DATABASE THEORY

Lecture 5: Complexity of FO Query Answering (II)

Markus Krötzsch

Knowledge-Based Systems

TU Dresden, 25 April 2022

Review: Query Complexity

Query answering as decision problem
$\leadsto$ consider Boolean queries

Various notions of complexity:

- Combined complexity (complexity w.r.t. size of query and database instance)
- Data complexity (worst case complexity for any fixed query)
- Query complexity (worst case complexity for any fixed database instance)

Various common complexity classes:

$$\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSpace} \subseteq \mathrm{ExpTime}$$

Review: FO Combined Complexity

Theorem 4.1 The evaluation of FO queries is PSpace-complete with respect to combined complexity.

We have actually shown something stronger:

Theorem 4.2 The evaluation of FO queries is PSpace-complete with respect to query complexity.

This also holds true when restricting to domain-independent queries.

Data Complexity of FO Query Answering

The algorithm showed that FO query evaluation is in L
$\leadsto$ can we do any better?

Data Complexity of FO Query Answering

The algorithm showed that FO query evaluation is in L
$\leadsto$ can we do any better?

What could be better than L?

$$? \subseteq \mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \ldots$$

Data Complexity of FO Query Answering

The algorithm showed that FO query evaluation is in L
$\leadsto$ can we do any better?

What could be better than L?

$$? \subseteq \mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \ldots$$

$\leadsto$ we need to define circuit complexities first

Boolean Circuits

Definition 5.1: A Boolean circuit is a finite, directed, acyclic graph where

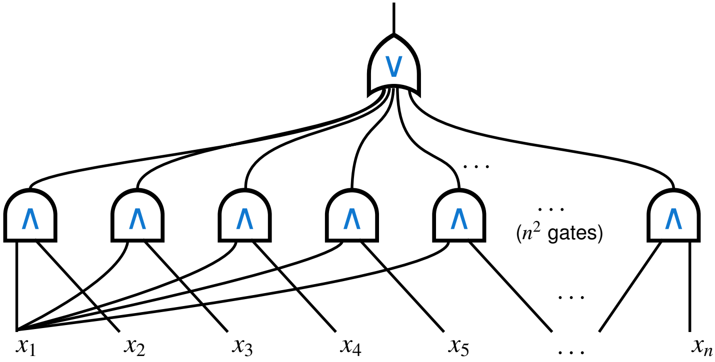
- each node that has no predecessors is an input node
- each node that is not an input node is one of the following types of logical gate: AND, OR, NOT
- one or more nodes are designated output nodes

$\leadsto$ we will only consider Boolean circuits with exactly one output

$\leadsto$ propositional logic formulae are Boolean circuits with one output and gates of fanout ≤ 1

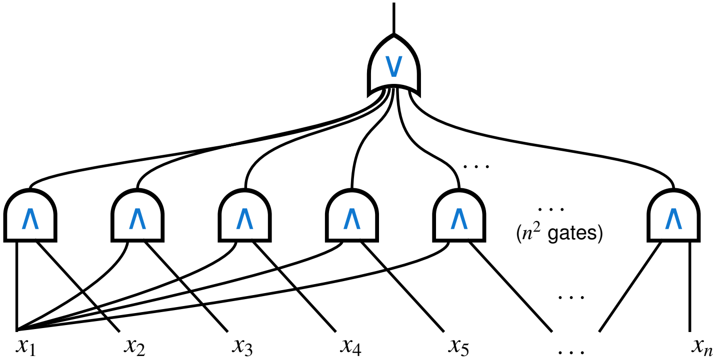
Example

A Boolean circuit over an input string $x_1 x_2 \ldots x_n$ of length n



Example

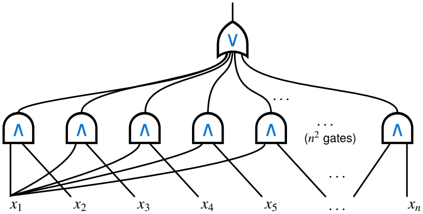
A Boolean circuit over an input string $x_1 x_2 \ldots x_n$ of length n



Corresponds to formula $(x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee \ldots \vee (x_{n-1} \wedge x_n)$
$\leadsto$ accepts all strings with at least two 1s

Circuits as a Model for Parallel Computation

Previous example:



$\leadsto$ n^2 processors working in parallel
$\leadsto$ computation finishes in 2 steps

- size: number of gates = total number of computing steps
- depth: longest path of gates = time for parallel computation

$\leadsto$ circuits as a refinement of polynomial time that takes parallelizability into account

Solving Problems With Circuits

Observation: the input size is "hard-wired" in circuits
$\leadsto$ each circuit only has a finite number of different inputs
$\leadsto$ not a computationally interesting problem

How can we solve interesting problems with Boolean circuits?

Solving Problems With Circuits

Observation: the input size is "hard-wired" in circuits
$\leadsto$ each circuit only has a finite number of different inputs
$\leadsto$ not a computationally interesting problem

How can we solve interesting problems with Boolean circuits?

Definition 5.2: A uniform family of Boolean circuits is a set of circuits C_n $(n \geq 0)$ that can easily[a] be computed from n.

A language $\mathcal{L} \subseteq \{0,1\}^*$ is decided by a uniform family $(C_n)_{n\geq 0}$ of Boolean circuits if for each word w of length $|w|$:

$$w \in \mathcal{L} \quad \text{if and only if} \quad C_{|w|}(w) = 1$$

[a] We don't discuss the details here; see course Complexity Theory.

Measuring Complexity with Boolean Circuits

How to measure the computing power of Boolean circuits?

Relevant metrics:

- size of the circuit: overall number of gates (as function of input size)
- depth of the circuit: longest path of gates (as function of input size)
- fan in: two inputs per gate or any number of inputs per gate?

Measuring Complexity with Boolean Circuits

How to measure the computing power of Boolean circuits?

Relevant metrics:

- size of the circuit: overall number of gates (as function of input size)
- depth of the circuit: longest path of gates (as function of input size)
- fan in: two inputs per gate or any number of inputs per gate?

Important classes of circuits: small-depth circuits

Definition 5.3: $(C_n)_{n\geq 0}$ is a family of small-depth circuits if

- the size of C_n is polynomial in n,
- the depth of C_n is poly-logarithmic in n, that is, $O(\log^k n)$.

The Complexity Classes NC and AC

Two important types of small-depth circuits:

Definition 5.4: NC^k is the class of problems that can be solved by uniform families of circuits $(C_n)_{n\geq 0}$ of fan-in ≤ 2, size polynomial in n, and depth in $O(\log^k n)$.

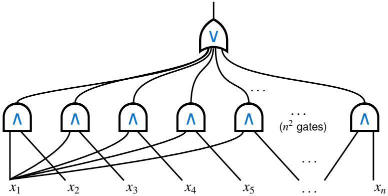
The class NC is defined as $\mathsf{NC} = \bigcup_{k\geq 0} \mathsf{NC}^k$.

("Nick's Class" named after Nicholas Pippenger by Stephen Cook)

Definition 5.5: AC^k and AC are defined like NC^k and NC, respectively, but for circuits with arbitrary fan-in.

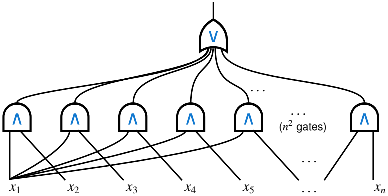
(A is for "Alternating": AND-OR gates alternate in such circuits)

Example



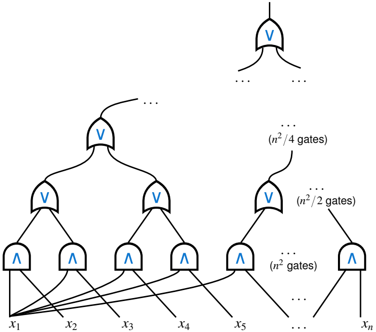
family of polynomial size,
constant depth,
arbitrary fan-in circuits
$\rightsquigarrow$ in AC^0

Example



family of polynomial size,
constant depth,
arbitrary fan-in circuits
$\leadsto$ in AC^0

We can eliminate arbitrary fan-ins by using more layers of gates:



family of polynomial size,
logarithmic depth,
bounded fan-in circuits
$\leadsto$ in NC^1

Relationships of Circuit Complexity Classes

The previous sketch can be generalised:

$$\mathsf{NC}^0 \subseteq \mathsf{AC}^0 \subseteq \mathsf{NC}^1 \subseteq \mathsf{AC}^1 \subseteq \ldots \subseteq \mathsf{AC}^k \subseteq \mathsf{NC}^{k+1} \subseteq \ldots$$

Only few inclusions are known to be proper: $\mathsf{NC}^0 \subset \mathsf{AC}^0 \subset \mathsf{NC}^1$

Relationships of Circuit Complexity Classes

The previous sketch can be generalised:

$$\mathsf{NC}^0 \subseteq \mathsf{AC}^0 \subseteq \mathsf{NC}^1 \subseteq \mathsf{AC}^1 \subseteq \ldots \subseteq \mathsf{AC}^k \subseteq \mathsf{NC}^{k+1} \subseteq \ldots$$

Only few inclusions are known to be proper: $\mathsf{NC}^0 \subset \mathsf{AC}^0 \subset \mathsf{NC}^1$

Direct consequence of above hierarchy: $\mathsf{NC} = \mathsf{AC}$

Interesting relations to other classes:

$$\mathsf{NC}^0 \subset \mathsf{AC}^0 \subset \mathsf{NC}^1 \subseteq \mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{AC}^1 \subseteq \ldots \subseteq \mathsf{NC} \subseteq \mathsf{P}$$

Intuition:

- Problems in NC are parallelisable (known from definition)
- Problems in $\mathsf{P} \setminus \mathsf{NC}$ are inherently sequential (educated guess)

However: it is not known if $\mathsf{NC} \neq \mathsf{P}$

Back to Databases . . .

Theorem 5.6: The evaluation of FO queries is complete for (logtime uniform) AC^0 with respect to data complexity.

Proof:

- Membership: For a fixed Boolean FO query, provide a uniform construction for a small-depth circuit based on the size of a database
- Hardness: Show that circuits can be transformed into Boolean FO queries in logarithmic time (not on a standard TM . . . not in this lecture)

From Query to Circuit

Assumptions:

- query and database schema is fixed
- database instance (and thus active domain) are variable

Construct circuit uniformly based on size of active domain

From Query to Circuit

Assumptions:

- query and database schema is fixed
- database instance (and thus active domain) are variable

Construct circuit uniformly based on size of active domain

Sketch of construction:

- one input node for each possible database tuple (over given schema and active domain)
  $\leadsto$ true or false depending on whether tuple is present or not
- Recursively, for each subformula, introduce a gate for each possible tuple (instantiation) of this formula
  $\leadsto$ true or false depending on whether the subformula holds for this tuple or not
- Logical operators correspond to gate types: basic operators obvious, $\forall$ as generalised conjunction, $\exists$ as generalised disjunction
- subformula with n free variables $\leadsto$ $|\textbf{adom}|^n$ gates
  $\leadsto$ especially: $|\textbf{adom}|^0 = 1$ output gate for Boolean query

Example

We consider the formula

$$\exists z.(\exists x.\exists y.R(x, y) \wedge S(y, z)) \wedge \neg R(a, z)$$

Over the database instance:

R:

a	a
a	b

S:

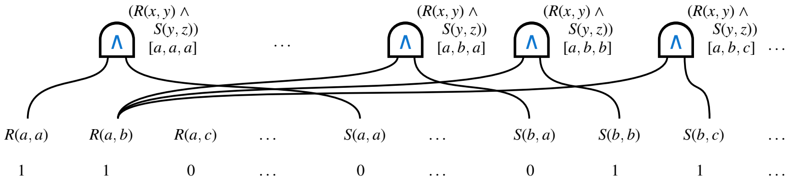
b	b
b	c

Active domain: $\{a, b, c\}$

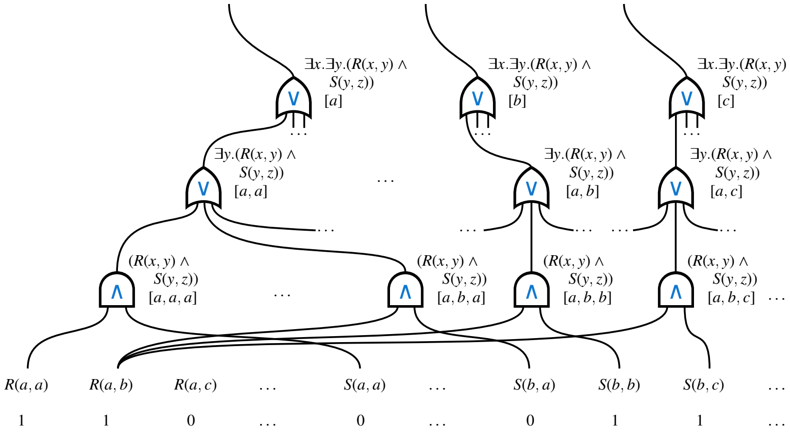
Example: $\exists z.(\exists x.\exists y.R(x, y) \wedge S(y, z)) \wedge \neg R(a, z)$

$R(a, a)$	$R(a, b)$	$R(a, c)$	$\ldots$	$S(a, a)$	$\ldots$	$S(b, a)$	$S(b, b)$	$S(b, c)$	$\ldots$
1	1	0	$\ldots$	0	$\ldots$	0	1	1	$\ldots$

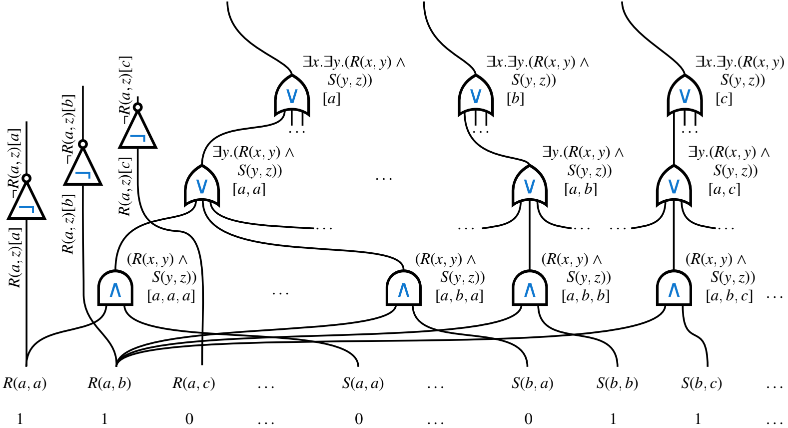
Example: $\exists z.(\exists x.\exists y.R(x, y) \wedge S(y, z)) \wedge \neg R(a, z)$



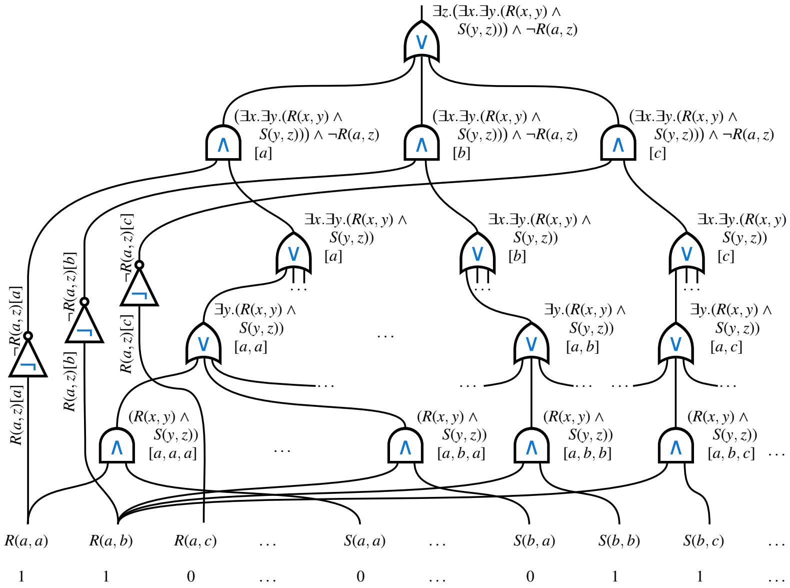
Example: $\exists z.(\exists x.\exists y.R(x,y) \wedge S(y,z)) \wedge \neg R(a,z)$



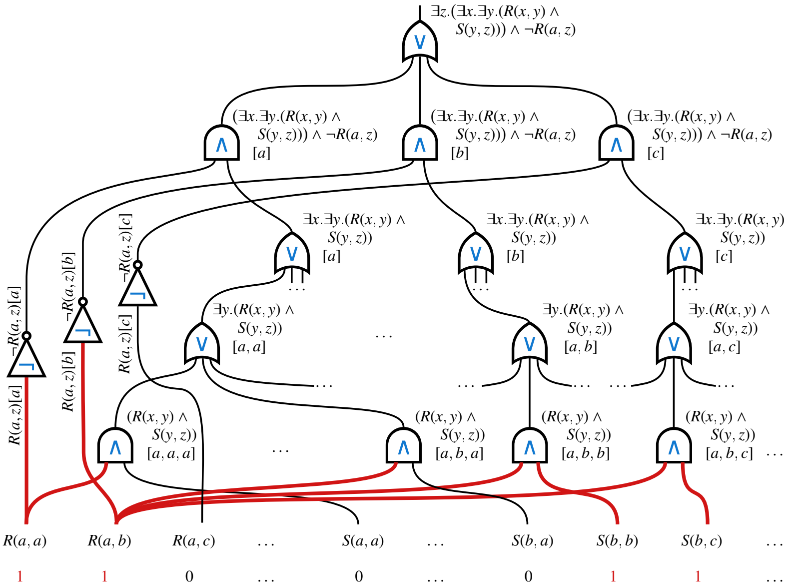
Example: $\exists z.(\exists x.\exists y.R(x,y) \wedge S(y,z)) \wedge \neg R(a,z)$



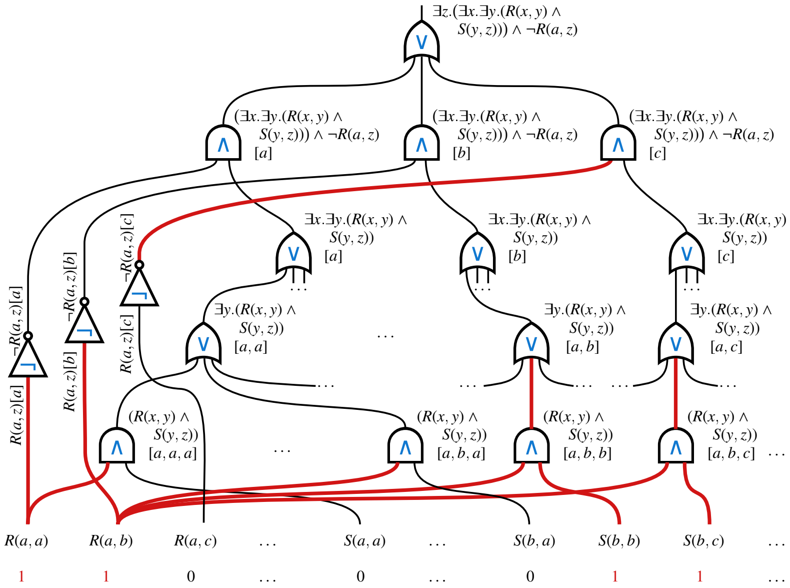
Example: $\exists z.(\exists x.\exists y.R(x,y) \land S(y,z)) \land \neg R(a,z)$



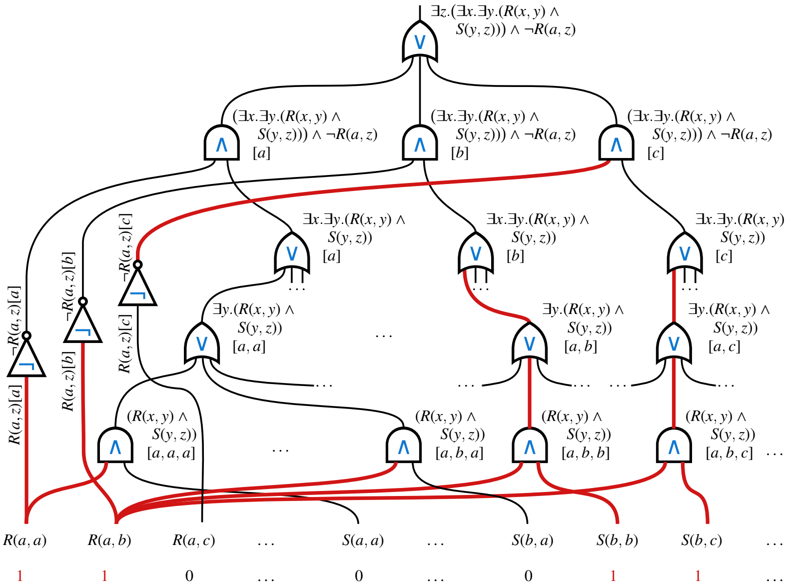
Example: $\exists z.(\exists x.\exists y.R(x,y) \wedge S(y,z)) \wedge \neg R(a,z)$



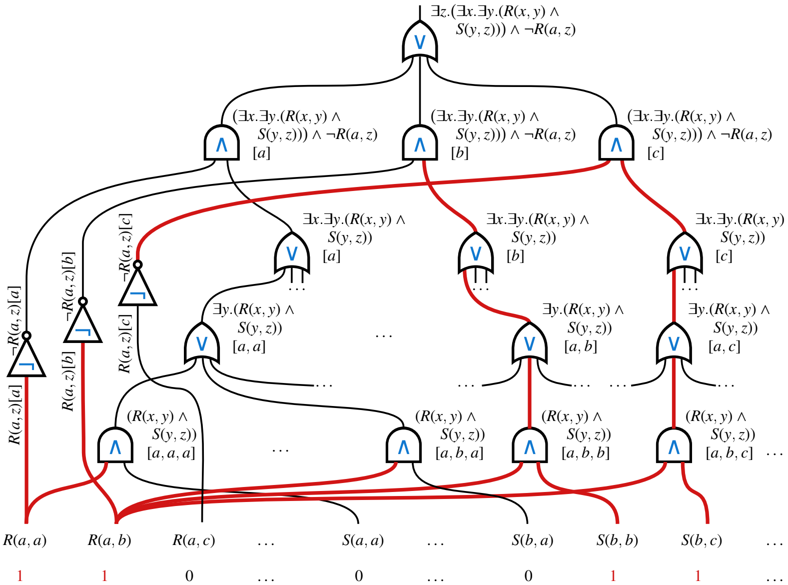
Example: $\exists z.(\exists x.\exists y.R(x,y) \land S(y,z)) \land \neg R(a,z)$



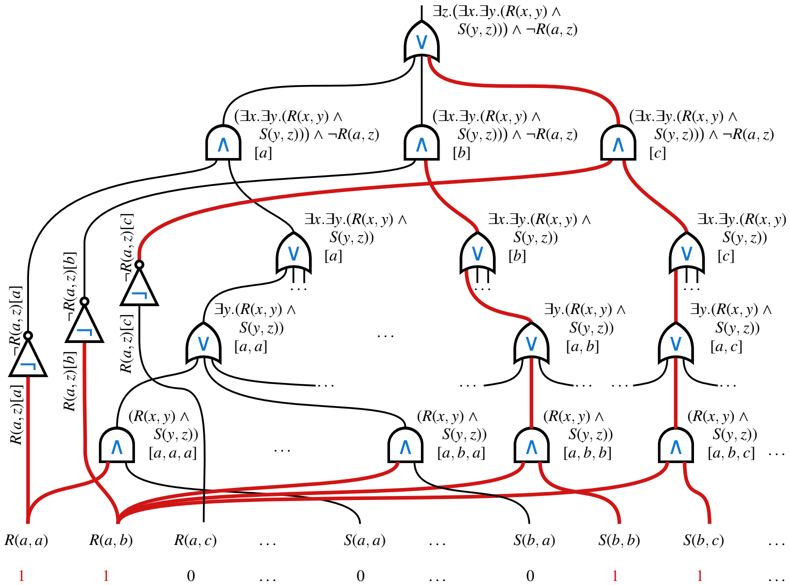
Example: $\exists z.(\exists x.\exists y.R(x,y) \land S(y,z)) \land \neg R(a,z)$



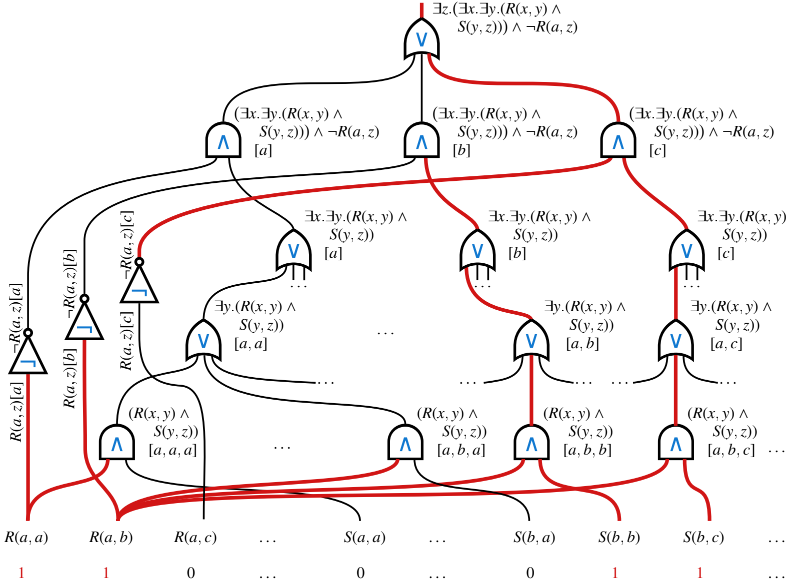
Example: $\exists z.(\exists x.\exists y.R(x,y) \land S(y,z)) \land \neg R(a,z)$



Example: $\exists z.(\exists x.\exists y.R(x,y) \wedge S(y,z)) \wedge \neg R(a,z)$



Example: $\exists z.(\exists x.\exists y.R(x,y) \wedge S(y,z)) \wedge \neg R(a,z)$



Summary and Outlook

The evaluation of FO queries is

- PSpace-complete for combined complexity
- PSpace-complete for query complexity
- AC^0-complete for data complexity

Circuit complexities help to identify highly parallelisable problems in P

Open questions:

- Are there query languages with lower complexities? (next lecture)
- Which other computing problems are interesting?
- How can we study the expressiveness of query languages?