsf{T}\in\mathscr{T}^{\mathsf{P}}(m-1,k)\\ \mathsf{T}'\in\mathcal{T}^{\mathsf{P}}(m-1,k)\\ (R'_{1,w_1'},w_1')\in R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}}} \{(w', w_1')\} \cup R'_{1,w_1'}.$$

Clearly. we have that $R_1' \subseteq R_1$. Moreover, from the properties of $(\star\star\star)$, it holds that for every $w_1' \in R_1'(w)$, $R_1'|_{w_1'} = R'_{1,w_1'}$. In order to conclude the proof, we need to show that

(1) $(W, R_1, V), w \leftrightarrows^{\mathsf{P}}_{m,k} (W', R_1', V'), w'$;
(2) if $R_1(w) = R(w)$ then $R_1'(w') = R'(w')$.

Let us first prove (2) by using the fifth property of $(\star\star\star)$. Suppose $R_1(w) = R(w)$ and hence $R(w) \setminus R_1(w) = \emptyset$. It is easy to see that $R(w) \setminus R_1(w)$ can also be written as $\bigcup_{\mathsf{T}\in\mathscr{T}^{\mathsf{P}}(m-1,k)}(R(w)|_{\mathsf{T}} \setminus R_1(w))$. We conclude that $|R(w)|_{\mathsf{T}} \setminus R_1(w)| = 0$ for every $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$. Similarly, $R'(w') \setminus R_1'(w')$ can be shown to be equivalent to $\bigcup_{\mathsf{T}\in\mathscr{T}^{\mathsf{P}}(m-1,k)}(R'(w')|_{\mathsf{T}} \setminus R_1'(w'))$. Notice that for every $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$, a world $w_1' \in R'(w')|_{\mathsf{T}} \setminus R_1'(w')$ cannot be inside a pair of $R'(w')|_{\mathsf{T}\rightsquigarrow\mathsf{T}'}$ (for any $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$). Indeed, if this was the case, then $(w', w_1') \in R_1'$ (see definition of R_1') in contradiction with $w_1' \in R'(w')|_{\mathsf{T}} \setminus R_1'(w')$. Then $w_1' \in \mathcal{G}_{\mathsf{T}}$ and we conclude that $R'(w')|_{\mathsf{T}} \setminus R_1'(w') = \mathcal{G}_{\mathsf{T}}$ and $R'(w') \setminus R_1'(w') = \bigcup_{\mathsf{T}\in\mathscr{T}^{\mathsf{P}}(m-1,k)} \mathcal{G}_{\mathsf{T}}$. By construction, every world $w_1' \in R'(w)$ can appear in at most one set in $\{\mathcal{G}_{\mathsf{T}} \mid \mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)\}$ and hence $|R'(w') \setminus R_1'(w')| = \sum_{\mathsf{T}\in\mathscr{T}^{\mathsf{P}}(m-1,k)}|\mathcal{G}_{\mathsf{T}}|$. We can now apply the fifth property of $(\star\star\star)$, i.e.

$$\min(|R(w)|_{\mathsf{T}} \setminus R_1(w)|, k) = \min(|\mathcal{G}_{\mathsf{T}}|, k)$$

so that together with $k \geq 1$ (see the beginning of the proof) and $|R(w)|_{\mathsf{T}} \setminus R_1(w)| = 0$ leads to $|R'(w') \setminus R_1'(w')| = 0$. As by definition $R_1'(w') \subseteq R'(w')$, this ends the proof of (2).

In order to conclude the proof, let us prove (1) and this is done by constructing a g-bisimulation $\mathcal{Z}^0, \dots, \mathcal{Z}^m$ up to (m, k, P) between (W, R_1, V) and (W', R_1', V') such that $\{w\}\mathcal{Z}_1^m\{w'\}$. Here, we iteratively construct the g-bisimulation starting from the sets $\mathcal{Z}_1^j = \{(w, w')\}$ (for every $j \in [0, m]$). During the construction we make sure to always preserve the satisfaction of the conditions (init), (refine), (size) and (atoms). Notice that these conditions hold for our initial sequence of relations. In particular, (atoms) holds as by hypothesis there is $\mathsf{T} \in \mathcal{T}^{\mathsf{P}}(m, k \cdot (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1))$ such that $\{(\mathfrak{M}, w), (\mathfrak{M}', w')\} \subseteq \mathsf{T}$ and hence $\mathfrak{M}, w \leftrightarrows^{\mathsf{P}}_{m,k\cdot(|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)} \mathfrak{M}', w'$. The construction can be split into four steps:

m-forth-step: Let $X \subseteq R_1(w)$ be a set such that $|X| \in [1,k]$. As required by the condition (m-forth), we want to pair this set with a suitable subset $Y \subseteq R'_1(w)$ of cardinality $|X|$ so that it is possible to then satisfy the conditions (g-forth) and (g-back). Let us consider the partition of X defined as $\{X_{\mathsf{T} \blacktriangleright \mathsf{T}'} \mid \mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k) \text{ and } \mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)\}$ where $X_{\mathsf{T} \blacktriangleright \mathsf{T}'} = X \cap R_1(w)|_{\mathsf{T} \blacktriangleright \mathsf{T}'}$. We consider the set $R'(w')|_{\mathsf{T} \rightsquigarrow \mathsf{T}'}$ and select $|X_{\mathsf{T} \blacktriangleright \mathsf{T}'}|$ worlds appearing in one of its pairs (which are of the form (R'_{1,w'_1}, w'_1)). Let $Y_{\mathsf{T} \rightsquigarrow \mathsf{T}'}$ be the set of these selected worlds. By $(\star\star\star)$ this set is guaranteed to exist and is such that every world w'_1 in it is also in $R'_1(w')$. Let $Y = \bigcup_{\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k), \mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)} Y_{\mathsf{T} \rightsquigarrow \mathsf{T}'}$. It is easy to see that $|X| = |Y|$. For every $j \in [0, m-1]$ we add (X, Y) to $\mathcal{Z}^j_{|X|}$.

m-back-step: Let $Y \subseteq R'_1(w)$ be a set such that $|Y| \in [1,k]$. Let us follow the condition (m-back) symmetrically to what was done for the condition (m-forth) in the previous step of the construction. Let us first consider the partition of Y defined as $\{Y_{\mathsf{T} \rightsquigarrow \mathsf{T}'} \mid \mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k) \text{ and } \mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)\}$ where

$$Y_{\mathsf{T} \rightsquigarrow \mathsf{T}'} = Y \cap \{w'_1 \mid (R'_{1,w'_1}, w'_1) \in R'(w')|_{\mathsf{T} \rightsquigarrow \mathsf{T}'} \text{ for some } R'_{1,w'_1}\}.$$

We select a subset $X_{\mathsf{T} \blacktriangleright \mathsf{T}'}$ of $R_1(w)|_{\mathsf{T} \blacktriangleright \mathsf{T}'}$ having cardinality $|Y_{\mathsf{T} \rightsquigarrow \mathsf{T}'}|$, which is guaranteed to exist by $(\star\star\star)$. Let $X = \bigcup_{\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k), \mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)} X_{\mathsf{T} \blacktriangleright \mathsf{T}'}$. It is easy to see that $|Y| = |X|$. For every $j \in [0, m-1]$ we add (X, Y) to $\mathcal{Z}^j_{|Y|}$.

g-forth-step: From the first two steps of the construction, the set $\mathcal{Z}^j_i$ was updated with new pairs (X, Y) where every element in X is from $R_1(w)$ and every element of Y is from $R'_1(w)$. Consider then one of these pairs (X, Y) and let $w_1 \in X$. There is $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$ and $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ such that $w_1 \in R_1(w)|_{\mathsf{T} \blacktriangleright \mathsf{T}'}$. By construction (first and second steps above), there is $w'_1 \in Y$ such that for some $R'_{1,w'_1} \subseteq R'_1$ it holds that $(R'_{1,w'_1}, w'_1) \in R'(w')|_{\mathsf{T} \rightsquigarrow \mathsf{T}'}$. Again, by applying $(\star\star\star)$ we obtain that $(W, R_1, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R_{1,w'_1}, V'), w'_1$. Since by definition $R'_{1,w'_1} = R'_1|_{w'_1}$ and from Lemma I.2 we obtain $(W, R_1, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R'_1, V'), w'_1$. Then, let $\mathcal{K}^0, \dots, \mathcal{K}^{m-1}$ be the g-bisimulation up to $(m-1, k, \mathsf{P})$ between (W, R_1, V) and (W', R'_1, V') such that $\{w_1\}\mathcal{K}^{m-1}_1\{w'_1\}$. For every $i \in [1,k]$ and every $j \in [0, m-1]$, update $\mathcal{Z}^j_i$ to $\mathcal{Z}^j_i \cup \mathcal{K}^j_i$.

g-back-step: Symmetrically to the previous point of the construction, let us consider again a pair (X, Y) introduced by one of the two steps (m-forth-step) and (m-back-step). Let $w'_1 \in Y$. Then there is $\mathsf{T} \in \mathscr{T}^{\mathsf{P}}(m-1,k)$ and $\mathsf{T}' \in \mathcal{T}^{\mathsf{P}}(m-1,k)$ and $R'_{1,w'_1} \subseteq R'_1$ such that $(R'_{1,w'_1}, w'_1) \in R'(w')|_{\mathsf{T} \rightsquigarrow \mathsf{T}'}$. By construction (steps (m-forth-step) and (m-back-step)), there is $w_1 \in X$ such that $w_1 \in R'(w)|_{\mathsf{T} \blacktriangleright \mathsf{T}'}$. Then by $(\star\star\star)$, we obtain that $(W, R_1, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R_{1,w'_1}, V'), w'_1$. Again, by definition $R'_{1,w'_1} = R'_1|_{w'_1}$ and from Lemma I.2 we obtain $(W, R_1, V), w_1 \leftrightarrows^{\mathsf{P}}_{m-1,k} (W', R'_1, V'), w'_1$. Then, let $\mathcal{K}^0, \dots, \mathcal{K}^{m-1}$ be the g-bisimulation up to $(m-1, k, \mathsf{P})$ between (W, R_1, V) and (W', R'_1, V') such that $\{w_1\}\mathcal{K}^{m-1}_1\{w'_1\}$. For every $i \in [1,k]$ and every $j \in [0, m-1]$, update $\mathcal{Z}^j_i$ to $\mathcal{Z}^j_i \cup \mathcal{K}^j_i$.

It is simple to see that this construction leads to a sequence of relations $\mathcal{Z}^0, \dots, \mathcal{Z}^m$ that is a g-bisimulation up to (m, k, P) between (W, R_1, V) and (W', R'_1, V') such that $\{w\}\mathcal{Z}^m_1\{w'\}$. Indeed, the conditions (init), (refine), (size) and (atoms) hold at any point during the construction. For the other condition, let (X, Y) be a pair in some $\mathcal{Z}^j_i$. If it was not introduced by the first two steps of the construction, then (X, Y) is a member of some set $\mathcal{K}^j_i \subseteq \mathcal{Z}^j_i$ that is used in a g-bisimulation whose elements are all used to construct $\mathcal{Z}^0, \dots, \mathcal{Z}^m$ (third and fourth point of the proof). Hence, w.r.t. (X, Y) no condition can be violated. If instead (X, Y) is added to the g-bisimulation during the first and second point of the construction, then by construction it is easy to check that it satisfies all the conditions. Therefore $(W, R_1, V), w \leftrightarrows^{\mathsf{P}}_{m,k} (W', R'_1, V'), w'$, which ends the proof of the lemma. □

K PROOF OF LEMMA 5.2

Proof. If $k = 0$, then the proof is by an easy verification as the formula φ from the statement is logically equivalent to a formula from the propositional calculus (each subformula $\Diamond_{\geq 0} \psi$ is logically equivalent to $\top$). Otherwise ($k \geq 1$), let $k^+ = k \times (|\mathscr{T}^{\mathsf{P}}(m-1,k)|+1)$. As, $\equiv^{\mathsf{P}}_{m,k^+}$ and $\leftrightarrows^{\mathsf{P}}_{m,k^+}$ are identical relations, there is a finite set $\{\chi_1, \ldots, \chi_Q\} \subseteq \mathsf{GML}[m, k^+, \mathsf{P}]$ such that

- $\chi_1 \vee \cdots \vee \chi_Q$ is valid, and each χ_i is satisfiable,
- for all $i \neq j \in [1, Q]$, $\chi_i \wedge \chi_j$ is unsatisfiable,
- $(\mathfrak{M}, w) \equiv^{\mathsf{P}}_{m,k^+} (\mathfrak{M}', w')$ iff there is i such that $(\mathfrak{M}, w) \models \chi_i$ and $(\mathfrak{M}', w') \models \chi_i$.

This is a direct consequence of Proposition I.1 containing results established in [22]. Let ψ be the formula $\bigvee\{\chi_i \mid \exists\, \mathfrak{M}, w \text{ s.t. } \mathfrak{M}, w \models \chi_i \wedge \blacklozenge\varphi\}$. An empty disjunction is understood as $\bot$.

Now, we show that ψ is logically equivalent to $\blacklozenge\varphi$. Suppose that $\mathfrak{M}, w \models \blacklozenge\varphi$. As $\chi_1 \vee \cdots \vee \chi_Q$ is valid, there is $i \in [1, Q]$ such that $\mathfrak{M}, w \models \chi_i$. Therefore χ_i occurs in ψ and consequently, $\mathfrak{M}, w \models \psi$.

Conversely, suppose that $\mathfrak{M}, w \models \psi$ with $\mathfrak{M} = (W, R, V)$. So, there is χ_i occuring in ψ such that $\mathfrak{M}, w \models \chi_i$ and there exist a model $\mathfrak{M}' = (W', R', V')$ and $w' \in W'$ such that $\mathfrak{M}', w' \models \chi_i \wedge \blacklozenge\varphi$. So, $(\mathfrak{M}, w) \equiv^{\mathsf{P}}_{m,k^+} (\mathfrak{M}', w')$. By the definition of the satisfaction relation $\models$, there is $R'_1 \subseteq R'$ such that $R'_1(w') = R'(w')$ and $(W', R'_1, V'), w' \models \varphi$. All the assumptions of Lemma 5.1 apply and therefore, there is $R_1 \subseteq R$ such that $R_1(w) = R(w)$, $(W, R_1, V), w \leftrightarrows^{\mathsf{P}}_{m,k} (W', R'_1, V'), w'$ and $(W, R_1, V), w \equiv^{\mathsf{P}}_{m,k} (W', R'_1, V'), w'$. As φ belongs to $\mathsf{GML}[m, k, \mathsf{P}]$, we also get that $(W, R_1, V), w \models \varphi$. But then by definition of $\models$, we conclude that $\mathfrak{M}, w \models \blacklozenge\varphi$. □

L PROOF OF (A) FOR LEMMA 5.5

Let us start by stating a few properties. Let us consider two models $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{M}$. We pinpoint three important properties of the models we are considering.

S1: Every world in $R(w)_{=0}$ is either in $R_1(w)_{=0}$ or $R_2(w)_{=0}$;

S2: Every world $w_1 \in R(w)_{=1}$ is in $R_1(w)_{=0}$, $R_2(w)_{=0}$, $R_1(w)_{=1}$ or in $R_2(w)_{=1}$. Indeed, suppose $(w, w_1) \in R_i$ (for some $i \in \{1, 2\}$). If w_1 is in the domain of the same relation R_i then $w_1 \in R_i(w)_{=1}$. Otherwise (w_1 is in the domain of R_{3-i}) then $w_1 \in R_i(w)_{=0}$.

S3: Every world in $R(w)_{=2}$ is in $R_1(w)_{=0}$, $R_2(w)_{=0}$, $R_1(w)_{=1}$, $R_2(w)_{=1}$, $R_1(w)_{=2}$ or $R_2(w)_{=2}$. The justification is similar to the one given above for $R(w)_{=1}$.

First, as worlds in our models do not satisfy any propositional symbol, the spoiler cannot win because of distinct propositional valuations. The proof is by cases on m and on the moves done by the spoiler, and by induction on s. First, suppose $m = 0$. Then it is easy to see that the duplicator has a winning strategy. Indeed, as $m = 0$, the spoiler cannot play the modal move and therefore cannot change the current worlds w and w'. Then, after s spatial moves the game will be in the state $(\mathfrak{M}_1, w)$ and $(\mathfrak{M}'_1, w')$ w.r.t. the rank $(0, 0, \mathsf{P})$. From I we conclude that the duplicator wins.

Suppose now $m \geq 1$ and the spoiler decides to perform a modal move. Notice that, in particular, this case also takes care of the case where $s = 0$ and the spoiler is forced to play a modal move. Moreover, suppose that the spoiler chooses $(\mathfrak{M}, w)$ (the case where it picks $(\mathfrak{M}', w')$ is analogous). We have to distinguish the following situations.

- Suppose that the spoiler chooses a world $w_1 \in R(w)_{=0}$. Then $|R(w)_{=0}| \geq 1$ and by hypothesis $\min(|R(w)_{=0}|, 2^s) = \min(|R'(w')_{=0}|, 2^s)$, it follows that $|R'(w')_{=0}| \geq 1$. It is then sufficient for the duplicator to choose $w_1 \in R'(w')_{=0}$ to guarantee him a victory, as the subtrees rooted in w_1 and w'_1 are isomorphic.
- Suppose that the spoiler chooses a world $w_1 \in R(w)_{=1}$. Then $|R(w)_{=1}| \geq 1$ and by hypothesis $\min(|R(w)_{=1}|, 2^s(s+1)) = \min(|R'(w')_{=1}|, 2^s(s+1))$, it follows that $|R'(w')_{=1}| \geq 1$. Then

again, it is sufficient for the duplicator to choose $w_1 \in R'(w')_{=1}$ to guarantee him a victory, as the subtrees rooted in w_1 and w'_1 are isomorphic.

- Suppose that the spoiler chooses a world $w_1 \in R(w)_{=2}$. Then $|R(w)_{=2}| \geq 1$ and by hypothesis $\min(|R(w)_{=2}|, 2^{s-1}(s+1)(s+2)) = \min(|R'(w')_{=2}|, 2^{s-1}(s+1)(s+2))$, it follows that $|R'(w')_{=2}| \geq 1$ (notice here that $2^{s-1}(s+1)(s+2) = 1$ for $s = 0$). Then again, it is sufficient for the duplicator to choose $w_1 \in R'(w')_{=2}$ to guarantee him a victory, as the subtrees rooted in w_1 and w'_1 are isomorphic.

As stated before, the case where the spoiler decides to perform a modal move also captures the base case of the induction on s. Then, it remains to show the case where $s \geq 1$ and the spoiler decides to do a spatial move. Again suppose that the spoiler chooses $(\mathfrak{M}, w)$ (the case where it picks $(\mathfrak{M}', w')$ is analogous). It then picks two structures $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{M}$. Notice that these two structures are such that both $(\mathfrak{M}_1, w)$ and $(\mathfrak{M}_2, w)$ satisfy I, II and III, as it is easy to see that these three properties are all preserved when taking submodels. The duplicator has now to pick two structures $\mathfrak{M}'_1 = (W', R'_1, V')$ and $\mathfrak{M}'_2 = (W', R'_2, V')$ such that $\mathfrak{M}'_1 + \mathfrak{M}'_2 = \mathfrak{M}'$ while guaranteeing him a victory. It does so by constructing R'_1 and R'_2 as follows (from the empty set):

Split of $R'(w)_{=0}$. We introduce the sets

$$R_1(w)|_{0\blacktriangleright 0} \stackrel{\text{def}}{=} R_1(w)_{=0} \cap R(w)_{=0}$$
$$R_2(w)|_{0\blacktriangleright 0} \stackrel{\text{def}}{=} R_2(w)_{=0} \cap R(w)_{=0}.$$

It is easy to see that these sets are pairwise disjoint. From (S1) it follows that

$$R(w)_{=0} = (R_1(w)_{=0} \cap R(w)_{=0}) \cup (R_2(w)_{=0} \cap R(w)_{=0}).$$

The duplicator starts by partitioning $R'(w)_{=0}$ into two sets Z_1 and Z_2 according to the cardinalities of the two components of $R(w)_{=0}$ highlighted above, namely the two sets $R_1(w)_{=0} \cap R(w)_{=0}$ and $R_2(w)_{=0} \cap R(w)_{=0}$.

- Suppose that $|R_1(w)|_{0\blacktriangleright 0}| < 2^{s-1}$ and $|R_2(w)|_{0\blacktriangleright 0}| < 2^{s-1}$. Hence, $|R(w)_{=0}| < 2^s$ and by hypothesis $|R'(w')_{=0}| = |R(w)_{=0}|$. Then the split of $R'(w)_{=0}$ into Z_1 and Z_2 is made so that $|Z_1| = |R_1(w)|_{0\blacktriangleright 0}|$ and $|Z_2| = |R_2(w)|_{0\blacktriangleright 0}|$.
- Suppose that there is $i \in \{1, 2\}$ such that $|R_i(w)|_{0\blacktriangleright 0}| < 2^{s-1}$ and $|R_j(w)|_{0\blacktriangleright 0}| \geq 2^{s-1}$, where $j = 3 - i$ is the index of the other set. Then the split of $R'(w)_{=0}$ into Z_i and Z_j is made so that $|Z_i| = |R_i(w)|_{0\blacktriangleright 0}|$. Notice that by hypothesis on the cardinality of $R'(w)_{=0}$ it holds that $|Z_j| \geq 2^{s-1}$ (otherwise $\min(|R(w)_{=0}|, 2^s) \neq \min(|R'(w')_{=0}|, 2^s)$).
- Suppose that $|R_1(w)|_{0\blacktriangleright 0}| \geq 2^{s-1}$ and $|R_2(w)|_{0\blacktriangleright 0}| \geq 2^{s-1}$. Then the split of $R'(w)_{=0}$ into Z_1 and Z_2 is made so that $|Z_1| = 2^{s-1}$. Notice that by hypothesis on the cardinality of $R'(w)_{=0}$ it holds that $|Z_j| \geq 2^{s-1}$.

For each $w'_1 \in Z_1$, the duplicator adds (w', w'_1) to R'_1. For each $w'_2 \in Z_2$, it adds (w', w'_2) to R'_2. Notice that by construction the two sets introduced are always such that

Z1: $\min(|R_1(w)|_{0\blacktriangleright 0}|, 2^{s-1}) = \min(|Z_1|, 2^{s-1})$

Z2: $\min(|R_2(w)|_{0\blacktriangleright 0}|, 2^{s-1}) = \min(|Z_2|, 2^{s-1})$.

Split of $R'(w)_{=1}$. We introduce the following sets:

$$R_1(w)|_{1\blacktriangleright 0} \stackrel{\text{def}}{=} R_1(w)_{=0} \cap R(w)_{=1} \qquad R_2(w)|_{1\blacktriangleright 0} \stackrel{\text{def}}{=} R_2(w)_{=0} \cap R(w)_{=1}$$
$$R_1(w)|_{1\blacktriangleright 1} \stackrel{\text{def}}{=} R_1(w)_{=1} \cap R(w)_{=1} \qquad R_2(w)|_{1\blacktriangleright 1} \stackrel{\text{def}}{=} R_2(w)_{=1} \cap R(w)_{=1}.$$

It is easy to see that these sets are pairwise disjoint. From (S2) it follows that

$$R(w)_{=1} = R_1(w)|_{1\blacktriangleright 0} \cup R_2(w)|_{1\blacktriangleright 0} \cup R_1(w)|_{1\blacktriangleright 1} \cup R_2(w)|_{1\blacktriangleright 1}.$$

The duplicator starts by partitioning $R'(w)_{=1}$ into four sets Z_1', Z_2', O_1 and O_2 according to the cardinalities of the four sets above ('Z' for 'zero', 'O' for 'one'). In order to shorten the presentation, instead of concretely make explicit all the cases as we did in the previous point of the construction, we treat them "schematically". Let $\mathcal{X} = \{R_1(w)|_{1\blacktriangleright 0}, R_2(w)|_{1\blacktriangleright 0}, R_1(w)|_{1\blacktriangleright 1}, R_2(w)|_{1\blacktriangleright 1}\}$ and let $\mathfrak{f}$ be the bijection

$$\mathfrak{f}(R_1(w)|_{1\blacktriangleright 0}) \stackrel{\text{def}}{=} Z_1', \quad \mathfrak{f}(R_2(w)|_{1\blacktriangleright 0}) \stackrel{\text{def}}{=} Z_2' \quad \mathfrak{f}(R_1(w)|_{1\blacktriangleright 1}) \stackrel{\text{def}}{=} O_1, \quad \mathfrak{f}(R_2(w)|_{1\blacktriangleright 1}) \stackrel{\text{def}}{=} O_2.$$

Moreover, we define ($\mathcal{B}$ stands for "bound")

$$\mathcal{B}(R_1(w)|_{1\blacktriangleright 0}) \stackrel{\text{def}}{=} \mathcal{B}(R_2(w)|_{1\blacktriangleright 0}) \stackrel{\text{def}}{=} 2^{s-1}$$
$$\mathcal{B}(R_1(w)|_{1\blacktriangleright 1}) \stackrel{\text{def}}{=} \mathcal{B}(R_2(w)|_{1\blacktriangleright 1}) \stackrel{\text{def}}{=} 2^{s-1}s.$$

So, these definitions (actually notations) are helpful at the metalevel. Besides, notice that, from $s \geq 1$, it holds that 2^{s-1} and $2^{s-1}s$ are both at least 1.

- Suppose that for every set $S \in \mathcal{X}$ it holds that $|S| < \mathcal{B}(S)$. Then, since it holds that
$$|R(w)_{=1}| = |R_1(w)|_{1\blacktriangleright 0}| + |R_2(w)|_{1\blacktriangleright 0}| + |R_1(w)|_{1\blacktriangleright 1}| + |R_2(w)|_{1\blacktriangleright 1}|$$
it holds that $|R(w)_{=1}| < 2^{s-1} + 2^{s-1} + 2^{s-1}s + 2^{s-1}s = 2^s(s+1)$ and therefore by hypothesis we conclude that $|R(w)_{=1}| = |R'(w')_{=1}|$. Then, the split of $R'(w')_{=1}$ into Z_1', Z_2', O_1 and O_2 is made so that for every $S \in \mathcal{X}$, $|\mathfrak{f}(S)| = |S|$.
- Suppose instead that there is $\widehat{S} \in \mathcal{X}$ such that $|\widehat{S}| \geq \mathcal{B}(\widehat{S})$. Then, the split of $R'(w')_{=1}$ into Z_1', Z_2', O_1 and O_2 can be made so that for every $S \in \mathcal{X} \setminus \{\widehat{S}\}$, $|\mathfrak{f}(S)| = \min(|S|, \mathcal{B}(S))$. From the hypothesis
$$\min(|R(w)_{=1}|, 2^s(s+1)) = \min(|R'(w')_{=1}|, 2^s(s+1))$$
we conclude that this construction can be effectively made and it is such that $|\mathfrak{f}(\widehat{S})| \geq \mathcal{B}(\widehat{S})$.

For each $w_1' \in Z_1'$, the duplicator adds (w', w_1') to R_1' and the only element of $R'|_{w_1'}$ to R_2'. For each $w_2' \in Z_2'$, it adds (w', w_2') to R_2' and the only element of $R'|_{w_2'}$ to R_1'. For each $w_1' \in O_1$, it adds (w', w_1') and the only element of $R'|_{w_1'}$ to R_1'. Lastly, for each $w_2' \in O_2$, it adds (w', w_2') and the only element of $R'|_{w_2'}$ to R_2'. Notice that by construction the four sets introduced are always such that

Z11: $\min(|R_1(w)|_{1\blacktriangleright 0}|, 2^{s-1}) = \min(|Z_1'|, 2^{s-1})$
Z21: $\min(|R_2(w)|_{1\blacktriangleright 0}|, 2^{s-1}) = \min(|Z_2'|, 2^{s-1})$
O1: $\min(|R_1(w)|_{1\blacktriangleright 1}|, 2^{s-1}s) = \min(|O_1|, 2^{s-1}s)$
O2: $\min(|R_2(w)|_{1\blacktriangleright 1}|, 2^{s-1}s) = \min(|O_2|, 2^{s-1}s)$

or, more schematically, for every $S \in \mathcal{X}$, $\min(|S|, \mathcal{B}(S)) = \min(|\mathfrak{f}(S)|, \mathcal{B}(S))$.

Split of $R'(w)_{=2}$. Similarly to the previous steps, we introduce the following sets:

$$R_1(w)|_{2\blacktriangleright 0} \stackrel{\text{def}}{=} R_1(w)_{=0} \cap R(w)_{=2} \qquad R_2(w)|_{2\blacktriangleright 0} \stackrel{\text{def}}{=} R_2(w)_{=0} \cap R(w)_{=2}$$
$$R_1(w)|_{2\blacktriangleright 1} \stackrel{\text{def}}{=} R_1(w)_{=1} \cap R(w)_{=2} \qquad R_2(w)|_{2\blacktriangleright 1} \stackrel{\text{def}}{=} R_2(w)_{=1} \cap R(w)_{=2}$$
$$R_1(w)|_{2\blacktriangleright 2} \stackrel{\text{def}}{=} R_1(w)_{=2} \cap R(w)_{=2} \qquad R_2(w)|_{2\blacktriangleright 2} \stackrel{\text{def}}{=} R_2(w)_{=2} \cap R(w)_{=2}.$$

It is easy to see that these sets are pairwise disjoint. From (S3) it follows that

$$R(w)_{=2} = R_1(w)|_{2\blacktriangleright 0} \cup R_2(w)|_{2\blacktriangleright 0} \cup R_1(w)|_{2\blacktriangleright 1} \cup R_2(w)|_{2\blacktriangleright 1} \cup R_1(w)|_{2\blacktriangleright 2} \cup R_2(w)|_{2\blacktriangleright 2}$$

The duplicator starts by partitioning $R'(w)_{=2}$ into six sets Z_1'', Z_2'', O_1', O_2', T_1 and T_2 according to the cardinalities of the six sets above ('T' for 'two'). Again, to shorten the presentation we introduce the set

$$\mathcal{X} = \{R_1(w)|_{2\blacktriangleright 0}, R_2(w)|_{2\blacktriangleright 0}, R_1(w)|_{2\blacktriangleright 1}, R_2(w)|_{2\blacktriangleright 1}, R_1(w)|_{2\blacktriangleright 2}, R_2(w)|_{2\blacktriangleright 2}\},$$

and the bijection $\mathfrak{f}$ such that

$$\mathfrak{f}(R_1(w)|_{2\blacktriangleright 0}) \stackrel{\text{def}}{=} Z_1'', \quad \mathfrak{f}(R_2(w)|_{2\blacktriangleright 0}) \stackrel{\text{def}}{=} Z_2'' \quad \mathfrak{f}(R_1(w)|_{2\blacktriangleright 1}) \stackrel{\text{def}}{=} O_1',$$
$$\mathfrak{f}(R_2(w)|_{2\blacktriangleright 1}) \stackrel{\text{def}}{=} O_2', \quad \mathfrak{f}(R_1(w)|_{2\blacktriangleright 2}) \stackrel{\text{def}}{=} T_1, \quad \mathfrak{f}(R_2(w)|_{2\blacktriangleright 2}) \stackrel{\text{def}}{=} T_2.$$

Moreover, we define

$$\mathcal{B}(R_1(w)|_{2\blacktriangleright 0}) \stackrel{\text{def}}{=} \mathcal{B}(R_2(w)|_{2\blacktriangleright 0}) \stackrel{\text{def}}{=} 2^{s-1}$$
$$\mathcal{B}(R_1(w)|_{2\blacktriangleright 1}) \stackrel{\text{def}}{=} \mathcal{B}(R_2(w)|_{2\blacktriangleright 1}) \stackrel{\text{def}}{=} 2^{s-1}s$$
$$\mathcal{B}(R_1(w)|_{2\blacktriangleright 2}) \stackrel{\text{def}}{=} \mathcal{B}(R_2(w)|_{2\blacktriangleright 2}) \stackrel{\text{def}}{=} 2^{s-2}s(s+1)$$

Notice that, from $s \geq 1$, it holds that 2^{s-1}, $2^{s-1}s$ and $2^{s-2}s(s+1)$ are all at least 1.

- Suppose that for every set $S \in \mathcal{X}$ it holds that $|S| < \mathcal{B}(S)$. Then, since $|R(w)_{=2}|$ is
$$|R_1(w)|_{2\blacktriangleright 0}| + |R_2(w)|_{2\blacktriangleright 0}| + |R_1(w)|_{2\blacktriangleright 1}| + |R_2(w)|_{2\blacktriangleright 1}| + |R_1(w)|_{2\blacktriangleright 2}| + |R_2(w)|_{2\blacktriangleright 2}|$$
it holds that
$$|R(w)_{=2}| < 2 \times 2^{s-1} + 2 \times 2^{s-1}s + 2 \times 2^{s-2}s(s+1) = 2^{s-1}(s+1)(s+2)$$
and therefore by hypothesis we conclude that $|R(w)_{=2}| = |R'(w')_{=2}|$. Then, the split of $R'(w')_{=2}$ into Z_1'', Z_2'', O_1', O_2', T_1 and T_2 is made so that for every $S \in \mathcal{X}$, $|\mathfrak{f}(S)| = |S|$.
- Suppose instead that there is $\widehat{S} \in \mathcal{X}$ such that $|\widehat{S}| \geq \mathcal{B}(\widehat{S})$. Then, the split of $R'(w')_{=2}$ into Z_1'', Z_2'', O_1', O_2', T_1 and T_2 is made so that for every $S \in \mathcal{X} \setminus \widehat{S}$, $|\mathfrak{f}(S)| = \min(|S|, \mathcal{B}(S))$. From the hypothesis
$$\min(|R(w)_{=2}|, 2^{s-1}(s+1)(s+2)) = \min(|R'(w')_{=2}|, 2^{s-1}(s+1)(s+2))$$
we conclude that this construction can be effectively made and it is such that $|\mathfrak{f}(\widehat{S})| \geq \mathcal{B}(\widehat{S})$.

Then, the duplicator updates R_1' and R_2' as follows:

- For each $w_1' \in Z_1''$, the duplicator adds (w', w_1') to R_1' and the two elements of $R'|_{w_1'}$ to R_2'.
- For each $w_2' \in Z_2''$, it adds (w', w_2') to R_2' and the two elements of $R'|_{w_2'}$ to R_1'.
- For each $w_1' \in O_1'$, it adds (w', w_1') and one of the two elements of $R'|_{w_1'}$ to R_1'. The other element of $R'|_{w_1'}$ is assigned to R_2'.
- For each $w_2' \in O_2'$, it adds (w', w_2') and one of the two elements of $R'|_{w_2'}$ to R_2'. The other element of $R'|_{w_2'}$ is assigned to R_1'.
- For each $w_1' \in T_1$, it adds (w', w_1') to R_1' and the two elements of $R'|_{w_1'}$ to R_1'.
- For each $w_2' \in T_2$, it adds (w', w_2') to R_2' and the two elements of $R'|_{w_2'}$ to R_2'.

Notice that by construction the six sets introduced are always such that

Z12: $\min(|R_1(w)|_{2\blacktriangleright 0}|, 2^{s-1}) = \min(|Z_1''|, 2^{s-1})$
Z22: $\min(|R_2(w)|_{2\blacktriangleright 0}|, 2^{s-1}) = \min(|Z_2''|, 2^{s-1})$
O11: $\min(|R_1(w)|_{2\blacktriangleright 1}|, 2^{s-1}s) = \min(|O_1'|, 2^{s-1}s)$
O21: $\min(|R_2(w)|_{2\blacktriangleright 1}|, 2^{s-1}s) = \min(|O_2'|, 2^{s-1}s)$
T1: $\min(|R_1(w)|_{2\blacktriangleright 2}|, 2^{s-2}s(s+1)) = \min(|T_1|, 2^{s-2}s(s+1))$
T2: $\min(|R_2(w)|_{2\blacktriangleright 2}|, 2^{s-2}s(s+1)) = \min(|T_2|, 2^{s-2}s(s+1))$

or, more schematically, for every $S \in \mathcal{X}$, $\min(|S|, \mathcal{B}(S)) = \min(|\mathfrak{f}(S)|, \mathcal{B}(S))$.

After these steps, since $(\mathfrak{M}', w')$ satisfies II and III, every element $(w_1', w_2') \in R'$ such that $w_1' \in R'^*(w)$ has been assigned to either R_1' or R_2'. Duplicator then concludes the construction of $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ by assigning the remaining elements of R' (i.e. the pairs $(w_1', w_2') \in R'$ such that $w_1' \notin R'^*(w)$) to either R_1' or R_2' (for example, it can put all these elements in R_1'). The two models $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$ are now defined and they trivially satisfy I, II and III (as they are submodels of $\mathfrak{M}'$). Moreover, by construction it is easy to verify that:

- $R_1'(w')_{=0} = Z_1 + Z_1' + Z_1''$
- $R_1'(w')_{=1} = O_1 + O_1'$
- $R_1'(w')_{=2} = T_1$
- for every $n > 2$, $R_1'(w')_{=n} = \emptyset$

- $R_2'(w')_{=0} = Z_2 + Z_2' + Z_2''$
- $R_2'(w')_{=1} = O_2 + O_2'$
- $R_2'(w')_{=2} = T_2$
- for every $n > 2$, $R_2'(w')_{=n} = \emptyset$

Indeed, we specifically built R_1' and R_2' so that these properties (which we later refer to with (†):) hold. Now, we end the proof of (A) by showing that for all $i \in \{1,2\}$,

zero: $\min(|R_i(w)_{=0}|, 2^{s-1}) = \min(|R_i'(w')_{=0}|, 2^{s-1})$;
one: $\min(|R_i(w)_{=1}|, 2^{s-1}s) = \min(|R_i'(w')_{=1}|, 2^{s-1}s)$;
two: $\min(|R_i(w)_{=2}|, 2^{s-2}s(s+1)) = \min(|R_i'(w')_{=2}|, 2^{s-2}s(s+1))$.

Indeed, once these three properties are shown we can apply the induction hypothesis to conclude that $(\mathfrak{M}_1, w) \approx^{\mathsf{P}}_{m,s-1} (\mathfrak{M}_1', w')$ and $(\mathfrak{M}_2, w) \approx^{\mathsf{P}}_{m,s-1} (\mathfrak{M}_2', w')$ and therefore, the play described with the construction above leads to a winning strategy for the duplicator on the game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$, i.e. $(\mathfrak{M}, w) \approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$. The proof of these three properties is quite easy (each case is similar to the others). Let $i \in \{1,2\}$. By using the definitions given during the construction of R_1' and R_2' it holds that

- $R_i(w)_{=0} = R_i(w)|_{0\blacktriangleright 0} \cup R_i(w)|_{1\blacktriangleright 0} \cup R_i(w)|_{2\blacktriangleright 0}$, and by definition for all $j,k \in [0,2]$ such that $j \neq k$ it holds that $R_i(w)|_{j\blacktriangleright 0} \cap R_i(w)|_{k\blacktriangleright 0} = \emptyset$.
- $R_i(w)_{=1} = R_i(w)|_{1\blacktriangleright 1} \cup R_i(w)|_{2\blacktriangleright 1}$, and by definition $R_i(w)|_{1\blacktriangleright 1} \cap R_i(w)|_{2\blacktriangleright 1} = \emptyset$.
- $R_i(w)|_{=2} = R_i(w)|_{2\blacktriangleright 2}$.

In what follows, we refer to these three properties with (‡):.

proof of (zero). By (‡), it holds that $|R_i(w)_{=0}| = |R_i(w)|_{0\blacktriangleright 0}| + |R_i(w)|_{1\blacktriangleright 0}| + |R_i(w)|_{2\blacktriangleright 0}|$. We divide the proof into two cases. For the first case, suppose $|R_i(w)|_{0\blacktriangleright 0}| < 2^{s-1}$, $|R_i(w)|_{1\blacktriangleright 0}| < 2^{s-1}$ and $|R_i(w)|_{2\blacktriangleright 0}| < 2^{s-1}$. Then,

(1) $|Z_i| = |R_i(w)|_{0\blacktriangleright 0}|$ (by (Z1) or (Z2), depending on whether $i = 1$ or $i = 2$)
(2) $|Z_i'| = |R_i(w)|_{1\blacktriangleright 0}|$ (by (Z11)/(Z21))
(3) $|Z_i''| = |R_i(w)|_{1\blacktriangleright 0}|$ (by (Z12)/(Z22))
(4) $|R_i'(w')_{=0}| = |R_i(w)|_{0\blacktriangleright 0}| + |R_i(w)|_{1\blacktriangleright 0}| + |R_i(w)|_{1\blacktriangleright 0}|$ (from (1), (2) and (3), by (†))
(5) $|R_i'(w')_{=0}| = |R_i(w)_{=0}|$ (from 4, by (‡)).

Otherwise, suppose that there is a set among $R_i(w)|_{0\blacktriangleright 0}$, $R_i(w)|_{1\blacktriangleright 0}$ and $R_i(w)|_{2\blacktriangleright 0}$ whose cardinality is at least 2^{s-1}. Then from (Z1)/(Z2), (Z11)/(Z21) or (Z12)/(Z22) (depending on whether $i = 1$ or $i = 2$ and on which set has at least 2^{s-1} elements) there is a set among Z_i, Z_i' and Z_i'' that has cardinality 2^{s-1}. Then, by (†) and (‡) we have that $R_i(w)_{=0}$ and $R_i'(w')_{=0}$ have both more than 2^{s-1} elements.

proof of (one). By (‡), it holds that $|R_i(w)_{=1}| = |R_i(w)|_{1\blacktriangleright 1}| + |R_i(w)|_{2\blacktriangleright 1}|$. We divide the proof into two cases. First, suppose $|R_i(w)|_{1\blacktriangleright 1}| < 2^{s-1}s$ and $|R_i(w)|_{2\blacktriangleright 1}| < 2^{s-1}s$. Then,

(1) $|O_i| = |R_i(w)|_{1\blacktriangleright 1}|$ (by (O1) or (O2), depending on whether $i = 1$ or $i = 2$)
(2) $|O_i'| = |R_i(w)|_{2\blacktriangleright 1}|$ (by (O11)/(O21))
(3) $|R_i'(w')_{=1}| = |R_i(w)|_{1\blacktriangleright 1}| + |R_i(w)|_{2\blacktriangleright 1}|$ (from (1) and (2), by (†))
(4) $|R_i'(w')_{=1}| = |R_i(w)_{=1}|$ (from 3, by (‡)).

Otherwise, suppose that there is a set among $R_i(w)|_{1\blacktriangleright 1}$ and $R_i(w)|_{2\blacktriangleright 1}$ whose cardinality is at least $2^{s-1}s$. Then from (O1)/(O2) or (O11)/(O21) (depending on whether $i = 1$ or $i = 2$ and on which set has at least $2^{s-1}s$ elements) there is a set among O_i, O_i' that has cardinality $2^{s-1}s$. Then, by (†) and (‡) we have that $R_i(w)_{=1}$ and $R_i'(w')_{=1}$ have both more than $2^{s-1}s$ elements.

proof of (two). By (‡), it holds that $|R_i(w)_{=2}| = |R_i(w)|_{2\blacktriangleright 2}|$. Again we divide the proof into two cases. First, suppose $|R_i(w)|_{2\blacktriangleright 2}| < 2^{s-2}s(s+1)$. Then,

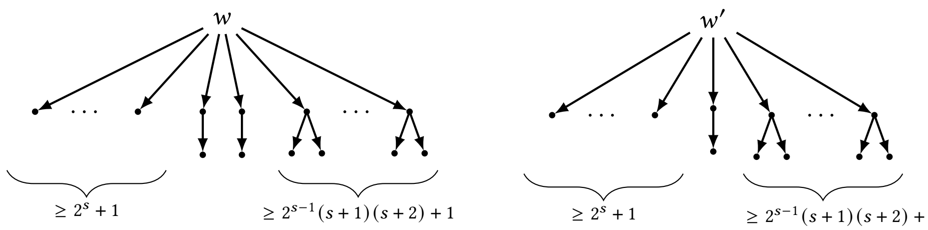
(1) $|T_i| = |R_i(w)|_{2\blacktriangleright 2}|$ (by (T1) or (T2), depending on whether $i = 1$ or $i = 2$)
(2) $|R_i'(w')_{=2}| = |R_i(w)|_{2\blacktriangleright 2}|$ (from (1), by (†))
(3) $|R_i'(w')_{=2}| = |R_i(w)_{=2}|$ (from 2, by (‡)).

Otherwise, suppose that $|R_i(w)|_{2\blacktriangleright 2}|$, and hence $|R_i(w)_{=2}|$, is at least $2^{s-2}s(s+1)$. Then,

(1) $|T_i| \geq 2^{s-2}s(s+1)$ (by (T1)/(T2))
(2) $|R_i'(w')_{=2}| \geq 2^{s-2}s(s+1)$ (from (1), by (†)).

M PROOF OF (B) FOR LEMMA 5.5

The two finite forests of the statement are schematically represented below, with $(\mathfrak{M}, w)$ on the left and $(\mathfrak{M}', w')$ on the right.



The proof of (B) is shown by cases on m, s and on the moves done by the spoiler. As in the proof of (A), if $m = 0$ then the duplicator has a winning strategy as after s spatial moves the game will be in the state $(\mathfrak{M}_1, w)$ and $(\mathfrak{M}'_1, w')$ (notice that w and w' do not change, since $m = 0$) w.r.t. the rank $(0, 0, \mathrm{P})$. From I, we conclude that the duplicator wins.

Now, suppose $m \geq 1$ and the spoiler decides to perform a modal move. Notice that, in particular, this case also takes care of the case where $s = 0$ and the spoiler is forced to play a modal move. Moreover, suppose that the spoiler chooses $(\mathfrak{M}, w)$ (the case where it picks $(\mathfrak{M}', w')$ is analogous). Then, suppose that the spoiler chooses a world $w_1 \in R(w)_{=n}$ for some $n \in \{0, 1, 2\}$. It is then sufficient for the duplicator to choose $w \in R'(w')_{=n}$ (which is a non-empty set by hypothesis) to guarantee him a victory, as the subtrees rooted in w_1 and w'_1 are isomorphic.

It remains to show the strategy for the duplicator when the spoiler decides to perform a spatial move (and therefore $s \geq 1$). The proof distinguishes several cases depending on the structure choosen by the spoiler.

The spoiler picks $(\mathfrak{M}, w)$. Notice that then the spoiler chooses the structure such that $|R(w)_{=1}| = 2$ and the duplicator has to reply in the structure $(\mathfrak{M}', w')$, where we recall that $|R'(w')_{=1}| = 1$. The idea is to make up for this discrepancy by using an element of $R'(w')_{=2}$. Let us see how. For a moment, consider the model obtained from $\mathfrak{M}'$ by removing from R' exactly one pair (w'_1, w'_2) where w'_1 is a world of $R'(w')_{=2}$. Formally, we are interested in a model $\widehat{\mathfrak{M}'} = (W', \widehat{R'}, V')$ such that $\widehat{R'} = R' \setminus \{(w'_1, w'_2)\}$ where $(w'_1, w'_2) \in R'$ and $w'_1 \in R'(w')_{=2}$. If the game was played on $(\mathfrak{M}, w)$ and $(\widehat{\mathfrak{M}'}, w')$ w.r.t. (m, s, P) then it is clear that the duplicator would have a winning strategy. Indeed, both $(\mathfrak{M}, w)$ and $(\widehat{\mathfrak{M}'}, w')$ satisfy I, II and III. Moreover,

- $|R(w)_{=0}|$ and $|\widehat{R'}(w')_{=0}|$ are both at least 2^s. Notice that by definition $\widehat{R'}(w')_{=0} = R'(w')_{=0}$.
- $|R(w)_{=1}| = 2$ and $|\widehat{R'}(w')_{=1}| = 2$. Here, by definition $\widehat{R'}(w')_{=1} = R'(w')_{=1} \cup \{w'_1\}$.
- $|R(w)_{=2}|$ and $|\widehat{R'}(w')_{=2}|$ are both at least $2^{s-1}(s+1)(s+2)$. Here, by definition $\widehat{R'}(w')_{=2} = R'(w')_{=2} \setminus \{w'_1\}$.

These properties allow us to apply (A) and conclude that $(\mathfrak{M}, w) \approx^{\mathrm{P}}_{m,s} (\widehat{\mathfrak{M}'}, w')$. In particular, in this game, if the spoiler picks $(\mathfrak{M}, w)$ and chooses $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{M}$, then the duplicator can apply the strategy described in (A) in order to construct two structures $\widehat{\mathfrak{M}'_1} = (W', \widehat{R'_1}, V')$ and $\widehat{\mathfrak{M}'_2} = (W', \widehat{R'_2}, V')$ such that $\widehat{\mathfrak{M}'_1} + \widehat{\mathfrak{M}'_2} = \widehat{\mathfrak{M}'}$ and for every $i \in \{1, 2\}$:

- $\min(|R_i(w)_{=0}|, 2^{s-1}) = \min(|\widehat{R'_i}(w')_{=0}|, 2^{s-1})$;
- $\min(|R_i(w)_{=1}|, 2^{s-1}s) = \min(|\widehat{R'_i}(w')_{=1}|, 2^{s-1}s)$;
- $\min(|R_i(w)_{=2}|, 2^{s-2}s(s+1)) = \min(|\widehat{R'_i}(w')_{=2}|, 2^{s-2}s(s+1))$.

Notice that these properties, which we later refer to with (††): are exactly (zero), (one) and (two) in the proof of (A).

Let us see how to use these pieces of information to derive a strategy for the duplicator in the original game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$. As the spoiler chooses $(\mathfrak{M}, w)$, it selects $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that $\mathfrak{M}_1 + \mathfrak{M}_2 = \mathfrak{M}$. Consider the two structures $\widehat{\mathfrak{M}'_1} = (W', \widehat{R'_1}, V')$ and $\widehat{\mathfrak{M}'_2} = (W', \widehat{R'_2}, V')$ choosen by the duplicator following the strategy, discussed above, for the game $((\mathfrak{M}, w), (\widehat{\mathfrak{M}'}, w'), (m, s, \mathsf{P}))$ in the case when the spoiler chooses $(\mathfrak{M}, w)$ and again selects $\mathfrak{M}_1$ and $\mathfrak{M}_2$. In particular these structures satisfy (††). Moreover, the two forests $\widehat{\mathfrak{M}'_1}$ and $\widehat{\mathfrak{M}'_2}$ are such that $\widehat{\mathfrak{M}'_1} + \widehat{\mathfrak{M}'_2} = \widehat{\mathfrak{M}}$ and therefore $\widehat{R'_1} \cup \widehat{R'_2} = \widehat{R'} = R' \setminus \{(w'_1, w'_2)\}$ where $(w'_1, w'_2) \in R'$ and $w'_1 \in R'(w')_{=2}$. We distinguish two cases.

- If $w'_1 \in \widehat{R'_1}(w')$ then in the original game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$, the duplicator replies to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with the two forests $\mathfrak{M}'_1 = (W', R'_1, V')$ and $\mathfrak{M}'_2 = (W', R'_2, V')$ such that $R'_1 = \widehat{R'_1}$ and $R'_2 = \widehat{R'_2} \cup \{(w'_1, w'_2)\}$.
- Otherwise $w'_1 \in \widehat{R'_2}(w')$ and in the game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$ the duplicator replies to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with the two forests $\mathfrak{M}'_1 = (W', R'_1, V')$ and $\mathfrak{M}'_2 = (W', R'_2, V')$ such that $R'_1 = \widehat{R'_1} \cup \{(w'_1, w'_2)\}$ and $R'_2 = \widehat{R'_2}$.

In both cases, as the pair (w', w'_1) is in one relation between R'_1 and R'_2 whereas (w'_1, w'_2) is in the other relation, the world w'_1 effectively behaves like if it was a member of the set $R'(w')_{=1}$ instead of $R'(w')_{=2}$, exactly as in the case of $\widehat{R'}$. In particular, it is easy to see that for $i \in \{1, 2\}$:

$$|R'_i(w')_{=0}| = |\widehat{R'_i}(w')_{=0}| \qquad |R'_i(w')_{=1}| = |\widehat{R'_i}(w')_{=1}| \qquad |R'_i(w')_{=2}| = |\widehat{R'_i}(w')_{=2}|$$

Hence, by (††) we have that

- $\min(|R_i(w)_{=0}|, 2^{s-1}) = \min(|R'_i(w')_{=0}|, 2^{s-1})$;
- $\min(|R_i(w)_{=1}|, 2^{s-1}s) = \min(|R'_i(w')_{=1}|, 2^{s-1}s)$;
- $\min(|R_i(w)_{=2}|, 2^{s-2}s(s+1)) = \min(|R'_i(w')_{=2}|, 2^{s-2}s(s+1))$.

Moreover, $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$ all satisfy I, II and III (as they are submodels of $\mathfrak{M}$ or $\mathfrak{M}'$), we can apply (A) and conclude that $(\mathfrak{M}_1, w) \approx^{\mathsf{P}}_{m,s-1} (\mathfrak{M}'_1, w')$ and $(\mathfrak{M}_2, w) \approx^{\mathsf{P}}_{m,s-1} (\mathfrak{M}'_2, w')$. Therefore, the play we just described leads to a winning strategy for the duplicator on the game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$, under the hypothesis that the spoiler chooses $(\mathfrak{M}, w)$.

The spoiler picks $(\mathfrak{M}', w')$. Then, the spoiler chooses the structure such that $|R'(w')_{=1}| = 1$ and the duplicator has to reply in the structure $(\mathfrak{M}, w)$ where $|R(w)_{=1}| = 2$. The proof is very similar to the previous case, but instead of choosing an element of $R'(w')_{=2}$ to make up for the discrepancy between $|R(w)_{=1}|$ and $|R'(w')_{=1}|$, the duplicator manipulates the additional element in $R(w)_{=1}$ so that it becomes a member of $R_1(w)_{=0}$ or $R_2(w)_{=0}$. Let us formalise this strategy.

For a moment, consider the model obtained from $\mathfrak{M}$ by removing from R exactly one pair (w_1, w_2) where w_1 is a world of $R(w)_{=1}$. Formally, we are interested in a model $\widehat{\mathfrak{M}} = (W, \widehat{R}, V)$ such that $\widehat{R} = R \setminus \{(w_1, w_2)\}$ where $(w_1, w_2) \in R$ and $w_1 \in R(w)_{=1}$. If the game was played on $(\widehat{\mathfrak{M}}, w)$ and $(\mathfrak{M}', w')$ w.r.t. (m, s, P) then it is clear than the duplicator would have a winning strategy. Indeed, both $(\widehat{\mathfrak{M}}, w)$ and $(\mathfrak{M}', w')$ satisfy I, II and III. Moreover,

- $|\widehat{R}(w)_{=0}|$ and $|R'(w')_{=0}|$ are both at least 2^s. Here, by definition, $\widehat{R}(w)_{=0} = R(w)_{=0} \cup \{w_1\}$.
- $|\widehat{R}(w)_{=1}| = 1$ and $|R'(w')_{=1}| = 1$. Here, by definition $\widehat{R}(w)_{=1} = R(w)_{=1} \setminus \{w_1\}$.
- $|\widehat{R}(w)_{=2}|$ and $|R'(w')_{=2}|$ are both at least $2^{s-1}(s+1)(s+2)$. Here, by definiton $\widehat{R}(w)_{=2} = R(w)_{=2}$.

These properties allow us to apply (A) and conclude that $(\widehat{\mathfrak{M}}, w) \approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$. In particular, in this game, if the spoiler picks $(\mathfrak{M}', w')$ and chooses $\mathfrak{M}'_1 = (W', R'_1, V')$ and $\mathfrak{M}'_2 = (W', R'_2, V')$ such that $\mathfrak{M}'_1 + \mathfrak{M}'_2 = \mathfrak{M}'$, then the duplicator can apply the strategy described in (A). Two structures $\widehat{\mathfrak{M}_1} = (W, \widehat{R_1}, V)$ and $\widehat{\mathfrak{M}_2} = (W, \widehat{R_2}, V)$ are constructed such that $\widehat{\mathfrak{M}_1} + \widehat{\mathfrak{M}_2} = \widehat{\mathfrak{M}}$ and for every $i \in \{1,2\}$:

- $\min(|\widehat{R_i}(w)_{=0}|, 2^{s-1}) = \min(|R'_i(w')_{=0}|, 2^{s-1})$;
- $\min(|\widehat{R_i}(w)_{=1}|, 2^{s-1}s) = \min(|R'_i(w')_{=1}|, 2^{s-1}s)$;
- $\min(|\widehat{R_i}(w)_{=2}|, 2^{s-2}s(s+1)) = \min(|R'_i(w')_{=2}|, 2^{s-2}s(s+1))$.

Again, notice that these properties, which we later refer to with (‡‡), are exactly (zero), (one) and (two) in the proof of (A). Let us see how to use these pieces of information to derive a strategy for the duplicator in the original game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$. As the spoiler chooses $(\mathfrak{M}', w')$, it selects $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$ such that $\mathfrak{M}'_1 + \mathfrak{M}'_2 = \mathfrak{M}'$. Consider the two structures $\widehat{\mathfrak{M}_1} = (W, \widehat{R_1}, V)$ and $\widehat{\mathfrak{M}_2} = (W, \widehat{R_2}, V)$ choosen by the duplicator following the strategy, discussed above, for the game $((\widehat{\mathfrak{M}}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$ in the case when the spoiler chooses $(\mathfrak{M}', w')$ and again select $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$. In particular these structures satisfy (‡‡). Moreover, the two forests $\widehat{\mathfrak{M}_1}$ and $\widehat{\mathfrak{M}_2}$ are such that $\widehat{\mathfrak{M}_1} + \widehat{\mathfrak{M}_2} = \widehat{\mathfrak{M}}$ and therefore $\widehat{R_1} \cup \widehat{R_2} = \widehat{R} = R \setminus \{(w_1, w_2)\}$ where $(w_1, w_2) \in R$ and $w_1 \in R(w)_{=1}$. We distinguish two cases.

- If $w_1 \in \widehat{R_1}(w)$ then in the original game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$, the duplicator replies to $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$ with the two structures $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $R_1 = \widehat{R_1}$ and $R_2 = \widehat{R_2} \cup \{(w_1, w_2)\}$.
- Otherwise $w_1 \in \widehat{R_2}(w)$ and in the game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$ the duplicator replies to $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$ with the two structures $\mathfrak{M}_1 = (W, R_1, V)$ and $\mathfrak{M}_2 = (W, R_2, V)$ such that $R_1 = \widehat{R_1} \cup \{(w_1, w_2)\}$ and $R_2 = \widehat{R_2}$.

In both cases, as the pair (w, w_1) is in one relation between R_1 and R_2 whereas (w_1, w_2) is in the other relation, the world w_1 effectively behaves as if it was a member of the set $R(w)_{=0}$ instead of $R(w)_{=1}$, exactly as in the case of $\widehat{R}$. In particular, it is easy to see that for $i \in \{1,2\}$:

$$|R_i(w)_{=0}| = |\widehat{R}_i(w)_{=0}| \qquad |R_i(w)_{=1}| = |\widehat{R}_i(w)_{=1}| \qquad |R_i(w)_{=2}| = |\widehat{R}_i(w)_{=2}|$$

Hence, by (‡‡) we have

- $\min(|R_i(w)_{=0}|, 2^{s-1}) = \min(|R'_i(w')_{=0}|, 2^{s-1})$;
- $\min(|R_i(w)_{=1}|, 2^{s-1}s) = \min(|R'_i(w')_{=1}|, 2^{s-1}s)$;
- $\min(|R_i(w)_{=2}|, 2^{s-2}s(s+1)) = \min(|R'_i(w')_{=2}|, 2^{s-2}s(s+1))$.

Moreover, $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$ all satisfy I, II and III (as they are submodels of $\mathfrak{M}$ or $\mathfrak{M}'$), we can apply (A) and conclude that $(\mathfrak{M}_1, w) \approx^{\mathsf{P}}_{m,s-1} (\mathfrak{M}'_1, w')$ and $(\mathfrak{M}_2, w) \approx^{\mathsf{P}}_{m,s-1} (\mathfrak{M}'_2, w')$. Therefore, the play we just described leads to a winning strategy for the duplicator on the game $((\mathfrak{M}, w), (\mathfrak{M}', w'), (m, s, \mathsf{P}))$, under the hypothesis that the spoiler chooses $(\mathfrak{M}', w')$.

As we constructed a strategy for the duplicator in both cases where the spoiler picks $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$, we have that $(\mathfrak{M}, w) \approx^{\mathsf{P}}_{m,s} (\mathfrak{M}', w')$ and therefore (B) holds.