TECHNISCHE UNIVERSITÄT DRESDEN

Computational Logic Group

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Cooperative Games

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Previously ...

- **General Game Playing** is concerned with computers learning to play previously unknown games without human intervention.
- The **game description language** (GDL) is used to declaratively specify (deterministic) games (with complete information about game states).
- The syntax of GDL game descriptions is that of **normal logic programs**; various restrictions apply to obtain a finite, unique interpretation.
- The semantics of GDL is given through a state transition system.
- GDL-II allows to represent moves by `Nature` and information sets.
- The semantics of GDL-II can be given through extensive-form games.
- Conversely, GDL-II can express any finite extensive-form game.

Written Exam

27th July 2023, 07:30–08:30hrs BAR/SCHOE/E

Overview

Cooperative Games with Transferable Utility

Solution Concept: The Core

Solution Concept: Stable Sets

Cooperative Games: Motivation

- In a noncooperative game, players cannot enter binding agreements.
- (Players can still cooperate if it pays off for them.)
- In a cooperative game, players form coalitions.
- The coalition gets some (overall) payoff, which is then to be distributed among the coalition's members.
- Players are still assumed to be rationally maximising their individual payoffs.

Example: Hospitals and X-Ray Machines

- Three hospitals (in the same city) are planning to buy x-ray machines.
- However, not every hospital necessarily needs its own machine.
- The smallest machine costs \$5m and could cover the needs of any two hospitals.
- A larger machine costs \$9m and could cover the needs of all three hospitals.
- Hospitals forming a coalition C can jointly save the difference to each individual hospital $i \in C$ buying its own \$5m machine.
- It is in society's interest to save money while covering patients' needs.

What should the hospitals do?

Cooperative Games with Transferable Utility

Cooperative Games with Transferable Utility

Definition

A **cooperative game with transferable utility** is a pair $G = (P, v)$ where

- $P = \{1, 2, \ldots, n\}$ is the set of players and
- $v\colon 2^P \to \mathbb{R}_{\geq 0}$ is the **characteristic function** of G.

- Intuition: Coalition $C \subseteq P$ earns $v(C)$ by cooperating.
- Terminology: We will occasionally omit "with transferable utility".

Assumption

For any cooperative game $G = (P, v)$, we have:

1. Normalisation: $v(\emptyset) = 0$.
2. Monotonicity: $C \subseteq D \subseteq P$ implies $v(C) \leq v(D)$.

Note that a cooperative game with n players requires a representation of a size that is exponential in n.

Cooperative Games: Example

Hospitals and X-Ray Machines

Three hospitals are planning to buy x-ray machines. However, not every hospital necessarily needs its own machine. A small machine costs \$5m and could cover the needs of any two hospitals. A larger machine costs \$9m and could cover the needs of all three hospitals. Hospitals forming a coalition C can jointly save the difference to each individual hospital $i \in C$ buying its own \$5m machine.

- $P = \{1, 2, 3\}$,
- $v(P) = 6$,
- $v(C) = 5$ for $|C| = 2$,
- $v(\{i\}) = 0$ for $i \in P$.

Coalition Structure

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility).
A **coalition structure** for G is a partition $\mathcal{C} = \{C_1, \ldots, C_k\}$ of P, that is,

- $C_1, \ldots, C_k \subseteq P$,
- $C_1 \cup \ldots \cup C_k = P$, and
- $C_i \cap C_j = \emptyset$ for all $1 \leq i \neq j \leq k$.

- The coalition structure $\mathcal{C} = \{P\}$ is called the **grand coalition**.
- $v(C)$ is the collective payoff of a coalition; it remains to be specified how to distribute the gains to the coalition's members.

Hospitals and X-Ray Machines

For $P = \{1, 2, 3\}$, some possible coalition structures are $\mathcal{C}_1 = \{\{1, 2, 3\}\}$, $\mathcal{C}_2 = \{\{1, 3\}, \{2\}\}$, and $\mathcal{C}_3 = \{\{1\}, \{2\}, \{3\}\}$.

Outcome of a Cooperative Game

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility).
An **outcome** of $G = (P, v)$ is a pair $(\mathfrak{C}, \mathbf{a})$ where

- $\mathfrak{C}$ is a coalition structure and
- $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is a payoff vector such that $a_i \geq 0$ for each $i \in P$ and

$$\sum_{i \in C} a_i = v(C) \quad \text{for each coalition } C \in \mathfrak{C}.$$

Efficiency: For each coalition $C \in \mathfrak{C}$, its payoff $v(C)$ is distributed completely.
Transferable Utility: Players within coalitions can transfer payoffs freely.

Hospitals and X-Ray Machines

Outcomes are $\mathfrak{C}_1$ with $\mathbf{a}_1 = (2, 2, 2)$, $\mathfrak{C}_2$ with $\mathbf{a}_2 = (2.5, 0, 2.5)$, and $\mathfrak{C}_3$ with $\mathbf{a}_3 = (0, 0, 0)$, but also $\mathfrak{C}_2$ with $\mathbf{a}_2' = (3, 0, 2)$. No outcome: $\mathfrak{C}_2$ with $(2, 1, 2)$.

Superadditive Games (1)

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility).
G is called **superadditive** iff for all coalitions $C, D \subseteq P$

$$C \cap D = \emptyset \quad \text{implies} \quad v(C \cup D) \geq v(C) + v(D).$$

Intuition: $C \cup D$ can achieve what C and D can achieve separately; there might be additional synergistic effects.

Non-Example

- A group C of emacs-using programmers achieves a part of a task T in $8h$.
- A (disjoint) group D of vi-using programmers achieves the rest of T in $8h$.
- The group $C \cup D$, attempting to work together, might not achieve T in $8h$.

We will only consider superadditive games unless specified otherwise.

Superadditive Games (2)

Observation

Let $G = (P, v)$ be a superadditive (cooperative) game.
For every coalition structure $\mathcal{C} = \{C_1, \ldots, C_k\}$, we have

$$v(P) \geq v(C_1) + \ldots + v(C_k)$$

$\rightsquigarrow$ In superadditive games, we can expect the grand coalition to form.

However, it does not automatically mean that the grand coalition is "stable":

Example

- The "Hospitals and X-Ray Machines" game is superadditive.
- In outcome $(\{\{1, 2, 3\}\}, (2, 2, 2))$, e.g. $\{1, 2\}$ have an incentive to deviate:
- in $(\{\{1, 2\}, \{3\}\}, (2.5, 2.5, 0))$, they would increase their individual payoff.

$\rightsquigarrow$ It remains to analyse how to distribute the grand coalition's payoff.

Solution Concept: The Core

Imputations

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility).

- A payoff vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is **individually rational** iff
$$a_i \geq v(\{i\}) \quad \text{for all } i \in P$$
- The **imputations for** G are the members of the following set:

$$Imp(G) := \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} a_i = v(P) \text{ and } a_i \geq v(\{i\}) \text{ for all } i \in P \right\}$$

Intuition: Imputations are efficient (w.r.t. to $\{P\}$) and individually rational.

Observation

1. $Imp(G) \neq \emptyset$ iff $v(P) \geq \sum_{i \in P} v(\{i\})$.
2. If G is superadditive, then $Imp(G) \neq \emptyset$.

The Core of a Cooperative Game

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility).
The **core of** G is the following set:

$$Core(G) := \left\{ (a_1, \ldots, a_n) \in Imp(G) \,\middle|\, \sum_{i \in C} a_i \geq v(C) \text{ for all coalitions } C \subseteq P \right\}$$

Intuition: No group C has an incentive to break off the grand coalition.

Example

In "Hospitals and X-Ray Machines", the core is empty:

- If $(a_1, a_2, a_3) \in Core(G)$, then $a_1 + a_2 + a_3 = 6$ by being an imputation.
- But for any $i, j \in \{1, 2, 3\}$ with $i \neq j$ we also have $a_i + a_j \geq v(\{a_i, a_j\}) = 5$.
- Let $a_i \leq a_j \leq a_k$, then $a_i + a_j \geq 5$, but $a_k \leq 1$ and $a_i + a_j \leq 2$, contradiction.

Cores of Cooperative Games: Example (1)

Chess Pairings

A group of $n \geq 3$ people want to play chess. Every pair of players appointed to play against each other receives \$1.

$$P = \{1, \ldots, n\}$$

$$v(C) = \begin{cases} \frac{|C|}{2} & \text{if } |C| \text{ is even,} \\ \frac{|C|-1}{2} & \text{otherwise} \end{cases}$$

- For $n \geq 4$ even, the payoff vector $\mathbf{a}_n := \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ is in the core:
  - deviation by an odd group $C \subseteq P$ would yield $v(C) = \frac{|C|-1}{2} < \frac{1}{2} \cdot |C|$;
  - deviation by an even group $C \subseteq P$ would yield $v(C) = \frac{|C|}{2} = \frac{1}{2} \cdot |C|$.
- In fact, for $n \geq 4$ even, we have $Core(G) = \{\mathbf{a}_n\}$:
  - Assume $\mathbf{a} \in Core(G)$, then for any $\{a_i, a_j\} \subseteq P$, it follows that $a_i + a_j \geq v(C) = 1$.
  - From $\mathbf{a} \in Imp(G)$, we get $a_1 + \ldots + a_n = \frac{n}{2}$, and we obtain $a_i = \frac{1}{2}$ for all $i \in P$.
- For $n \geq 3$ odd, the core is empty: (One player remains without a partner.)
  - For $n = 3$ and $\mathbf{a} \in Core(G)$, we get $a_1 + a_2 + a_3 = 1$, so e.g. $a_1 > 0$.
  - But then $a_2 + a_3 = 1 - a_1 < 1$ although $v(\{a_2, a_3\}) = 1$, contradicting $\mathbf{a} \in Core(G)$.

Cores of Cooperative Games: Example (2)

Shoe Makers

Of 201 shoe makers, (the first) 100 have made one left shoe each, (the remaining) 101 have made one right shoe each. A pair of shoes consists of one left and one right shoe (ignoring sizes), and can be sold for \$10.

$$P = \{1, 2, \ldots, 201\}$$
$$v(C) = 10 \cdot \min\{|C_L|, |C_R|\}$$
where
$$C_L := \{c \in C \mid c \leq 100\}$$
$$C_R := \{c \in C \mid c \geq 101\}$$

- The grand coalition makes a total of \$1000 from selling all 100 pairs.
- The core of this game contains as only imputation $\mathbf{a} = (a_1, a_2, \ldots, a_{201})$ with $a_1 = a_2 = \ldots = a_{100} = 10$ and $a_{101} = a_{102} = \ldots = a_{201} = 0$:
- For any imputation $\mathbf{b}$ with $b_i > 0$ for some $101 \leq i \leq 201$, the coalition $P \setminus \{i\}$ would obtain $v(P \setminus \{i\}) = v(P) > \sum_{j \in C, j \neq i} b_j$ on their own.
- Intuitively: Left shoes are scarce, right shoes are overabundant.

Linear Programming (in a Nutshell)

Definition

- A **linear program** is of the form

$$\begin{aligned} \text{maximise} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq 0, \\ \text{and} \quad & \mathbf{x} \in \mathbb{R}^k \end{aligned}$$

  where $\mathbf{x}$ is a vector of **decision variables**, and $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ are a matrix and two vectors of real values; the expression $\mathbf{c}^T\mathbf{x}$ is the **objective function**.
- If there is no objective function the program is a **feasibility problem**.
- A **solution** is a variable-value assignment that satisfies all constraints.

- A linear program is a special case of a mixed integer program (Lecture 2).
- Linear programming problems can be solved in polynomial time.

Computing the Core

For a given cooperative game $G = (P, v)$, its core is given by the feasible region of the following linear program over variables $a_1, \ldots, a_n$:

$$
\begin{array}{lll}
\text{find} & a_1, \ldots, a_n & \\
\text{subject to} & a_i \geq 0 & \text{for all } i \in P \\
 & \sum_{i \in P} a_i = v(P) & \\
 & \sum_{i \in C} a_i \geq v(C) & \text{for all } C \subseteq P
\end{array}
$$

Observe: The problem specification contains $2^n + n + 1$ constraints.

Corollary

For a cooperative game $G = (P, v)$ whose characteristic function v is explicitly represented, its core can be computed in deterministic polynomial time.

The ε-Core

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility) and $\varepsilon \in \mathbb{R}$.

1. The set of **pre-imputations of** G is

$$PreImp(G) := \left\{(a_1, \ldots, a_n) \in \mathbb{R}^n \;\middle|\; \textstyle\sum_{i \in P} a_i = v(P)\right\}$$

2. The ε**-core of** G is the following set:

$$\varepsilon\text{-}Core(G) := \left\{(a_1, \ldots, a_n) \in PreImp(G) \;\middle|\; \sum_{i \in C} a_i \geq v(C) - \varepsilon \text{ for all } C \subseteq P\right\}$$

- Intuition: Coalitions $C \subsetneq P$ that leave P have to pay a penalty of at least ε.
- For $\varepsilon = 0$, we have $0\text{-}Core(G) = Core(G)$.
- If $Core(G) = \emptyset$, then there is some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, for which $\varepsilon\text{-}Core(G) \neq \emptyset$.
- If $Core(G) \neq \emptyset$, then there is some $\varepsilon \in \mathbb{R}$, $\varepsilon < 0$, for which $\varepsilon\text{-}Core(G) = \emptyset$.

The Least Core

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility).
The **least core of** G is the intersection of all non-empty ε-cores of G.

Alternatively: The least core of G is $\tilde{\varepsilon}$-$Core(G)$ for $\tilde{\varepsilon} \in \mathbb{R}$ such that $\tilde{\varepsilon}$-$Core(G) \neq \emptyset$ and ε-$Core(G) = \emptyset$ for all $\varepsilon < \tilde{\varepsilon}$.

The value of the least core can be computed via linear programming:

$$
\begin{array}{lll}
\text{minimise} & \varepsilon & \\
\text{subject to} & a_i \geq 0 & \text{for all } i \in P \\
& \sum_{i \in P} a_i = v(P) & \\
& \sum_{i \in C} a_i \geq v(C) - \varepsilon & \text{for all } C \subseteq P
\end{array}
$$

The Cost of Stability

Idea: If $Core(G) = \emptyset$, stabilise G by subsidising the grand coalition.

Modelling Assumptions

- Some external authority has an interest in a stable grand coalition.
- The supplemental payment y gets distributed among P along with $v(P)$.

Definition

Let $G = (P, v)$ be a cooperative game (with transferable utility).

1. For a supplemental payment $y \geq 0$, the **adjusted game** $G_y = (P, v')$ has

$$v'(C) := \begin{cases} v(P) + y & \text{if } C = P, \\ v(C) & \text{otherwise.} \end{cases}$$

2. The **cost of stability of** G is $\inf \left\{ y \in \mathbb{R} \;\middle|\; y \geq 0 \text{ and } Core(G_y) \neq \emptyset \right\}$.

Computing the Cost of Stability

Example: Hospitals and X-Ray Machines

The cost of stability is $\gamma = 1.5$: In G_γ, we have $v'(\{1,2,3\}) = 6 + 1.5 = 7.5$ whence for no $C \subseteq \{1,2,3\}$ with $|C| = 2$ it would pay to deviate (as $v'(C) = 5$).

The cost of stability can be computed by linear programming:

$$
\begin{array}{lll}
\text{minimise} & \gamma & \\
\text{subject to} & \gamma \geq 0 & \\
 & a_i \geq 0 & \text{for all } i \in P \\
 & \sum_{i \in P} a_i = v(P) + \gamma & \\
 & \sum_{i \in C} a_i \geq v(C) & \text{for all } C \subseteq P
\end{array}
$$

Least Core vs. Cost of Stability

Observation

For any cooperative game G, the following are equivalent:

1. $Core(G) = \emptyset$.
2. The value ε of the least core is strictly positive.
3. The cost γ of stability is strictly positive.

What is the relationship between the values ε and γ?

- Least core: Punish undesired behaviour
  $\rightsquigarrow$ a fine for leaving the grand coalition.
- Cost of stability: Encourage desired behaviour
  $\rightsquigarrow$ a subsidy for staying in the grand coalition.

Least Core v. Cost of Stability: Examples

Let $n \geq 2$ and consider the following two games (i.e. where $P = \{1, \ldots, n\}$):

$G_1 = (P, v_1)$

$$v_1(C) = \begin{cases} n-1 & \text{if } C \cap \{1,2\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

$G_2 = (P, v_2)$

$$v_2(C) = \begin{cases} 1 & \text{if } C \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

$G_3 = (P, v_3)$

$$v_3(C) = \begin{cases} \frac{2n-2}{n} & \text{if } C \cap \{1,2\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

- In both games G_1 and G_2, the core is empty.
- The cost of stability in both games is $y = n - 1$:

$$\mathbf{a}_1 = (n-1, n-1, 0, \ldots, 0) \quad \text{vs.} \quad \mathbf{a}_2 = (1, 1, 1, \ldots, 1)$$

- The value of the least core in G_1 is $\varepsilon_1 = \frac{n-1}{2}$, via $\left(\frac{n-1}{2}, \frac{n-1}{2}, 0, \ldots, 0\right)$.
- The value of the least core in G_2 is $\varepsilon_2 = \frac{n-1}{n}$, via $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$.
- For G_3, we have $\varepsilon_3 = \frac{n-1}{n}$ via $\mathbf{a}_3 = \left(\frac{n-1}{n}, \frac{n-1}{n}, 0, \ldots, 0\right)$ and $y_3 = \frac{2n-2}{n}$ via $\mathbf{a}'_3 = \left(\frac{2n-2}{n}, \frac{2n-2}{n}, 0, \ldots, 0\right)$.

Solution Concept: Stable Sets

Stable Sets

Definition [von Neumann and Morgenstern, 1941]

Let $G = (P, v)$ be a cooperative game, and let $\mathbf{a}$ and $\mathbf{b}$ be imputations for G.

- **a dominates b via a coalition** C with $\emptyset \subsetneq C \subseteq P$, written $\mathbf{a} \succ_C \mathbf{b}$, iff
  - $a_i < b_i$ for all $i \in C$, and
  - $\sum_{i \in C} a_i \leq v(C)$.
- **a dominates b**, written $\mathbf{a} \succ \mathbf{b}$, iff $\mathbf{a}$ dominates $\mathbf{b}$ via some coalition $C \subseteq P$.
- A set $S \subseteq Imp(G)$ of imputations is a **stable set of** G iff
  - Internal stability: For any two $\mathbf{a}, \mathbf{b} \in S$, we have $\mathbf{a} \not\succ \mathbf{b}$.
  - External stability: For every $\mathbf{b} \in Imp(G) \setminus S$, there is some $\mathbf{a} \in S$ with $\mathbf{a} \succ \mathbf{b}$.

- If $a_i > b_i$ for all $i \in C$, then every member of C is better off in $\mathbf{a}$ than in $\mathbf{b}$.
- If $\sum_{i \in C} a_i \leq v(C)$, then C can plausibly threaten to leave the grand coalition.
- Internal stability: No imputations need to be removed from S.
- External stability: No imputations can be added to S.

Stable Sets: Example

Recall Hospitals and X-Ray Machines with $P = \{1, 2, 3\}$ and

$$v(C) = \begin{cases} 6 & \text{if } C = P, \\ 5 & \text{if } |C| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$S = \{(1, x, 5 - x) \mid x \in [0, 5]\}$ is a stable set of $G = (P, v)$:

- Internal stability:
  - Consider $(1, x, 5 - x) \in S$ and $(1, y, 5 - y) \in S$.
  - If $x > y$, then $5 - x < 5 - y$, thus $(1, x, 5 - x) \not\succ_{\{2,3\}} (1, y, 5 - y)$.
- External stability:
  - Consider $\mathbf{b} = (b_1, b_2, b_3) \in Imp(G) \setminus S$. Then $b_1 + b_2 + b_3 = 6$ and $b_1 \neq 1$.
  - If $b_1 < 1$, then $\min\{b_2, b_3\} \leq 3$ whence $(1, 4, 1) \succ_{\{1,2\}} \mathbf{b}$ or $(1, 1, 4) \succ_{\{1,3\}} \mathbf{b}$.
  - If $b_1 > 1$, then $b_2 + b_3 < 5$, whence we can choose $\mathbf{a} \in S$ such that $\mathbf{a} \succ_{\{2,3\}} \mathbf{b}$.

The Core vs. Stable Sets (1)

Proposition

Let $G = (P, v)$ be a cooperative game.

1. $Core(G)$ is contained in every (if any) stable set of G.
2. If $Core(G)$ is a stable set of G, then it is the only stable set of G.

Proof.

1. – Let $\mathbf{a} \in Core(G)$ and $\mathbf{b} \in Imp(G)$.
   – Assume (for contradiction) that for some $C \subseteq P$, we have $\mathbf{b} \succ_C \mathbf{a}$.
   – Then $a_i > b_i$ for all $i \in C$ and $\sum_{i \in C} b_i \leq v(C)$.
   – But then $\sum_{i \in C} a_i < \sum_{i \in C} b_i \leq v(C)$.
   – But $\mathbf{a} \in Core(G)$ means that $\sum_{i \in C} a_i \geq v(C)$. Contradiction.
   – Thus $\mathbf{b} \not\succ \mathbf{a}$ and $\mathbf{a}$ is contained in every (if any) stable set of G.
2. – No stable set can be a proper subset of another stable set:
   – If $S_1 \subsetneq S_2$ and both are stable then $\mathbf{b} \in S_2 \setminus S_1$ is dominated by some $\mathbf{a} \in S_1$.
   – But then $\mathbf{a} \in S_2$ and S_2 does not satisfy internal stability, contradiction. $\square$

The Core vs. Stable Sets (2)

Proposition

For any superadditive cooperative game $G = (P, v)$, we have
$Core(G) = \{\mathbf{a} \in Imp(G) \mid \text{there is no } \mathbf{b} \in Imp(G) \text{ with } \mathbf{b} \succ \mathbf{a}\}$.

Proof.

- Direction $\subseteq$ follows from the previous slide, so it remains to show $\supseteq$.
- Let $\mathbf{b} \in Imp(G) \setminus Core(G)$. Then $\sum_{i \in P} b_i = v(P)$ and $b_i \geq v(\{i\})$ for all $i \in P$.
- Since $\mathbf{b} \notin Core(G)$, there is a $C \subseteq P$ such that $v(C) > \sum_{i \in C} b_i$, whence $C \neq \emptyset$.
- Denote $\delta := v(C) - \sum_{i \in C} b_i$ and define $\mathbf{a} \in Imp(G)$ with $\mathbf{a} \succ_C \mathbf{b}$ by setting

$$a_i := \begin{cases} b_i + \frac{1}{|C|} \cdot \delta & \text{if } i \in C, \\ b_i - \frac{d_i}{\sum_{j \in P \setminus C} d_j} \cdot \delta & \text{otherwise,} \end{cases} \quad \text{where } d_j := b_j - v(\{j\}) \text{ for each } j \in P \setminus C.$$

- Note that $\sum_{j \in P \setminus C} d_j = \sum_{j \in P \setminus C} b_j - \sum_{j \in P \setminus C} v(\{j\}) \geq \delta = v(C) - \sum_{i \in C} b_i$ because v is superadditive: $\sum_{j \in P \setminus C} b_j + \sum_{i \in C} b_i = v(P) \geq v(C) + \sum_{j \in P \setminus C} v(\{j\})$. $\square$

The Core vs. Stable Sets: Example

G^1

$$P = \{1, 2, 3\}$$

$$v(C) = \begin{cases} 1 & \text{if } 1 \in C \text{ and } |C| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

- The core of G^1, $Core(G^1) = \{(1, 0, 0)\}$, is not a stable set of G:
- We have $(1, 0, 0) \not\succ (0, 0.5, 0.5)$ since $(1, 0, 0) \not\succ_{\{1\}} (0, 0.5, 0.5)$.

$\rightsquigarrow$ The core does not necessarily satisfy external stability.

- One stable set of G^1 is $S_{1,2} = \{(x, 1 - x, 0) \mid x \in [0, 1]\}$:
  - If $(x, 1 - x, 0), (y, 1 - y, 0) \in S_{1,2}$, then $x > y$ would imply $1 - x < 1 - y$.
  - If $(x, y, z) \in Imp(G^1)$ with $z > 0$, then $\left(x + \frac{z}{2}, y + \frac{z}{2}, 0\right) \succ_{\{1,2\}} (x, y, z)$.
- Likewise, $S_{1,3} = \{(x, 0, 1 - x) \mid x \in [0, 1]\}$ is a stable set of G^1.

Exercise: Find additional stable sets, if any.

Convex Games

Definition

1. A function $v\colon 2^P \to \mathbb{R}^+$ is **supermodular** iff for all $C, D \subseteq P$:

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$$

2. A cooperative game $G = (P, v)$ is **convex** iff v is supermodular.

Observation

Function $v\colon 2^P \to \mathbb{R}^+$ is supermodular iff for all $C \subseteq D \subseteq P$ and all $i \in P \setminus D$:

$$v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D) \tag{1}$$

where $v(C \cup \{i\}) - v(C)$ is player i's **marginal contribution** to coalition C.

- A supermodular function is superadditive (via $v(\emptyset) = 0$),
- but not vice versa.

Cores of Convex Games (1)

Theorem [Shapley, 1971]

Every convex game has a nonempty core.

Proof (1/2).

- Given $G = (P, v)$ with $P = \{1, \ldots, n\}$, we construct $\mathbf{a} = (a_1, \ldots, a_n) \in \mathit{Core}(G)$.
- Define $a_1 := v(\{1\})$, $a_2 := v(\{1,2\}) - v(\{1\})$, …, $a_n := v(P) - v(P \setminus \{n\})$.
- Payoff vector $\mathbf{a}$ is individually rational: For all $i \in P$, inequality (1) yields

$$a_i = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\}) \geq v(\{i\}) - v(\emptyset) = v(\{i\})$$

- $\mathbf{a}$ is also efficient:

$$\sum_{i \in P} a_i = v(\{1\}) + v(\{1,2\}) - v(\{1\}) + \ldots + v(P) - v(P \setminus \{n\}) = v(P)$$

- Thus $\mathbf{a} \in \mathit{Imp}(G)$. It remains to show $\mathbf{a} \in \mathit{Core}(G)$.

Cores of Convex Games (2)

Theorem [Shapley, 1971]

Every convex game has a nonempty core.

Proof (2/2).

- Consider any coalition $C = \{i, j, \ldots, k\}$ with $1 \leq i < j < \ldots < k \leq n$.
- We have $v(C) = v(\{i\}) - v(\emptyset) + v(\{i, j\}) - v(\{i\}) + \ldots + v(C) - v(C \setminus \{k\})$.
- Due to v being supermodular, inequality (1) yields

$$\begin{aligned} v(\{i\}) - v(\emptyset) &\leq v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\}) &= a_i \\ v(\{i, j\}) - v(\{i\}) &\leq v(\{1, \ldots, j\}) - v(\{1, \ldots, j-1\}) &= a_j \\ &\vdots \\ v(C) - v(C \setminus \{k\}) &\leq v(\{1, \ldots, k\}) - v(\{1, \ldots, k-1\}) &= a_k \end{aligned}$$

- Therefore $v(C) \leq a_i + a_j + \ldots + a_k$ and since C was arbitrary, $\mathbf{a} \in Core(G)$. □

Every convex game $G = (P, v)$ also has a unique stable set $S = Core(G)$.

Reprise: Solution Concepts

We have seen the following solution concepts for cooperative games:

- core [Gillies, 1959]
  - A unique set of imputations, but may be empty.
- ε-core [Shapley and Shubik, 1966]
  - A unique set of imputations, (non-)empty depending on $\varepsilon \in \mathbb{R}$.
- stable sets [von Neumann and Morgenstern, 1941] (called "solutions")
  - There can be zero, one, or more stable sets; every stable set is non-empty.

There are further solution concepts for cooperative games:

- Shapley value [Shapley, 1953]
  - A unique payoff vector that is efficient, symmetric, and additive.
  - For superadditive games, it is also individually rational (thus an imputation).
- kernel [Davis and Maschler, 1965]
  - A set of imputations stating that no player has "bargaining power" over another.
- nucleolus [Schmeidler, 1969]
  - A unique payoff vector that is contained in both core and kernel.

Conclusion

Summary

- In **cooperative** games, players P form explicit **coalitions** $C \subseteq P$.
- Coalitions receive payoffs, which are distributed among its members.
- We concentrate on **superadditive** games, where disjoint coalitions can never decrease their payoffs by joining together.
- Of particular interest is the **grand coalition** $\{P\}$ and whether it is *stable*.
- An **imputation** is an outcome that is efficient and individually rational.
- Various solution concepts formalise stability of the grand coalition:
  - the **core** contains all imputations where no coalition has an incentive to leave;
  - the ε**-core** disincentivises leaving the grand coalition via a fine of ε;
  - the **cost of stability** incentivises staying in the grand coalition;
  - **stable sets** are sets of imputations that do not dominate each other and dominate every imputation not in the set.
- A **convex** game has a non-empty core that equals its unique stable set.