The Complexity of Subsumption in Fuzzy $\mathcal{EL}$

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Abstract

Fuzzy Description Logics (DLs) are used to represent and reason about vague and imprecise knowledge that is inherent to many application domains. It was recently shown that the complexity of reasoning in finitely valued fuzzy DLs is often not higher than that of the underlying classical DL. We show that this does not hold for fuzzy extensions of the light-weight DL $\mathcal{EL}$, which is used in many biomedical ontologies, under the Łukasiewicz semantics. The complexity of reasoning increases from PTIME to EXPTIME, even if only one additional truth value is introduced. The same lower bound holds also for infinitely valued Łukasiewicz extensions of $\mathcal{EL}$.

1 Introduction

Description Logics (DLs) are a family of knowledge representation formalisms that are successfully applied in many application domains. They provide the logical foundation for the Direct Semantics of the standard web ontology language OWL 2.[1] The light-weight DL $\mathcal{EL}$, underlying the OWL 2 EL profile, is of particular interest since all common reasoning problems are polynomial in this logic, and it is used in many prominent biomedical ontologies like SNOMED CT[2] and the Gene Ontology.[3] Knowledge is represented by a set of *general concept inclusions (GCIs)* like

$$\exists \mathsf{hasDisease.Flu} \sqsubseteq \exists \mathsf{hasSymptom.Headache} \sqcap \exists \mathsf{hasSymptom.Fever} \quad (1)$$

which states that every patient with a flu must also show headache and fever as symptoms. Reasoning in $\mathcal{EL}$ is a polynomial problem [Baader *et al.*, 2005].

An important problem for AI practical applications is to represent and reason with vague or imprecise knowledge in a formal way. Fuzzy Description Logics (FDLs) [Straccia, 2001; Hájek, 2005] were introduced with this goal in mind. The main premise of fuzzy logics is the use of more than two truth degrees to allow a more fine-grained analysis of dependencies between concepts. Usually, these degrees are arranged in a totally ordered algebra, or *chain*, in the interval $[0, 1]$. A patient having a body temperature of $37.5\,°\mathrm{C}$ can have a degree of fever of 0.5, whereas a temperature of $39.2\,°\mathrm{C}$ may be interpreted as a fever with degree of 0.9. Considering the GCI (1), the severity of the symptoms certainly influences the severity of the disease, and thus truth degrees can be transferred between concepts. Depending on the granularity one wants to have, one can choose to allow 10 or 100 truth degrees, or even admit the whole interval $[0, 1]$. Another degree of freedom in FDLs comes from the choice of possible semantics for the logical constructors. The most general semantics are based on *triangular norms (t-norms)* that are used to interpret conjunctions. Among these, the most prominent ones are the *Gödel*, *Łukasiewicz*, and *product* t-norms. All (continuous) t-norms over chains can be expressed as combinations of these three basic ones.

Unfortunately, reasoning in many infinitely valued FDLs becomes undecidable [Baader and Peñaloza, 2011; Cerami and Straccia, 2013]. For a systematic study on this topic, see [Borgwardt *et al.*, 2015b]. On the other hand, every finitely valued FDL that has been recently studied has not only been proved to be decidable, but even to belong to the same complexity class as the corresponding classical DL [Borgwardt and Peñaloza, 2013; 2014; Bou *et al.*, 2012].

A question that naturally arises is whether the finitely valued fuzzy framework always yields the same computational complexity as the corresponding classical formalisms. A common opinion is that everything that can be expressed in finitely valued FDLs can be reduced to the corresponding classical DLs without any serious loss of efficiency. Indeed, although some known direct translations of finitely valued FDLs into classical DLs are exponential [Bobillo and Straccia, 2011], more efficient reasoning can be achieved through direct algorithms [Borgwardt and Peñaloza, 2013]. The problem of finding a complexity gap between classical and finitely valued logics has already been considered. In [Cerami and Straccia, 2014], the authors analyze different constructors that could cause an increase in the complexity, but no specific answer is found. In [Borgwardt *et al.*, 2014] it is shown that the Łukasiewicz t-norm is a source of nondeterminism able to cause a significant increase in expressivity in very simple propositional languages. In this work, we build on the meth-

[1] http://www.w3.org/TR/owl2-overview/
[2] http://www.ihtsdo.org/snomed-ct/
[3] http://geneontology.org/

ods of [Borgwardt *et al.*, 2014] to show even more dramatic increases in complexity for finitely valued extensions of $\mathcal{EL}$.

The question about the computational complexity of $\mathcal{EL}$ under infinitely valued semantics has been already considered. Borgwardt and Peñaloza show that reasoning in $\mathcal{EL}$ under semantics including the Łukasiewicz t-norm is co-NP-hard [2013], but the proof does not apply to the finitely valued case. In contrast, infinitely valued Gödel semantics do not increase the complexity of reasoning [Mailis *et al.*, 2012].

In this work, we prove that $\mathcal{EL}$ under finitely valued semantics is ExpTime-complete whenever the Łukasiewicz t-norm is included in the semantics. This proves a dichotomy similar to one that exists for infinitely valued FDLs [Borgwardt *et al.*, 2015b] since, for all other finitely valued chains of truth values, reasoning in fuzzy $\mathcal{EL}$ can be shown to be in PTime using the methods from [Mailis *et al.*, 2012]. The relevance of our result goes beyond the computational aspect. Indeed, this is so far the first instance of a finitely valued DL that is more complex than the same language under classical semantics. In this way, we obtain an answer to the open problem whether finitely valued FDLs and classical DLs are equally powerful, at least from a computational complexity point of view. As a side benefit, we obtain the same (ExpTime) lower bound for the complexity of infinitely valued fuzzy extensions of $\mathcal{EL}$ that use the Łukasiewicz t-norm, improving the lower bound from [Borgwardt and Peñaloza, 2013]. Full proofs of all results appear in the technical report [Borgwardt *et al.*, 2015a].

2 Preliminaries

Fuzzy DLs extend classical DLs by allowing more than two truth degrees in the semantics. We first introduce the classes of truth degrees relevant for this paper and then recall the logics $\mathcal{ELU}$ and L-$\mathcal{EL}$.

2.1 Chains of Truth Values

A *chain* is an algebra L whose elements are linearly ordered; that is, for every $x, y \in \mathsf{L}$ it holds that either $x \leqslant y$ or $y \leqslant x$. We consider here chains of the form $(\mathsf{L}, *_\mathsf{L}, \Rightarrow_\mathsf{L})$, where

- L is a subset of $[0,1]$ that contains 0 and 1;
- the *t-norm* $*_\mathsf{L}$ is a binary operator on L that is associative, commutative, monotone, and has 1 as unit element; and
- the *residuum* $\Rightarrow_\mathsf{L}$ of $*_\mathsf{L}$ is a binary operator on L that satisfies $(x *_\mathsf{L} y) \leqslant z$ iff $y \leqslant (x \Rightarrow_\mathsf{L} z)$ for all $x, y, z \in \mathsf{L}$.

An element $x \in \mathsf{L}$ is *idempotent* if $x *_\mathsf{L} x = x$. For ease of presentation, we will often identify L and $(\mathsf{L}, *_\mathsf{L}, \Rightarrow_\mathsf{L})$ and omit the subscript L if the chain we use is clear from the context.

We consider in particular the two cases where (i) L is defined over the interval $[0,1]$ of real numbers, or (ii) L is a *finite* chain. In the former case, we always make the assumption that the operator $*_\mathsf{L}$ is *continuous* as a function from $[0,1] \times [0,1]$ to $[0,1]$. One reason for this assumption is that it ensures that the residuum is uniquely determined by the t-norm [Klement *et al.*, 2000]. In case (ii), we similarly assume that $*_\mathsf{L}$ is *smooth*, i.e. for every $x, y, z \in \mathsf{L}$, whenever x and y are direct neighbors in L, with $x < y$, then there is no $w \in \mathsf{L}$ such that $x *_\mathsf{L} z < w < y *_\mathsf{L} z$ [Mayor and Torrens, 1993]. If $*_\mathsf{L}$ is continuous (smooth), then we call L *continuous (smooth)*.

Table 1: Basic t-norms and their residua

	t-norm $x * y$	residuum $x \Rightarrow y$
G_n/ G	$\min(x, y)$	$\begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{if } x > y \end{cases}$
$\mathsf{Ł}_n$/ $\mathsf{Ł}$	$\max(x + y - 1, 0)$	$\min(1 - x + y, 1)$
$\mathsf{\Pi}$	$x \cdot y$	$\begin{cases} 1 & \text{if } x \leqslant y \\ \frac{y}{x} & \text{if } x > y \end{cases}$

By restricting the algebra of truth values to two elements, the classical Boolean algebra of truth and falsity $\mathsf{B} = (\{0,1\}, *_\mathsf{B}, \Rightarrow_\mathsf{B})$ is obtained. In this case, $*_\mathsf{B}$ and $\Rightarrow_\mathsf{B}$ are the classical conjunction and the material implication, respectively.

The most interesting chains with continuous or smooth t-norms are the ones given by the Gödel (G), Łukasiewicz ($\mathsf{Ł}$), and product ($\mathsf{\Pi}$) t-norms. The finitely valued versions of the former two, denoted by $\mathsf{Ł}_n$ and G_n for $n \geqslant 2$, are defined over the n-element chain $0 < \frac{1}{n-1} < \cdots < \frac{n-2}{n-1} < 1$. These operators and their residua are defined in Table 1. Notice that a finitely valued version of the product t-norm $\mathsf{\Pi}$ cannot exist: the chain L needs to be closed under the t-norm, but for any $x \in (0,1)$, the set $\{x^m \mid m \geqslant 0\}$ is infinite.

The following properties, which are crucial for our reductions, follow directly from the previous definitions.

Fact 1. *For all $x, y \in \mathsf{L}$ and $T \subseteq \mathsf{L}$, it holds that*

- $x \Rightarrow_\mathsf{L} y = 1$ *iff* $x \leqslant y$*;*
- $1 \Rightarrow_\mathsf{L} x = x$*;*
- $x \Rightarrow_\mathsf{L} y \geqslant y$*;*
- *if* $\mathsf{L} = \mathsf{Ł}_n$*, then* $x *_{\mathsf{Ł}_n} y \geqslant \frac{n-2}{n-1}$ *iff either* $x = 1$ *or* $y = 1$*;*
- *if* $\mathsf{L} = \mathsf{Ł}_n$ *and* $x < 1$*, then for all* $m \geqslant n-1$ *we have* $\underbrace{x *_{\mathsf{Ł}_n} \dots *_{\mathsf{Ł}_n} x}_{m \text{ times}} = 0$.

The t-norms defined so far can be used to build all other continuous t-norms over $[0,1]$, and all smooth t-norms over finite chains, using the following construction.

Definition 2. Let L be a chain, $(\mathsf{L}_i)_{i \in I}$ be a family of chains, and $(\lambda_i)_{i \in I}$ be isomorphisms between intervals $[a_i, b_i] \subseteq \mathsf{L}$ and L_i such that the intersection of any two intervals contains at most one element. L is the *ordinal sum* of the family $(\mathsf{L}_i, \lambda_i)_{i \in I}$ if, for all $x, y \in \mathsf{L}$,

$$x *_\mathsf{L} y = \begin{cases} \lambda_i^{-1}\big(\lambda_i(x) *_{\mathsf{L}_i} \lambda_i(y)\big) & \text{if } x, y \in (a_i, b_i), \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Intuitively, the ordinal sum of the chains L_i is a chain whose domain is built up by appending the domains of the chains L_i and whose operation $*_\mathsf{L}$ is $*_{\mathsf{L}_i}$ when the operands belong to the same chain L_i and it is min otherwise.

Every chain over $[0,1]$ with a continuous t-norm is isomorphic to an ordinal sum of infinitely valued Łukasiewicz and product chains [Hájek, 2001; Mostert and Shields, 1957].

Similarly, every smooth finite chain is an ordinal sum of chains of the form $Ł_n$ with $n \geqslant 3$ [Mayor and Torrens, 2005]. All elements that are not contained strictly within one such Łukasiewicz or product component are idempotent and can be thought of as belonging to a (finite or infinite) Gödel chain. We say that a (finite or infinite) chain *contains* the Łukasiewicz t-norm if its ordinal sum representation contains at least one Łukasiewicz component; similarly, it *starts with* the Łukasiewicz t-norm if it contains a Łukasiewicz component in an interval $[0, b]$. Note that every chain that contains the Łukasiewicz t-norm can be represented as the ordinal sum of an arbitrary chain L_1 and another chain L_2 that starts with the Łukasiewicz t-norm.

Another way to view these characterizations is to observe that every smooth finite chain is either a Gödel chain or contains at least one Łukasiewicz component, and every continuous chain over $[0, 1]$ is either a Gödel chain or contains at least one Łukasiewicz or product component. As we will see, analyzing the properties of the basic t-norms from Table 1 provides an insight into the general case, with arbitrary (continuous or smooth) t-norms.

2.2 $\mathcal{ELU}$ and L-$\mathcal{EL}$

A *description signature* is a tuple $(\mathsf{N_C}, \mathsf{N_R})$, where $\mathsf{N_C}$ and $\mathsf{N_R}$ are disjoint countable sets of *concept names* and *role names*, respectively. $\mathcal{EL}$ *concepts* are built inductively from concept and role names through the grammar rule $C, D ::= A \mid \top \mid C \sqcap D \mid \exists r.C$ where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. $\mathcal{ELU}$ concepts are formed by adding the option $C \sqcup D$ to the previous rule. In the rest of the paper we will use the abbreviation C^m, $m \geqslant 1$, for the m-ary conjunction; i.e. $C^1 := C$ and $C^{m+1} := C^m \sqcap C$.

There is often no difference between the syntax of classical and fuzzy languages. The differences between both frameworks begin when the *semantics* of concepts and roles is introduced. As remarked in Section 2.1, it suffices to restrict the semantics to the two-element chain B to obtain the classical semantics. However, we define both semantics to aid understanding and readability of the proofs.

Fuzzy Semantics of L-$\mathcal{EL}$.

Consider an arbitrary but fixed chain $\mathsf{L} = (\mathsf{L}, *, \Rightarrow)$. An L-*interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of:

- a nonempty (classical) set $\Delta^\mathcal{I}$ (called *domain*), and
- a *fuzzy interpretation function* $\cdot^\mathcal{I}$ that assigns
  - to each $A \in \mathsf{N_C}$ a fuzzy set $A^\mathcal{I} \colon \Delta^\mathcal{I} \to \mathsf{L}$, and
  - to each $r \in \mathsf{N_R}$ a fuzzy relation $r^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to \mathsf{L}$.

The interpretation function is extended to $\mathcal{EL}$ concepts inductively by defining, for all $x \in \mathsf{L}$,

$$\begin{aligned} \top^\mathcal{I}(x) &:= 1, \\ (C \sqcap D)^\mathcal{I}(x) &:= C^\mathcal{I}(x) * D^\mathcal{I}(x), \\ (\exists r.C)^\mathcal{I}(x) &:= \sup_{y \in \Delta^\mathcal{I}} r^\mathcal{I}(x, y) * C^\mathcal{I}(y). \end{aligned}$$

Classical semantics of $\mathcal{ELU}$.

A *classical interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where

- $\Delta^\mathcal{I}$ is a nonempty (classical) set (called *domain*), and
- $\cdot^\mathcal{I}$ is an *interpretation function* that assigns:
  - to each $A \in \mathsf{N_C}$ a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and
  - to each $r \in \mathsf{N_R}$ a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

This function is extended to $\mathcal{ELU}$ concepts by setting

$$\begin{aligned} \top^\mathcal{I} &:= \Delta^\mathcal{I}, \\ (C \sqcap D)^\mathcal{I} &:= C^\mathcal{I} \cap D^\mathcal{I}, \\ (C \sqcup D)^\mathcal{I} &:= C^\mathcal{I} \cup D^\mathcal{I}, \\ (\exists r.C)^\mathcal{I} &:= \{x \in \Delta^\mathcal{I} \mid \exists\, y \in \Delta^\mathcal{I} \colon (x, y) \in r^\mathcal{I} \wedge y \in C^\mathcal{I}\}. \end{aligned}$$

Clearly, by replacing the relation $\in$ by its characteristic function $\chi_\in \colon \Delta^\mathcal{I} \to \{0, 1\}$, we obtain a special case of fuzzy semantics. Whenever L is one of the specific chains introduced in the previous section, e.g. $Ł_n$, then we denote the resulting logic by $Ł_n$-$\mathcal{EL}$ instead of L-$\mathcal{EL}$.

In infinite chains, interpretations are often restricted to be *witnessed* [Hájek, 2005], which means that for every existential restriction $\exists r.C$ and $x \in \Delta^\mathcal{I}$ there is an element $y \in \Delta^\mathcal{I}$ that realizes the supremum in the semantics of $\exists r.C$ at x, i.e. $(\exists r.C)^\mathcal{I}(x) = r^\mathcal{I}(x, y) * C^\mathcal{I}(y)$. Under finitely valued and classical semantics this property is always satisfied, and it corresponds to the intuition that an existential restriction actually forces the existence of a single domain element that satisfies it, instead of infinitely many that only satisfy the restriction in the limit. We also adopt the restriction to witnessed interpretations in what follows.

In DLs, the domain knowledge is represented by axioms that restrict the class of interpretations under consideration. In the fuzzy framework, these axioms are assigned a minimum degree of truth to which they must be satisfied. Graded *general concept inclusions (GCIs)* are expressions of the form $\langle C \sqsubseteq D \geqslant \ell \rangle$, where $\ell \in \mathsf{L}$. The L-interpretation $\mathcal{I}$ *satisfies* this axiom if $C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \geqslant \ell$ holds for all $x \in \Delta^\mathcal{I}$. As usual, a TBox is a finite set of GCIs, and an L-interpretation $\mathcal{I}$ *satisfies* a TBox if it satisfies every axiom in it.

We consider the problem of deciding whether a concept C is ℓ-*subsumed* by another concept D with respect to a TBox $\mathcal{T}$ for a value $\ell \in \mathsf{L} \setminus \{0\}$. That is, whether every L-interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$ also satisfies $\langle C \sqsubseteq D \geqslant \ell \rangle$. In the classical case, we talk simply about *subsumption*, and for $\ell = 1$ the problem simplifies to the question whether $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds in all interpretations $\mathcal{I}$ that satisfy $\mathcal{T}$.

In the particular case of G_n-$\mathcal{EL}$, subsumption is decidable in polynomial time. This can be shown by generalizing the proof from [Baader *et al.*, 2005] (for $n = 2$), as it was done in [Mailis *et al.*, 2012] for the infinitely valued G-$\mathcal{EL}$.

Proposition 3. *Deciding ℓ-subsumption with respect to a TBox in G_n-$\mathcal{EL}$ is* PTIME*-complete.*

In this paper, we show that for all other finite chains the subsumption problem becomes EXPTIME-complete. As a first step, we show that this problem is EXPTIME-hard for all finite Łukasiewicz chains with at least three elements, and then use this result in Section 4 to show EXPTIME-hardness

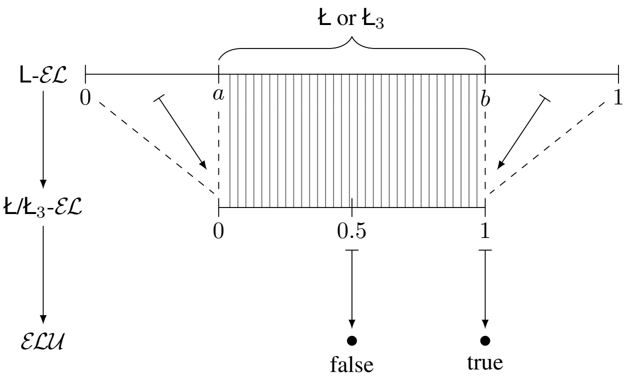

Figure 1: Illustration of the reductions

under any finite chain that contains a Łukasiewicz component, i.e. is not of the form G_n (see Definition 2). A matching EXPTIME upper bound was shown in [Borgwardt and Peñaloza, 2013]. In Section 5, we adapt our reduction to show EXPTIME-hardness of Ł-$\mathcal{EL}$, and even for every continuous chain over $[0,1]$ containing a Łukasiewicz component.

The idea behind the reductions is illustrated in Figure 1 for chains L containing either an Ł$_3$-component or an infinitely valued Ł-component. To simulate the semantics of $\mathcal{ELU}$, the values 0.5 and 1 in Ł$_3$-$\mathcal{EL}$ (or Ł-$\mathcal{EL}$) are used to simulate the truth values *false* and *true*, respectively. The chain Ł$_3$ (Ł) is then embedded into L as depicted.

3 Finite Łukasiewicz Chains

We now reduce the subsumption problem of the classical DL $\mathcal{ELU}$ to the subsumption problem of Ł$_n$-$\mathcal{EL}$, where $n \geqslant 3$. Since concept subsumption in $\mathcal{ELU}$ is EXPTIME-complete [Baader *et al.*, 2005], this reduction shows that the subsumption problem is EXPTIME-hard already for Ł$_3$-$\mathcal{EL}$; i.e. for Łukasiewicz chains containing three truth degrees.

Note that it suffices to consider subsumption problems between two *concept names* since an $\mathcal{ELU}$ concept C is subsumed by another $\mathcal{ELU}$ concept D w.r.t. an $\mathcal{ELU}$ TBox $\mathcal{T}$ iff A is subsumed by B w.r.t. $\mathcal{T} \cup \{\langle A \sqsubseteq C\rangle, \langle D \sqsubseteq B\rangle\}$, for two new concept names A, B [Baader *et al.*, 2005]. Furthermore, we can restrict our attention to $\mathcal{ELU}$ TBoxes in *normal form*, which only contain axioms of the form

$$A_1 \sqcap A_2 \sqsubseteq B,\ \exists r.A \sqsubseteq B,\ A \sqsubseteq \exists r.B, \text{or } A \sqsubseteq B_1 \sqcup B_2,$$

where A, A_1, A_2, B, B_1 and B_2 are concept names or $\top$. As shown in [Baader *et al.*, 2005], every $\mathcal{ELU}$ TBox can be transformed in linear time into an equivalent one (w.r.t. the original signature) in normal form.

The main idea of our reduction is to simulate a classical concept name in Ł$_n$-$\mathcal{EL}$ by considering all values below $\frac{n-2}{n-1}$ to be equivalent to 0, and thus only the value 1 can be used to express that a domain element belongs to the concept name. We can then express a classical disjunction of the form $B_1 \sqcup B_2$ by restricting the value of the fuzzy conjunction $B_1 \sqcap B_2$ to be $\geqslant \frac{n-2}{n-1}$: according to Fact 1, the latter is equivalent to B_1 *or* B_2 having value 1. Furthermore, we reformulate classical subsumption between C and D as 1-subsumption between C^{n-1} and D^{n-1}: again by Fact 1, the latter two concepts can take only the values 0 or 1. Notice that the conjunctions C^{n-1} and D^{n-1} are fundamental for this reduction to work; their purpose is to produce a crisp version of the concepts C and D.

More formally, let $n \geqslant 3$, $\mathcal{T}$ be an $\mathcal{ELU}$ TBox in normal form, and $C, D \in \mathsf{N_C}$. We construct an Ł$_n$-$\mathcal{EL}$ TBox $\rho_n(\mathcal{T})$ such that C is subsumed by D w.r.t. $\mathcal{T}$ if and only if C^{n-1} is 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$. Since $\mathcal{T}$ is in normal form, we can define the reduction ρ_n for each of the four kinds of axioms listed above:

$$\begin{aligned}
\rho_n(A_1 \sqcap A_2 \sqsubseteq B) &:= \langle A_1 \sqcap A_2 \sqsubseteq B \geqslant 1\rangle \\
\rho_n(\exists r.A \sqsubseteq B) &:= \langle \exists r.A \sqsubseteq B \geqslant 1\rangle \\
\rho_n(A \sqsubseteq \exists r.B) &:= \langle A \sqsubseteq (\exists r.B)^{n-1} \geqslant \tfrac{1}{n-1}\rangle \\
\rho_n(A \sqsubseteq B_1 \sqcup B_2) &:= \langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \tfrac{n-2}{n-1}\rangle
\end{aligned}$$

Finally, we set $\rho_n(\mathcal{T}) := \{\rho_n(\alpha) \mid \alpha \in \mathcal{T}\}$. Notice that $\rho_n(\mathcal{T})$ has as many axioms as $\mathcal{T}$, and the size of each axiom is increased by a factor of at most n. Hence, the translation $\rho_n(\mathcal{T})$ can be performed in polynomial time. The translation of the axiom $A \sqsubseteq \exists r.B$ deserves special attention. Notice that $\rho_n(A \sqsubseteq \exists r.B)$ uses a conjunction of the concept $\exists r.B$ on the right-hand side. This is necessary to guarantee that we consider cases where both, the role relation, and the membership to B have degree 1. We show that this translation satisfies the properties described above.

First we show that if C is subsumed by D w.r.t. $\mathcal{T}$, then C^{n-1} is 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$. In order to achieve this result, for any Ł$_n$-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ we define the classical interpretation $\mathcal{I}_{cr} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}_{cr}})$, where:

- $x \in A^{\mathcal{I}_{cr}}$ iff $A^{\mathcal{I}}(x) = 1$ for every $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$,
- $(x, y) \in r^{\mathcal{I}_{cr}}$ iff $r^{\mathcal{I}}(x, y) = 1$ for every $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Recall that for every $x \in \Delta^{\mathcal{I}}$ it holds that $x \in \top^{\mathcal{I}_{cr}}$ and $\top^{\mathcal{I}}(x) = 1$. This means that $\top$ behaves exactly like a concept name in this reduction.

It can be shown that if the Ł$_n$-interpretation $\mathcal{I}$ satisfies $\rho_n(E \sqsubseteq F)$ for some $\mathcal{ELU}$ GCI $E \sqsubseteq F$ in normal form, then $\mathcal{I}_{cr}$ satisfies $E \sqsubseteq F$.

Lemma 4. *Let $\mathcal{I}$ be an Ł$_n$-interpretation that satisfies $\rho_n(\mathcal{T})$. Then $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$.*

Proof Sketch. We consider only the two most interesting kinds of axioms here. Take first any $A \sqsubseteq \exists r.B \in \mathcal{T}$ and assume that $\langle A \sqsubseteq (\exists r.B)^{n-1} \geqslant \frac{1}{n-1}\rangle$ is satisfied by $\mathcal{I}$. For every element $x \in A^{\mathcal{I}_{cr}}$, we need to show that $x \in (\exists r.B)^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have $A^{\mathcal{I}}(x) = 1$. By our assumption, this implies that $((\exists r.B)^{n-1})^{\mathcal{I}}(x) \geqslant \frac{1}{n-1}$, and thus $(\exists r.B)^{\mathcal{I}}(x) = 1$ by Fact 1. Hence, $1 = \sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) *_{Ł_n} B^{\mathcal{I}}(z)$, i.e. there exists $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) = 1$ and $B^{\mathcal{I}}(y) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we have $(x, y) \in r^{\mathcal{I}_{cr}}$ and $y \in B^{\mathcal{I}_{cr}}$, and hence $x \in (\exists r.B)^{\mathcal{I}_{cr}}$.

Consider now any $A \sqsubseteq B_1 \sqcup B_2 \in \mathcal{T}$ and assume that $\langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \frac{n-2}{n-1}\rangle$ is satisfied by $\mathcal{I}$. If $x \in A^{\mathcal{I}_{cr}}$, then $A^{\mathcal{I}}(x) = 1$. By our assumption, this implies that

$(B_1 \sqcap B_2)^{\mathcal{I}}(x) \geqslant \frac{n-2}{n-1}$, and hence either $B_1^{\mathcal{I}}(x) = 1$ or $B_2^{\mathcal{I}}(x) = 1$ must hold by Fact 1. Again by the definition of $\mathcal{I}_{cr}$, we get that either $x \in B_1^{\mathcal{I}_{cr}}$ or $x \in B_2^{\mathcal{I}_{cr}}$. $\square$

Suppose now that C^{n-1} is *not* 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$. Then, there exists an $Ł_n$-interpretation $\mathcal{I}$ that satisfies $\rho_n(\mathcal{T})$ and an $x \in \Delta^{\mathcal{I}}$ such that $(C^{n-1})^{\mathcal{I}}(x) > (D^{n-1})^{\mathcal{I}}(x)$. By Fact 1, $(C^{n-1})^{\mathcal{I}}(x) = 1$ and $(D^{n-1})^{\mathcal{I}}(x) = 0$. This in particular means that $x \in C^{\mathcal{I}_{cr}} \setminus D^{\mathcal{I}_{cr}}$, yielding the following.

Proposition 5. *If C is subsumed by D w.r.t. $\mathcal{T}$, then C^{n-1} is 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$.*

To prove the converse, we construct from a classical interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ the $Ł_n$-interpretation $\mathcal{I}_n = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}_n})$, where:

- $A^{\mathcal{I}_n}(x) := 1$ if $x \in A^{\mathcal{I}}$ and $A^{\mathcal{I}_n}(x) := \frac{n-2}{n-1}$ otherwise, for every $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$,
- $r^{\mathcal{I}_n}(x, y) := 1$ if $(x, y) \in r^{\mathcal{I}}$ and $r^{\mathcal{I}_n}(x, y) := \frac{n-2}{n-1}$ otherwise, for every $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

As before, this transformation preserves the satisfaction of the TBox w.r.t. the operator ρ_n.

Lemma 6. *If a classical interpretation $\mathcal{I}$ satisfies $\mathcal{T}$, then $\mathcal{I}_n$ satisfies $\rho_n(\mathcal{T})$.*

Proof Sketch. We again consider only two cases. Consider first any $\langle A \sqsubseteq (\exists r.B)^{n-1} \geqslant \frac{1}{n-1}\rangle \in \rho_n(\mathcal{T})$ and $x \in \Delta^{\mathcal{I}}$. If $((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = 0$, then

$$1 > (\exists r.B)^{\mathcal{I}_n}(x) = \sup_{z \in \Delta^{\mathcal{I}_n}} r^{\mathcal{I}_n}(x, z) *_{Ł_n} B^{\mathcal{I}_n}(z).$$

Therefore, every $y \in \Delta^{\mathcal{I}}$ must satisfy either $r^{\mathcal{I}_n}(x, y) < 1$ or $B^{\mathcal{I}_n}(y) < 1$. By the definition of $\mathcal{I}_n$, we get $x \notin (\exists r.B)^{\mathcal{I}}$, and thus $x \notin A^{\mathcal{I}}$ by assumption. Again by the definition of $\mathcal{I}_n$, we have $A^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$, and hence $A^{\mathcal{I}_n}(x) \Rightarrow_{Ł_n} ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = \frac{1}{n-1}$.

In the case that $((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) > 0$, Fact 1 yields that $((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = 1$ and

$$A^{\mathcal{I}_n}(x) \Rightarrow_{Ł_n} ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = 1 \geqslant \tfrac{1}{n-1}.$$

Consider now any $\langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \frac{n-2}{n-1}\rangle \in \rho_n(\mathcal{T})$. If $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) < \frac{n-2}{n-1}$, then by the definition of $\mathcal{I}_n$ we have $B_1^{\mathcal{I}_n}(x) = B_2^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$, and thus $x \notin B_1^{\mathcal{I}} \cup B_2^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this implies $x \notin A^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, $A^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. Since $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) = \frac{n-3}{n-1}$, we can conclude that $A^{\mathcal{I}_n}(x) \Rightarrow_{Ł_n} (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$.

In the case that $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant \frac{n-2}{n-1}$, we also have

$$A^{\mathcal{I}_n}(x) \Rightarrow_{Ł_n} (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant \tfrac{n-2}{n-1}. \ \square$$

Using arguments analogous to those sketched above, we obtain the following proposition.

Proposition 7. *If C is not subsumed by D w.r.t. $\mathcal{T}$, then C^{n-1} is not 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$.*

We have thus reduced classical subsumption in $\mathcal{ELU}$ to 1-subsumption in $Ł_n$-$\mathcal{EL}$. Since the former is EXPTIME-hard [Baader *et al.*, 2005], we obtain an EXPTIME lower bound for the complexity of the latter. An exponential-time algorithm for solving subsumption in the more expressive language $Ł_n$-$\mathcal{ALC}$, which provides a matching upper bound, was presented in [Borgwardt and Peñaloza, 2013].

Theorem 8. *For any $n \geqslant 3$, deciding ℓ-subsumption with respect to a TBox in $Ł_n$-$\mathcal{EL}$ is* EXPTIME*-complete.*

4 Arbitrary Finite Chains

We now show that the complexity result from the previous section can be transferred to almost all logics of the form L-$\mathcal{EL}$ where L is a finite chain. More precisely, subsumption in L-$\mathcal{EL}$ is EXPTIME-complete for all finite chains except those of the form G_n. For the latter, this problem can be shown to be tractable.

As detailed in Section 2, any finite chain L that is not of the form G_n must contain a finite Łukasiewicz chain in an interval $[a, b]$ with at least three elements. We use this fact to reduce subsumption in $Ł_n$-$\mathcal{EL}$ to subsumption in L-$\mathcal{EL}$, where n is the cardinality of the interval $[a, b]$ in L that is isomorphic to $Ł_n$. We extend the bijection $\lambda\colon [a, b] \to Ł_n$ to the chain L:

- $\lambda(x) := 0$ if $x < a$ and
- $\lambda(x) := 1$ if $x > b$.

We also use the inverse $\lambda^{-1}\colon Ł_n \to 2^{\mathsf{L}}$ of this function, for which we have $\lambda^{-1}(0) = [0, a]$, and $\lambda^{-1}(1) = [b, 1]$. When it is clear from the context, we will also use λ^{-1} to denote the inverse of the original bijection; i.e. $\lambda^{-1}\colon Ł_n \to [a, b]$. As shown in the following lemma, these operators are compatible with all the operators that are relevant to fuzzy $\mathcal{EL}$.

Lemma 9. *1. For all $p, q \in \mathsf{L}$, we have*

- *$\lambda(p *_{\mathsf{L}} q) = \lambda(p) *_{Ł_n} \lambda(q)$, and*
- *if $q \geqslant a$, then $\lambda(p \Rightarrow_{\mathsf{L}} q) = \lambda(p) \Rightarrow_{Ł_n} \lambda(q)$.*

2. For all values $p, q \in Ł_n$, $p' \in \lambda^{-1}(p) \cap [a, 1]$, and $q' \in \lambda^{-1}(q) \cap [a, 1]$, we have

- *$p' *_{\mathsf{L}} q' \in \lambda^{-1}(p *_{Ł_n} q) \cap [a, 1]$, and*
- *$p' \Rightarrow_{\mathsf{L}} q' \in \lambda^{-1}(p \Rightarrow_{Ł_n} q) \cap [a, 1]$.*

We can now describe the reduction from $Ł_n$-$\mathcal{EL}$ to L-$\mathcal{EL}$. Let $\mathcal{T}$ be an $Ł_n$-$\mathcal{EL}$ TBox, $\ell \in Ł_n \setminus \{0\}$, and A, B two concept names for which we want to check whether A is ℓ-subsumed by B w.r.t. $\mathcal{T}$. We define the L-$\mathcal{EL}$ TBox $\mathcal{T}'$ as follows.

$$\begin{aligned}\mathcal{T}' := {} & \{\langle C \sqsubseteq D \geqslant \lambda^{-1}(p)\rangle \mid \langle C \sqsubseteq D \geqslant p\rangle \in \mathcal{T}\} \cup \\ & \{\langle \top \sqsubseteq D \geqslant a\rangle \mid \langle C \sqsubseteq D \geqslant p\rangle \in \mathcal{T}\} \cup \\ & \{\langle \top \sqsubseteq B \geqslant a\rangle\}.\end{aligned}$$

Restricting all right-hand side of GCIs in $\mathcal{T}$ to have values $\geqslant a$ is necessary in light of Lemma 9.

We prove that if A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$, then A is ℓ-subsumed by B w.r.t. $\mathcal{T}$. Given an $Ł_n$-interpretation $\mathcal{I}$, we define the L-interpretation $\mathcal{I}_L = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}_L})$ for all $C \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $x, y \in \Delta^{\mathcal{I}}$ as follows:

- $C^{\mathcal{I}_L}(x) := \lambda^{-1}(C^{\mathcal{I}}(x))$ and
- $r^{\mathcal{I}_L}(x, y) := \lambda^{-1}(r^{\mathcal{I}}(x, y))$.

Using the properties from Lemma 9, it is easy to see that if $\mathcal{I}$ satisfies $\mathcal{T}$, then $\mathcal{I}_L$ satisfies $\mathcal{T}'$.

Suppose now that A is not ℓ-subsumed by B w.r.t. $\mathcal{T}$. Then there is an $Ł_n$-interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$, and an $x \in \Delta^{\mathcal{I}}$ such that $A^{\mathcal{I}}(x) \Rightarrow_{Ł_n} B^{\mathcal{I}}(x) < \ell$. But then also $A^{\mathcal{I}_L}(x) \Rightarrow_{Ł_n} B^{\mathcal{I}_L}(x) < \lambda^{-1}(\ell)$ since λ^{-1} is strictly monotone. Since $\mathcal{I}_L$ is a model of $\mathcal{T}'$, this proves the claim.

Proposition 10. *If A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$, then A is ℓ-subsumed by B w.r.t. $\mathcal{T}$.*

Conversely, given an L-interpretation $\mathcal{I}$, we can construct the $Ł_n$-interpretation $\mathcal{I}_n = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}_n})$ where

- $C^{\mathcal{I}_n}(x) := \lambda(C^{\mathcal{I}}(x))$ for all $C \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$, and
- $r^{\mathcal{I}_n}(x, y) := \lambda(r^{\mathcal{I}}(x, y))$ for all $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Using arguments similar to those presented before, it is possible to prove that if $\mathcal{I}$ satisfies $\mathcal{T}'$, then $\mathcal{I}_n$ satisfies $\mathcal{T}$. Thus, subsumption w.r.t. $\mathcal{T}'$ can be decided by checking the subsumption w.r.t. the original TBox $\mathcal{T}$.

Proposition 11. *If A is ℓ-subsumed by B w.r.t. $\mathcal{T}$, then A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$.*

This shows that subsumption in $Ł_n$-$\mathcal{EL}$ is polynomially reducible to subsumption in L-$\mathcal{EL}$, for any finite chain L containing an interval $Ł_n$. By Theorem 8, the latter problem is ExpTime-hard. A matching upper bound is also a consequence of the results from [Borgwardt and Peñaloza, 2013]. As discussed before, every finite chain L is either of the form G_n or contains an interval isomorphic to $Ł_n$, for some n. Overall, this yields the desired complexity result.

Theorem 12. *Let L be a finite chain that is not of the form G_n. Deciding ℓ-subsumption with respect to a TBox in L-$\mathcal{EL}$ is ExpTime-complete.*

Together with Proposition 3, we thus obtain a full characterization of the complexity of reasoning in fuzzy $\mathcal{EL}$ over finite chains, depending on the t-norm that defines the semantics. If all elements of the chain are idempotent w.r.t. the t-norm, then subsumption can be decided in PTime. Otherwise (i.e. if there is at least one non-idempotent element), this problem becomes ExpTime-hard. In the following section, we show that the exponential lower bound holds also for *infinite* chains that contain a Łukasiewicz component.

5 The Infinite Łukasiewicz T-norm

We now consider the infinite chain $[0, 1]$, and show ExpTime-hardness for deciding subsumption in L-$\mathcal{EL}$, for any t-norm that contains a Łukasiewicz component (Definition 2). As shown in [Borgwardt and Peñaloza, 2013], it suffices to prove this result for any t-norm that *starts* with the Łukasiewicz t-norm. Thus, for the rest of this section we consider a continuous t-norm that is isomorphic to the infinitely valued Łukasiewicz t-norm in the interval $[0, b]$ for some $b \in (0, 1]$.

To obtain the ExpTime lower bound, we reduce subsumption in $\mathcal{ELU}$ to subsumption in L-$\mathcal{EL}$. This reduction is very similar to the construction from Section 3. The main difference is that, in order to guarantee that the constructed L-$\mathcal{EL}$ TBox can be used to decide the original $\mathcal{ELU}$ subsumption problem, we need to restrict its models in such a way that all relevant concepts can only take the values $\frac{b}{2}$ or $\geqslant b$.

Given a concept C, let $\mathcal{T}_C$ be the L-$\mathcal{EL}$ TBox

$$\mathcal{T}_C := \{\langle C^2 \sqsubseteq C^3 \geqslant 1\rangle,\ \langle \top \sqsubseteq C \geqslant \tfrac{b}{2}\rangle\}.$$

Every model $\mathcal{I}$ of this TBox must satisfy $C^{\mathcal{I}}(x) \geqslant \frac{b}{2}$ for every $x \in \Delta^{\mathcal{I}}$ due to the second axiom. The first axiom additionally guarantees that $C^{\mathcal{I}}(x) \notin (\frac{b}{2}, b)$ holds: if $\frac{b}{2} < C^{\mathcal{I}}(x) < b$, then $(C^2)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) + C^{\mathcal{I}}(x) - b > 0$, and thus $(C^3)^{\mathcal{I}}(x) < (C^2)^{\mathcal{I}}(x)$, violating the axiom.

Similar to the reduction in Section 3, we will use the truth degree $\frac{b}{2} \in \mathsf{L}$ to stand for "false" in $\mathcal{ELU}$ and any degree greater or equal to b to represent "true." We define the function $\rho_Ł$ for every $\mathcal{ELU}$ GCI in normal form (cf. Section 3):

$$\begin{aligned}
\rho_Ł(A_1 \sqcap A_2 \sqsubseteq B) &:= \langle A_1 \sqcap A_2 \sqsubseteq B \geqslant b\rangle \\
\rho_Ł(\exists r.A \sqsubseteq B) &:= \langle \exists r.A \sqsubseteq B \geqslant b\rangle \\
\rho_Ł(A \sqsubseteq \exists r.B) &:= \langle A \sqsubseteq (\exists r.B)^2 \geqslant \tfrac{b}{2}\rangle \\
\rho_Ł(A \sqsubseteq B_1 \sqcup B_2) &:= \langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \tfrac{b}{2}\rangle.
\end{aligned}$$

Given an $\mathcal{ELU}$ TBox $\mathcal{T}$ in normal form, let $\mathsf{AC}(\mathcal{T})$ be the set of all concept names and existential restrictions appearing in $\mathcal{T}$. We extend the mapping $\rho_Ł$ to $\mathcal{ELU}$ TBoxes as follows:

$$\rho_Ł(\mathcal{T}) := \{\rho_Ł(C \sqsubseteq D) \mid C \sqsubseteq D \in \mathcal{T}\} \cup \bigcup_{C \in \mathsf{AC}(\mathcal{T})} \mathcal{T}_C.$$

Let now $A, B \in \mathsf{N_C}$. One can show that A is subsumed by B w.r.t. $\mathcal{T}$ iff A is b-subsumed by B w.r.t. $\rho_Ł(\mathcal{T}) \cup \mathcal{T}_A \cup \mathcal{T}_B$. The proof follows the same ideas presented in Section 3. The TBoxes $\mathcal{T}_C$ ensure that only three values are relevant for the models, and hence L behaves like $Ł_3$ on them.

From the previous arguments, we see that for any continuous chain L that starts with Łukasiewicz, subsumption in L-$\mathcal{EL}$ is ExpTime-hard. As shown in [Borgwardt and Peñaloza, 2013], if L is the ordinal sum of L_1 and L_2 over the intervals $[0, a]$ and $[a, 1]$, respectively, for some $a \in (0, 1)$, then subsumption in L-$\mathcal{EL}$ is at least as hard as subsumption in L_2-$\mathcal{EL}$. Additionally, every chain L that contains a Łukasiewicz component can be described as such an ordinal sum, where L_2 starts with Łukasiewicz. This means that the ExpTime-hardness holds for all such continuous chains.

Theorem 13. *If L is defined using any continuous t-norm over $[0, 1]$ containing a Łukasiewicz component, then deciding ℓ-subsumption w.r.t. a TBox in L-$\mathcal{EL}$ is ExpTime-hard.*

This improves the co-NP lower bound from [Borgwardt and Peñaloza, 2013]. It is unknown whether a similar lower bound holds for t-norms containing only product components. An upper bound is known only for G-$\mathcal{EL}$, where subsumption can be decided in PTime [Mailis *et al.*, 2012].

6 Conclusions

We have shown that reasoning in finitely valued extensions of fuzzy $\mathcal{EL}$ becomes exponentially harder than in classical $\mathcal{EL}$ even if only one additional truth value interpreted under Łukasiewicz semantics is considered. This provides the first example of a finitely valued DL that exhibits an increased complexity compared to the underlying classical DL. The same complexity lower bound holds for any infinitely valued t-norm over $[0, 1]$ that contains a Łukasiewicz component.

Although EXPTIME-complete, we believe that subsumption in finitely valued $\mathcal{EL}$ can be solved more efficiently than by the algorithms developed for expressive finitely valued DLs [Borgwardt and Peñaloza, 2013; 2014]. We plan to look at adaptations of consequence-based algorithms for classical DLs [Baader *et al.*, 2005; Kazakov, 2009]. On the theoretical side, we will investigate whether other inexpressive DLs like $\mathcal{FL}_0$ [Baader, 1990] or *DL-Lite* [Calvanese *et al.*, 2005] also exhibit an increase in complexity under Łukasiewicz semantics. We also want to study the effect of the product semantics on the complexity of these logics.

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