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Consistency in Fuzzy Description Logics over Residuated De Morgan Lattices

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Abstract

Fuzzy description logics can be used to model vague knowledge in application domains. This paper analyses the consistency and satisfiability problems in the description logic $\mathcal{SHI}$ with semantics based on a complete residuated De Morgan lattice. The problems are undecidable in the general case, but can be decided by a tableau algorithm when restricted to finite lattices. For some sublogics of $\mathcal{SHI}$, we provide upper complexity bounds that match the complexity of crisp reasoning.

1 Introduction

Description Logics (DLs) [1] are a family of knowledge representation formalisms that are widely used to model application domains. In DLs, knowledge is represented with the help of *concepts* (unary predicates) and *roles* (binary predicates) that express the relationships between concepts. They have been successfully employed to formulate ontologies–especially in the medical domain–like Galen[1] and serve as the underpinning for the current semantic web language OWL 2.[2] Standard reasoning in these logics includes *concept satisfiability* (is a given concept non-contradictory?) and *ontology consistency* (does a given ontology have a model?). These and other reasoning problems have been studied for DLs, and several algorithms have been proposed and implemented.

One of the main challenges in knowledge representation is the correct modeling and use of imprecise or vague knowledge. For example, medical diagnoses from experts are rarely clear-cut and usually depend on concepts like HighBloodPressure that are necessarily vague. Fuzzy variants of description logics were introduced in the nineties as a means to tackle this challenge. Their applicability to the representation of medical knowledge was studied in [19].

[1] http://www.opengalen.org/
[2] http://www.w3.org/TR/owl2-overview/

Fuzzy DLs generalize (crisp) DLs by providing a *membership degree* semantics for their concepts. Thus, e.g. 130/85 belongs to the concept HighBloodPressure with a lower degree than, say 140/80. In their original form, membership degrees are elements of the real-number interval $[0, 1]$, but this was later generalized to lattices [18, 23]. The papers [18, 23] consider only a limited kind of semantics over lattices, where conjunction and disjunction are interpreted through the lattice operators meet and join, respectively.

In this paper, we consider a more general lattice-based semantics that uses a *triangular norm* (t-norm) and its residuum as interpretation functions for the logical constructors. We study fuzzy variants of the standard reasoning problems like concept satisfiability and ontology consistency in this setting.

We show that concept satisfiability in $\mathcal{ALC}$ under this semantics is undecidable in general, even if we restrict ourselves to a very simple class of infinite lattices. However, we show with the help of a tableaux-based algorithm that decidability of reasoning can be regained—even for the more expressive DL $\mathcal{SHI}$—if the underlying lattice is required to be finite. Moreover, we describe a black-box method that can be used to transform any decision algorithm for (a small generalization of) satisfiability into a decision procedure for consistency.

2 Preliminaries

We start with a short introduction to residuated lattices, which will be the base for the semantics of the fuzzy DL L-$\mathcal{SHI}$. For a more comprehensive view on these lattices, we refer the reader to [13, 15].

2.1 Lattices

A *lattice* is a triple $(L, \vee, \wedge)$, consisting of a *carrier set* L and two idempotent, associative, and commutative binary operators *join* $\vee$ and *meet* $\wedge$ on L that satisfy the absorption laws $\ell_1 \vee (\ell_1 \wedge \ell_2) = \ell_1 = \ell_1 \wedge (\ell_1 \vee \ell_2)$ for all $\ell_1, \ell_2 \in L$. These operations induce a partial order $\leq$ on L: $\ell_1 \leq \ell_2$ iff $\ell_1 \wedge \ell_2 = \ell_1$. As usual, we write $\ell_1 < \ell_2$ if $\ell_1 \leq \ell_2$ and $\ell_1 \neq \ell_2$. A subset $T \subseteq L$ is called an *antichain (in L)* if there are no two elements $\ell_1, \ell_2 \in T$ with $\ell_1 < \ell_2$. Whenever it is clear from the context, we will use the carrier set L to represent the lattice $(L, \vee, \wedge)$.

The lattice L is *distributive* if $\vee$ and $\wedge$ distribute over each other, *finite* if L is finite, and *bounded* if it has a *minimum* and a *maximum* element, denoted as $\mathbf{0}$ and $\mathbf{1}$, respectively. It is *complete* if joins and meets of arbitrary subsets $T \subseteq L$, $\bigvee_{t \in T} t$ and $\bigwedge_{t \in T} t$, respectively, exist. Clearly, every finite lattice is also complete, and every complete lattice is bounded.

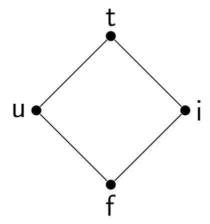

Figure 1: The De Morgan residuated lattice L_4 with $\sim\mathsf{u} = \mathsf{u}$ and $\sim\mathsf{i} = \mathsf{i}$.

A *De Morgan lattice* is a bounded distributive lattice L extended with an involutive and anti-monotonic unary operation $\sim$, called *(De Morgan) negation*, satisfying the De Morgan laws $\sim(\ell_1 \vee \ell_2) = \sim\ell_1 \wedge \sim\ell_2$ and $\sim(\ell_1 \wedge \ell_2) = \sim\ell_1 \vee \sim\ell_2$ for all $\ell_1, \ell_2 \in L$.

Given a lattice L, a *t-norm* is an associative and commutative binary operator on L that is monotonic and has $\mathbf{1}$ as its unit. A *residuated lattice* is a lattice L with a t-norm $\otimes$ and a binary operator $\Rightarrow$ (called *residuum*) such that for all $\ell_1, \ell_2, \ell_3 \in L$ we have $\ell_1 \otimes \ell_2 \leq \ell_3$ iff $\ell_2 \leq \ell_1 \Rightarrow \ell_3$. A simple consequence is that for all $\ell_1, \ell_2 \in L$ we have $\mathbf{1} \Rightarrow \ell_1 = \ell_1$, and $\ell_1 \leq \ell_2$ iff $\ell_1 \Rightarrow \ell_2 = \mathbf{1}$.

A t-norm $\otimes$ over a complete lattice L is *continuous* if for all $\ell \in L$ and $T \subseteq L$ we have $\ell \otimes (\bigvee_{\ell' \in T} \ell') = \bigvee_{\ell' \in T}(\ell \otimes \ell')$. Every continuous t-norm $\otimes$ has the unique residuum $\Rightarrow$ defined by $\ell_1 \Rightarrow \ell_2 = \bigvee\{x \mid \ell_1 \otimes x \leq \ell_2\}$ for all $\ell_1, \ell_2 \in L$. If L is a distributive lattice, then the meet operator $\ell_1 \wedge \ell_2$ always defines a continuous t-norm, called the *Gödel t-norm*. In a residuated De Morgan lattice L, the *t-conorm* $\oplus$ is defined as as $\ell_1 \oplus \ell_2 := \sim(\sim\ell_1 \otimes \sim\ell_2)$. The t-conorm of the Gödel t-norm is the join operator $\ell_1 \vee \ell_2$.

For example, consider the finite lattice L_4, with the elements f, u, i, and t as shown in Figure 1. This lattice has been used for reasoning about incomplete and contradictory knowledge [5] and as a basis for a paraconsistent rough DL [25]. In our blood pressure scenario, the two degrees i and u may be used to express readings that are *potentially* and *partially* high blood pressures, respectively. The incomparability of these degrees reflects the fact that none of them can be stated to belong *more* to the concept $\mathsf{HighBloodPressure}$ than the other.

For the rest of this paper, L denotes a complete residuated De Morgan lattice with t-norm $\otimes$ and residuum $\Rightarrow$, unless explicitely stated otherwise.

2.2 The Fuzzy DL L-$\mathcal{SHI}$

The fuzzy DL L-$\mathcal{SHI}$ is a generalization of the crisp DL $\mathcal{SHI}$ that uses the elements of L as truth values, instead of just the Boolean *true* and *false*. The syntax of L-$\mathcal{SHI}$ is the same as in $\mathcal{SHI}$ with the addition of the constructor $\to$.

Definition 1 (syntax of L-$\mathcal{SHI}$)**.** Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be pairwise disjoint sets of *concept-*, *role-*, and *individual names*, respectively, and $\mathsf{N_R^+} \subseteq \mathsf{N_R}$ a set of *transitive role names*. The set of *(complex) roles* is $\mathsf{N_R} \cup \{r^- \mid r \in \mathsf{N_R}\}$. The set of *(complex) concepts* C is obtained through the following syntactic rule, where $A \in \mathsf{N_C}$ and s is a complex role:

$$C ::= A \mid C_1 \sqcap C_2 \mid C_1 \sqcup C_2 \mid C_1 \to C_2 \mid \neg C \mid \exists s.C \mid \forall s.C \mid \top \mid \bot.$$

The *inverse* of a complex role s (denoted by $\overline{s}$) is s^- if $s \in \mathsf{N_R}$ and r if $s = r^-$. A complex role s is *transitive* if either s or $\overline{s}$ belongs to $\mathsf{N_R^+}$.

The semantics of this logic is based on functions specifying the *membership degree* of every domain element in a concept C.

Definition 2 (semantics of L-$\mathcal{SHI}$)**.** An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty *domain*, and $\cdot^{\mathcal{I}}$ is a function that assigns to every individual name a an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, to every concept name A a function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to L$, and to every role name r a function $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to L$, where $r^{\mathcal{I}}(x, y) \otimes r^{\mathcal{I}}(y, z) \leq r^{\mathcal{I}}(x, z)$ holds for all $r \in \mathsf{N_R^+}$ and $x, y, z \in \Delta^{\mathcal{I}}$.

The function $\cdot^{\mathcal{I}}$ is extended to L-$\mathcal{SHI}$ concepts as follows for every $x \in \Delta^{\mathcal{I}}$:

- $\top^{\mathcal{I}}(x) = \mathbf{1}$,
- $\bot^{\mathcal{I}}(x) = \mathbf{0}$,
- $(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$,
- $(C \sqcup D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$,
- $(C \to D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$,
- $(\neg C)^{\mathcal{I}}(x) = \sim C^{\mathcal{I}}(x)$,
- $(\exists s.C)^{\mathcal{I}}(x) = \bigvee_{y \in \Delta^{\mathcal{I}}} \big(s^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)\big)$,
- $(\forall s.C)^{\mathcal{I}}(x) = \bigwedge_{y \in \Delta^{\mathcal{I}}} \big(s^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)\big)$,

where $(r^-)^{\mathcal{I}}(x, y) = r^{\mathcal{I}}(y, x)$ for all $x, y \in \Delta^{\mathcal{I}}$ and $r \in \mathsf{N_R}$.

Notice that, unlike in crisp $\mathcal{SHI}$, existential and universal quantifiers are not dual to each other, i.e. in general, $(\neg\exists s.C)^{\mathcal{I}}(x) = (\forall s.\neg C)^{\mathcal{I}}(x)$ does not hold. Likewise, the implication constructor $\to$ cannot be expressed in terms of the negation $\neg$ and conjunction $\sqcap$.

The axioms of this logic are those of crisp $\mathcal{SHI}$, but with associated lattice values, which express the degree to which the restrictions must be satisfied.

Definition 3 (axioms)**.** An *assertion* can be a *concept assertion* of the form $\langle a : C \triangleright \ell\rangle$ or a *role assertion* of the form $\langle (a, b) : s \triangleright \ell\rangle$, where C is a concept, s is a complex role, a, b are individual names, $\ell \in L$, and $\triangleright \in \{=, \geq\}$. If $\triangleright$ is $=$, then it is called an *equality assertion.* A *general concept inclusion (GCI)* is of the form $\langle C \sqsubseteq D, \ell\rangle$, where C, D are concepts, and $\ell \in L$. A *role inclusion* is of the form $s \sqsubseteq s'$, where s and s' are complex roles.

An *ontology* $(\mathcal{A}, \mathcal{T}, \mathcal{R})$ consists of a finite set $\mathcal{A}$ of assertions (*ABox*), a finite set $\mathcal{T}$ of GCIs (*TBox*), and a finite set $\mathcal{R}$ of role inclusions (*RBox*). The ABox $\mathcal{A}$ is called *local* if there is an individual $a \in \mathsf{N_I}$ such that all assertions in $\mathcal{A}$ are of the form $\langle a : C = \ell\rangle$, for some concept C and $\ell \in L$.

An interpretation $\mathcal{I}$ *satisfies* the assertion $\langle a : C \triangleright \ell\rangle$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \ell$ and the assertion $\langle (a, b) : s \triangleright \ell\rangle$ if $s^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright \ell$. It satisfies the GCI $\langle C \sqsubseteq D, \ell\rangle$ if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell$ holds for every $x \in \Delta^{\mathcal{I}}$. It satisfies the role inclusion $s \sqsubseteq s'$ if for all $x, y \in \Delta^{\mathcal{I}}$ we have $s^{\mathcal{I}}(x, y) \leq s'^{\mathcal{I}}(x, y)$.

$\mathcal{I}$ is a *model* of the ontology $(\mathcal{A}, \mathcal{T}, \mathcal{R})$ if it satisfies all axioms in $\mathcal{A}$, $\mathcal{T}$, $\mathcal{R}$.

Given an RBox $\mathcal{R}$, the *role hierarchy* $\sqsubseteq_{\mathcal{R}}$ on the set of complex roles is the reflexive and transitive closure of the relation

$$\{(s, s') \mid s \sqsubseteq s' \in \mathcal{R} \text{ or } \overline{s} \sqsubseteq \overline{s'} \in \mathcal{R}\}.$$

Using reachability algorithms, the role hierarchy can be computed in polynomial time in the size of $\mathcal{R}$. An RBox $\mathcal{R}$ is called *acyclic* if it contains no cycles of the form $s \sqsubseteq_{\mathcal{R}} s'$, $s' \sqsubseteq_{\mathcal{R}} s$ for two roles $s \neq s'$.

The fuzzy DL L-$\mathcal{ALC}$ is the sublogic of L-$\mathcal{SHI}$ where no role inclusions, transitive roles, or inverse roles are allowed. $\mathcal{SHI}$ is the sublogic of L-$\mathcal{SHI}$ where the underlying lattice contains only the elements $\mathbf{0}$ and $\mathbf{1}$, which may be interpreted as *false* and *true*, respectively, and the t-norm and t-conorm are conjunction and disjunction, respectively.

Recall that the semantics of the quantifiers require the computation of a supremum or infimum of the membership degrees of a possibly infinite set of elements of the domain. To obtain effective decision procedures, reasoning is usually restricted to a special kind of models, called witnessed models [16].

Definition 4 (witnessed model)**.** Let $n \in \mathbb{N}$. A model $\mathcal{I}$ of an ontology $\mathcal{O}$ is *n-witnessed* if for every $x \in \Delta^{\mathcal{I}}$, every role s and every concept C there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that

$$(\exists s.C)^{\mathcal{I}}(x) = \bigvee_{i=1}^{n} \left(s^{\mathcal{I}}(x, x_i) \otimes C^{\mathcal{I}}(x_i)\right), \quad (\forall s.C)^{\mathcal{I}}(x) = \bigwedge_{i=1}^{n} \left(s^{\mathcal{I}}(x, y_i) \Rightarrow C^{\mathcal{I}}(y_i)\right).$$

In particular, if $n = 1$, the suprema and infima from the semantics of $\exists s.C$ and $\forall s.C$ are maxima and minima, respectively, and we say that $\mathcal{I}$ is *witnessed.*

The reasoning problems for $\mathcal{SHI}$ generalize to the fuzzy semantics of L-$\mathcal{SHI}$.

Definition 5 (decision problems)**.** Let $\mathcal{O}$ be an ontology, C, D be two concepts, $a \in \mathsf{N_I}$, and $\ell \in L$. $\mathcal{O}$ is *consistent* if it has a (witnessed) model. C is *strongly ℓ-satisfiable* if there is a (witnessed) model $\mathcal{I}$ of $\mathcal{O}$ and $x \in \Delta^{\mathcal{I}}$ with $C^{\mathcal{I}}(x) \geq \ell$. The individual a is an *ℓ-instance* of C if $\langle a : C \geq \ell \rangle$ is satisfied by all (witnessed) models of $\mathcal{O}$. C is *ℓ-subsumed* by D if $\langle C \sqsubseteq D, \ell \rangle$ is satisfied by all (witnessed) models of $\mathcal{O}$.

Example 6. It is known that coffee drinkers and salt consumers tend to have a higher blood pressure. On the other hand, bradycardia is highly correlated with a lower blood pressure. This knowledge can be expressed through the TBox

$$\{\langle \mathsf{CoffeeDrinker} \sqsubseteq \mathsf{HighBloodPressure}, \mathsf{i}\rangle,\ \langle \mathsf{SaltConsumer} \sqsubseteq \mathsf{HighBloodPressure}, \mathsf{i}\rangle,\\ \langle \mathsf{Bradycardia} \sqsubseteq \neg\mathsf{HighBloodPressure}, \mathsf{i}\rangle\},$$

over the lattice L_4 from Figure 1. The degree i in these axioms expresses that the relation between the causes and $\mathsf{HighBloodPressure}$ is not absolute. Consider the patients ana, who is a coffee drinker, and bob, a salt consumer with bradycardia, as expressed by the ABox

$$\{\langle \mathsf{ana} : \mathsf{CoffeeDrinker} = \mathsf{t}\rangle,\ \langle \mathsf{bob} : \mathsf{SaltConsumer} \sqcap \mathsf{Bradycardia} = \mathsf{t}\rangle\}.$$

We can deduce that both patients are an i-instance of $\mathsf{HighBloodPressure}$, but only bob is an i-instance of $\neg\mathsf{HighBloodPressure}$. Notice that if we changed all the degrees from the GCIs to the value t, the ontology would be inconsistent.

We will focus first on a version of the consistency problem where the ABox is required to be a local ABox; we call this problem *local consistency*. We show in Section 5 that local consistency can be used for solving other reasoning problems in L-$\mathcal{SHI}$ if L is finite. Before that, we show that satisfiability and (local) consistency are undecidable in L-$\mathcal{ALC}$, and hence also in L-$\mathcal{SHI}$, in general.

3 Undecidability

To show undecidability, we use a reduction from the Post Correspondence Problem [21] to strong satisfiability in L-$\mathcal{ALC}$ over a specific infinite lattice. The reduction uses ideas that have been successfully applied to showing undecidability of reasoning for several fuzzy description logics [2, 3, 12].

Although the basic idea of the proof is not new, it is interesting for several reasons. First, previous incarnations of the proof idea focused on decidability of ontology consistency [3, 11, 12], while we are concerned with strong ℓ-satisfiability. Second, most of the previous undecidability results only hold for reasoning w.r.t.

witnessed models, but the current proof works for both witnessed and general models. Finally, in contrast to an earlier version of this proof [10], the employed lattice has a quite simple structure in the sense that it is a total order that has only the two limit points $-\infty$ and ∞ instead of infinitely many. Note that any distributive lattice without limit points is already finite and reasoning in finite residuated De Morgan lattices is decidable (see Sections 4 and 5).

Definition 7 (PCP)**.** Let $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ be a finite set of pairs of words over the alphabet $\Sigma = \{1, \ldots, s\}$ with $s > 1$. The *Post Correspondence Problem (PCP)* asks for a finite non-empty sequence $i_1 \ldots i_k \in \{1, \ldots, n\}^+$ such that $v_{i_1} \ldots v_{i_k} = w_{i_1} \ldots w_{i_k}$. If this sequence exists, it is called a *solution* for $\mathcal{P}$.

For $\nu = i_1 \cdots i_k \in \{1, \ldots, n\}^*$, we define $v_\nu := v_{i_1} \cdots v_{i_k}$ and $w_\nu := w_{i_1} \cdots w_{i_k}$.

We consider the lattice $\mathbb{Z}_\infty$ whose domain is $\mathbb{Z} \cup \{-\infty, \infty\}$ with the usual ordering over the integers and $-\infty$ and ∞ as the minimal and maximal element, respectively. Its De Morgan negation is $\sim\ell = -\ell$ if $\ell \in \mathbb{Z}$, $\sim\infty = -\infty$, and $\sim(-\infty) = \infty$. The t-norm $\otimes$ is defined as follows for all $\ell, m \in \mathbb{Z}_\infty$:

$$\ell \otimes m := \begin{cases} \ell + m & \text{if } \ell, m \in \mathbb{Z} \text{ and } \ell, m \leq 0 \\ \min\{\ell, m\} & \text{otherwise.} \end{cases}$$

This is in fact a residuated lattice with the following residuum:

$$\ell \Rightarrow m := \begin{cases} \infty & \text{if } \ell \leq m \\ m & \text{if } \ell > m \text{ and } \ell \geq 0 \\ m - \ell & \text{if } \ell > m \text{ and } \ell < 0. \end{cases}$$

Given an instance $\mathcal{P}$ of the PCP, we will construct a TBox $\mathcal{T}_\mathcal{P}$ such that the designated concept name S is strongly ∞-satisfiable iff $\mathcal{P}$ has no solution. Recall that the alphabet Σ consists of the first s positive integers. Thus, every word in Σ^+ can be seen as a positive integer written in base $s+1$; we extend this intuition and denote the empty word by 0. We encode each word $u \in \Sigma^*$ with the number $-u \leq 0$.

The idea is that the TBox $\mathcal{T}_\mathcal{P}$ describes the *search tree* of $\mathcal{P}$ with the nodes $\{1, \ldots, n\}^*$. At its root ε, it encodes the value $v_\varepsilon = w_\varepsilon = \varepsilon$, which is represented by 0, using the concept names V and W. These concept names are used throughout the tree to express the values v_ν and w_ν at every node $\nu \in \{1, \ldots, n\}^*$. Additionally, we will use the auxiliary concept names V_i and W_i to encode the constant words v_i and w_i, respectively, for each $i \in \{1, \ldots, n\}$. These will be used to compute the concatenation $v_{\nu i} = v_\nu v_i$ at each node.

To simplify the reduction, we will use some abbreviations. Given two L-$\mathcal{ALC}$ concepts C and D and $r \in \mathsf{N_R}$, $\langle C \equiv D \rangle$ abbreviates the axioms $\langle C \sqsubseteq D, \infty \rangle$, $\langle D \sqsubseteq C, \infty \rangle$; and $\langle C \stackrel{r}{\rightsquigarrow} D \rangle$ stands for the axioms $\langle C \sqsubseteq \forall r.D, \infty \rangle$, $\langle \exists r.D \sqsubseteq C, \infty \rangle$. For $n \geq 1$, the concept C^n is inductively defined by $C^1 := C$ and $C^{n+1} := C^n \sqcap C$.

Proposition 8. *Let $\mathcal{I}$ be an interpretation and $x \in \Delta^{\mathcal{I}}$.*

- *If $\mathcal{I}$ satisfies $\langle C \equiv D \rangle$, then $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(x)$.*
- *If $\mathcal{I}$ satisfies $\langle C \overset{r}{\leadsto} D \rangle$ and $C^{\mathcal{I}}(x) \leq 0$, then $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(y)$ holds for all $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) \geq 1$.*
- *If $C^{\mathcal{I}}(x) \in \mathbb{Z}$, $C^{\mathcal{I}}(x) \leq 0$, and $n \geq 1$, then $(C^n)^{\mathcal{I}}(x) = n \cdot C^{\mathcal{I}}(x)$.*

We now introduce the TBox $\mathcal{T}_0 := \bigcup_{i=0}^{n} \mathcal{T}_{\mathcal{P}}^{i}$ that encodes the search tree of the instance $\mathcal{P}$ of the PCP:

$$\begin{aligned}
\mathcal{T}_{\mathcal{P}}^{0} :=\ & \{\ \langle S \sqsubseteq V, 0\rangle, \langle S \sqsubseteq \neg V, 0\rangle, \langle S \sqsubseteq W, 0\rangle, \langle S \sqsubseteq \neg W, 0\rangle\},\\
\mathcal{T}_{\mathcal{P}}^{i} :=\ & \{\ \langle \top \sqsubseteq \exists r_i.\top, 1\rangle,\\
& \quad \langle \top \sqsubseteq V_i, -v_i\rangle, \langle \top \sqsubseteq \neg V_i, v_i\rangle, \langle \top \sqsubseteq W_i, -w_i\rangle, \langle \top \sqsubseteq \neg W_i, w_i\rangle,\\
& \quad \langle (V^{(s+1)^{|v_i|}} \sqcap V_i) \overset{r_i}{\leadsto} V\rangle, \langle (W^{(s+1)^{|w_i|}} \sqcap W_i) \overset{r_i}{\leadsto} W\rangle\}
\end{aligned}$$

The TBox $\mathcal{T}_{\mathcal{P}}^{0}$ initializes the search tree by ensuring for every model $\mathcal{I}$ and every domain element $x \in \Delta^{\mathcal{I}}$ that satisfies $S^{\mathcal{I}}(x) = \infty$ that the values of V and W are both 0, which is the encoding of the empty word. Each TBox $\mathcal{T}_{\mathcal{P}}^{i}$ ensures the existence of an r_i-successor for every domain element and describes the constant pair (v_i, w_i) using the concepts V_i and W_i, i.e. it forces that $V_i^{\mathcal{I}}(x) = -v_i$ and $W_i^{\mathcal{I}}(x) = -w_i$ for every $x \in \Delta^{\mathcal{I}}$. Using the last two axioms, the search tree is then extended by concatenating the words v and w produced so far with v_i and w_i, respectively. In the following, we will describe this in more detail.

Consider the interpretation $\mathcal{I}_{\mathcal{P}}$ over the domain $\Delta^{\mathcal{I}_{\mathcal{P}}} = \{1, \ldots, n\}^*$, where for all $\nu, \nu' \in \{1, \ldots, n\}^*$ and $i \in \{1, \ldots, n\}$,

- $V^{\mathcal{I}_{\mathcal{P}}}(\nu) = -v_\nu, \quad W^{\mathcal{I}_{\mathcal{P}}}(\nu) = -w_\nu,$
- $V_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = -v_i, \quad W_i^{\mathcal{I}_{\mathcal{P}}}(\nu) = -w_i,$
- $r_i^{\mathcal{I}_{\mathcal{P}}}(\nu, \nu i) = \infty$ and $r_i^{\mathcal{I}_{\mathcal{P}}}(\nu, \nu') = -\infty$ if $\nu' \neq \nu i$,
- $S^{\mathcal{I}_{\mathcal{P}}}(\varepsilon) = \infty$ and $S^{\mathcal{I}_{\mathcal{P}}}(\nu') = -\infty$ if $\nu' \neq \varepsilon$.

It is easy to see that $\mathcal{I}_{\mathcal{P}}$ is in fact a model of $\mathcal{T}_0$ and it strongly satisfies S with degree ∞. Moreover, *every* model of this TBox that strongly ∞-satisfies S must "include" $\mathcal{I}_{\mathcal{P}}$ in the following sense.

Lemma 9. *Let $\mathcal{I}$ be a model of $\mathcal{T}_0$ such that $S^{\mathcal{I}}(x_0) = \infty$ for some $x_0 \in \Delta^{\mathcal{I}}$. Then there exists a function $g : \Delta^{\mathcal{I}_{\mathcal{P}}} \to \Delta^{\mathcal{I}}$ such that $A^{\mathcal{I}_{\mathcal{P}}}(\nu) = A^{\mathcal{I}}(g(\nu))$ and $r_i(g(\nu), g(\nu i)) \geq 1$ hold for every concept name $A \in \{V, W, V_1, W_1, \ldots, V_n, W_n\}$, every $\nu \in \Delta^{\mathcal{I}_{\mathcal{P}}}$, and every $i \in \{1, \ldots, n\}$.*

Proof. We construct the function g by induction on ν and set $g(\varepsilon) := x_0$. Since $\mathcal{I}$ is a model of $\mathcal{T}_\mathcal{P}^0$ and $S^\mathcal{I}(x_0) = \infty$, we have $V^\mathcal{I}(x_0) \geq 0$ and $\sim V^\mathcal{I}(x_0) \geq 0$, i.e. $V^\mathcal{I}(x_0) = 0$, and similarly $W^\mathcal{I}(x_0) = 0$. In the same way, for every $i \in \{1, \ldots, n\}$, $V_i^\mathcal{I}(x_0)$ and $W_i^\mathcal{I}(x_0)$ are restricted by $\mathcal{T}_\mathcal{P}^i$ to be $-v_i$ and $-w_i$, respectively.

Let now $\nu \in \{1, \ldots, n\}^*$ and assume that $g(\nu)$ already satisfies the condition. For each $i \in \{1, \ldots, n\}$, the first axiom of $\mathcal{T}_\mathcal{P}^i$ ensures that $\bigvee_{y \in \Delta^\mathcal{I}} r_i^\mathcal{I}(g(\nu), y) \geq 1$. Thus, there is $y_i \in \Delta^\mathcal{I}$ such that $r_i^\mathcal{I}(g(\nu), y_i) \geq 1$. We define $g(\nu i) := y_i$. By Proposition 8, we have

$$V^\mathcal{I}(y_i) = (V^{(s+1)^{|v_i|}} \sqcap V_i)^\mathcal{I}(g(\nu)) = -\left((s+1)^{|v_i|} v_\nu + v_i\right) = -v_\nu v_i = -v_{\nu i},$$

and similarly for $W^\mathcal{I}(y_i)$. The claim for V_i and W_i can be shown as above. $\square$

This proposition shows that every model of $\mathcal{T}_0$ encodes a description of the search tree for a solution of $\mathcal{P}$. Thus, to decide the PCP, it suffices to detect whether there is a node $\nu \in \{1, \ldots, n\}^+$ of $\mathcal{I}_\mathcal{P}$ where $V^{\mathcal{I}_\mathcal{P}}(\nu) = W^{\mathcal{I}_\mathcal{P}}(\nu)$. We accomplish this using the TBox

$$\mathcal{T}' := \{\langle \top \sqsubseteq \forall r_i.\neg((V \to W) \sqcap (W \to V)), 0\rangle \mid 1 \leq i \leq n\}.$$

The interpretation $\mathcal{I}_\mathcal{P}$ is a model of $\mathcal{T}'$ iff $V^{\mathcal{I}_\mathcal{P}}(\nu) \neq W^{\mathcal{I}_\mathcal{P}}(\nu)$ holds for every $\nu \in \{1, \ldots, n\}^+$.

Lemma 10. *$\mathcal{P}$ has a solution iff S is not ∞-satisfiable w.r.t. $\mathcal{T}_\mathcal{P} := \mathcal{T}_0 \cup \mathcal{T}'$.*

Proof. For any two values $\ell, m \leq 0$, we have $\ell \neq m$ iff $(\ell \Rightarrow m) \otimes (m \Rightarrow \ell) \leq 0$.

Assume now that S is not ∞-satisfiable w.r.t. $\mathcal{T}_\mathcal{P}$. Then, in particular, $\mathcal{I}_\mathcal{P}$ does not satisfy $\mathcal{T}'$, i.e. we have $(\forall r_i.\neg((V \to W) \sqcap (W \to V)))^{\mathcal{I}_\mathcal{P}}(\nu) < 0$ for some $\nu \in \{1, \ldots, n\}^*$ and $i \in \{1, \ldots, n\}$. There must be a $\nu \in \{1, \ldots, n\}^+$ with $(\neg((V \to W) \sqcap (W \to V)))^{\mathcal{I}_\mathcal{P}}(\nu) < 0$; thus, $-v_\nu = V^{\mathcal{I}_\mathcal{P}}(\nu) = W^{\mathcal{I}_\mathcal{P}}(\nu) = -w_\nu$. This shows that $v_\nu = w_\nu$, i.e. $\mathcal{P}$ has a solution.

For the other direction, let $\mathcal{I}$ be a model of $\mathcal{T}_\mathcal{P}$ and $x_0 \in \Delta^\mathcal{I}$ such that $S^\mathcal{I}(x_0) = \infty$. In particular, we have

$$r_i^\mathcal{I}(g(\nu), g(\nu i)) \Rightarrow (\neg((V \to W) \sqcap (W \to V)))^\mathcal{I}(g(\nu i)) \geq 0$$

for every $\nu \in \{1, \ldots, n\}^*$ and $i \in \{1, \ldots, n\}$, where g is the function constructed in Lemma 9. Thus, $((V \to W) \sqcap (W \to V))^\mathcal{I}(g(\nu)) \leq 0$ for every $\nu \in \{1, \ldots, n\}^+$, which implies $-v_\nu = V^\mathcal{I}(g(\nu)) \neq W^\mathcal{I}(g(\nu)) = -w_\nu$. This shows that $v_\nu \neq w_\nu$ for all $\nu \in \{1, \ldots, n\}^+$, i.e. $\mathcal{P}$ has no solution. $\square$

As mentioned before, since the interpretation $\mathcal{I}_\mathcal{P}$ is witnessed, undecidability holds even if we restrict reasoning to n-witnessed models, for any $n \in \mathbb{N}$.

Theorem 11. *Strong satisfiability is undecidable in L-$\mathcal{ALC}$, even if L is a countable total order with at most two limit points and reasoning is restricted to n-witnessed models.*

This theorem also shows that (local) consistency is undecidable in L-$\mathcal{ALC}$ since S is strongly ∞-satisfiable w.r.t. $\mathcal{T}_\mathcal{P}$ iff $(\{\langle a : S = \infty\rangle\}, \mathcal{T}_\mathcal{P})$ is locally consistent, where a is an arbitrary individual name. Notice that these do not exclude the existence of classes of infinite lattices for which reasoning in L-$\mathcal{SHI}$ is decidable. If we restrict to finite lattices, then a tableau algorithm can be used for reasoning.

4 A Tableaux Algorithm for Local Consistency

Before presenting a tableau algorithm [4] that decides local consistency by constructing a model of a given L-$\mathcal{SHI}$ ontology, we discuss previous approaches to deciding consistency of fuzzy DLs over finite residuated De Morgan lattices in the presence of GCIs.

A popular method is the reduction of fuzzy ontologies into crisp ones, which has so far only been done for finite total orders [7, 8, 23]. Reasoning can then be performed through existing optimized reasoners for crisp DLs. The main idea is to translate every concept name A into finitely many crisp concept names $A_{\geq\ell}$, one for each truth value ℓ, where $A_{\geq\ell}$ collects all those individuals that belong to A with a truth degree $\geq \ell$. The lattice structure is expressed through GCIs of the form $A_{\geq\ell_2} \sqsubseteq A_{\geq\ell_1}$, where ℓ_2 is a minimal element above ℓ_1, and analogously for the role names. All axioms are then recursively translated into crisp axioms that use only the introduced crisp concept and role names. The resulting crisp ontology is consistent iff the original fuzzy ontology is consistent.

In general such a translation is exponential in the size of the concepts that occur in the fuzzy ontology. The reason is that, depending on the t-norm used, there may be many possible combinations of values ℓ_1, ℓ_2 for C, D, respectively, that lead to $C \sqcap D$ having the value $\ell = \ell_1 \otimes \ell_2$, and similarly for the other constructors. All these possibilities have to be expressed in the translation. If after the translation one uses a crisp DL reasoner, which usually implement tableaux algorithms with a worst-case complexity above NExpTime, one gets a 2-NExpTime reasoning procedure. In contrast, our tableau algorithm has a worst-case complexity of NExpTime, matching the complexity of crisp $\mathcal{SHI}$.

To the best of our knowledge, at the moment there exists only one (correct) tableaux algorithm that can deal with a finite total order of truth values and GCIs [22],[3] but it is restricted to the Gödel t-norm. The main difference between this algorithm and ours is that we non-deterministically guess the degree

[3]Several tableau algorithms for fuzzy DLs exist, but they are either restricted to acyclic TBoxes or are not correct in the presence of GCIs, as shown in [2, 6].

of membership of each individual to every relevant concept, while the approach from [22] sets only lower and upper bounds for these degrees; this greatly reduces the amount of non-determinism, but introduces several complications when a t-norm different from the Gödel t-norm is used.

We present a straightforward tableaux algorithm with a larger amount of non-determinism that nevertheless matches the theoretical worst-case complexity of tableaux algorithms for crisp $\mathcal{SHI}$. It is loosely based on the crisp tableaux algorithm in [17]. A first observation that simplifies the algorithm is that since L is finite, we can w.l.o.g. restrict reasoning to n-witnessed models.

Proposition 12. *If the maximal cardinality of an antichain of L is n, then every interpretation in L-$\mathcal{SHI}$ is n-witnessed.*

For simplicity, we consider only the case $n = 1$. For $n > 1$, the construction is similar, but several witnesses have to be produced for satisfying each existential and value restriction. The necessary changes in the algorithm are described at the end of this section. We can also assume w.l.o.g. that the RBox is acyclic. The proof of this follows similar arguments as for crisp $\mathcal{SHI}$ [24].

Proposition 13. *Deciding local consistency in L-$\mathcal{SHI}$ is polynomially equivalent to deciding local consistency in L-$\mathcal{SHI}$ w.r.t. acyclic RBoxes.*

In the following, let $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ be an ontology where $\mathcal{A}$ is a local ABox that contains only the individual name a and $\mathcal{R}$ is an acyclic RBox. We first show that $\mathcal{O}$ has a model if we can find a *tableau*; intuitively, a possibly infinite "completed version" of $\mathcal{A}$. Later we describe an algorithm for constructing a finite representation of such a tableau.

Definition 14. A *tableau* for $\mathcal{O}$ is a set $\mathbf{T}$ of equality assertions over a set Ind of individuals such that $a \in \mathsf{Ind}$, $\mathcal{A} \subseteq \mathbf{T}$, and the following conditions are satisfied for all $C, C_1, C_2 \in \mathsf{sub}(\mathcal{O})$, $x, y \in \mathsf{Ind}$, $r, s \in \mathsf{N_R}$, and $\ell \in L$:

Clash-free: If $\langle x : C = \ell \rangle \in \mathbf{T}$ or $\langle (x, y) : r = \ell \rangle \in \mathbf{T}$, then there is no $\ell' \in L$ such that $\ell' \neq \ell$ and $\langle x : C = \ell' \rangle \in \mathbf{T}$ or $\langle (x, y) : r = \ell' \rangle \in \mathbf{T}$, respectively.

Complete: For every row of Table 1, the following condition holds:

"If $\langle \mathsf{trigger} \rangle$ is in $\mathbf{T}$, there are $\langle \mathsf{values} \rangle$ such that $\langle \mathsf{assertions} \rangle$ are in $\mathbf{T}$."

These conditions help to abstract from the interplay between transitive roles and existential and value restrictions. It suffices to satisfy the above conditions to make certain that $\mathcal{O}$ has a model.

Lemma 15. *$\mathcal{O}$ is locally consistent iff it has a tableau.*

	$\langle$trigger$\rangle$	$\langle$values$\rangle$	$\langle$assertions$\rangle$
$\sqcap$	$\langle x : C_1 \sqcap C_2 = \ell \rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \otimes \ell_2 = \ell$	$\langle x : C_1 = \ell_1 \rangle$, $\langle x : C_2 = \ell_2 \rangle$
$\sqcup$	$\langle x : C_1 \sqcup C_2 = \ell \rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \oplus \ell_2 = \ell$	$\langle x : C_1 = \ell_1 \rangle$, $\langle x : C_2 = \ell_2 \rangle$
$\to$	$\langle x : C_1 \to C_2 = \ell \rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 = \ell$	$\langle x : C_1 = \ell_1 \rangle$, $\langle x : C_2 = \ell_2 \rangle$
$\neg$	$\langle x : \neg C = \ell \rangle$		$\langle x : C = \sim \ell \rangle$
$\exists$	$\langle x : \exists r.C = \ell \rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \otimes \ell_2 = \ell$, individual y	$\langle (x, y) : r = \ell_1 \rangle$, $\langle y : C = \ell_2 \rangle$
$\exists_\leq$	$\langle x : \exists r.C = \ell \rangle$, $\langle (x, y) : r = \ell_1 \rangle$	$\ell_2 \in L$ with $\ell_1 \otimes \ell_2 \leq \ell$	$\langle y : C = \ell_2 \rangle$
$\exists_+$	$\langle x : \exists s.C = \ell \rangle$, $\langle (x, y) : r = \ell_1 \rangle$ with r transitive and $r \sqsubseteq_\mathcal{R} s$	$\ell_2 \in L$ with $\ell_1 \otimes \ell_2 \leq \ell$	$\langle y : \exists r.C = \ell_2 \rangle$
$\forall$	$\langle x : \forall r.C = \ell \rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 = \ell$, individual y	$\langle (x, y) : r = \ell_1 \rangle$, $\langle y : C = \ell_2 \rangle$
$\forall_\geq$	$\langle x : \forall r.C = \ell \rangle$, $\langle (x, y) : r = \ell_1 \rangle$	$\ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 \geq \ell$	$\langle y : C = \ell_2 \rangle$
$\forall_+$	$\langle x : \forall s.C = \ell \rangle$, $\langle (x, y) : r = \ell_1 \rangle$ with r transitive and $r \sqsubseteq_\mathcal{R} s$	$\ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 \geq \ell$	$\langle y : \forall r.C = \ell_2 \rangle$
inv	$\langle (x, y) : r = \ell_1 \rangle$		$\langle (y, x) : \overline{r} = \ell_1 \rangle$
$\sqsubseteq_\mathcal{R}$	$\langle (x, y) : r = \ell_1 \rangle$, $r \sqsubseteq_\mathcal{R} s$	$\ell_2 \in L$ with $\ell_1 \leq \ell_2$	$\langle (x, y) : s = \ell_2 \rangle$
$\sqsubseteq_\mathcal{T}$	individual x, $\langle C_1 \sqsubseteq C_2, \ell \rangle$ in $\mathcal{T}$	$\ell_1, \ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 \geq \ell$	$\langle x : C_1 = \ell_1 \rangle$, $\langle x : C_2 = \ell_2 \rangle$

Table 1: The tableau conditions for L-$\mathcal{SHI}$.

Proof. Let $\mathbf{T}$ be a tableau for $\mathcal{O}$ over the set Ind of individuals. We define $C^\mathbf{T}(x) = \ell$ if $\langle x : C = \ell \rangle \in \mathbf{T}$ and $r^\mathbf{T}(x, y) = \ell$ if $\langle (x, y) : r = \ell \rangle \in \mathbf{T}$. Note that these values are either unique or undefined since $\mathbf{T}$ is clash-free. In this way, $\mathbf{T}$ immediately defines a rudimentary interpretation. However, transitive roles are not yet interpreted by transitive fuzzy relations. In the following, we denote by $r^\mathbf{T}(z_1, \dots, z_n)$ the value $r^\mathbf{T}(z_1, z_2) \otimes \dots \otimes r^\mathbf{T}(z_{n-1}, z_n)$ for any sequence $z_1, \dots, z_n \in \mathsf{Ind}$. This value is $\mathbf{1}$ if $n = 1$ since $\mathbf{1}$ is the unit element for $\otimes$.

We now define a proper model $\mathcal{I}$ of $\mathcal{O}$ by setting $\Delta^\mathcal{I} := \mathsf{Ind}$, $A^\mathcal{I}(x) = A^\mathbf{T}(x)$ for all concept names A and $x \in \mathsf{Ind}$,

$$r^{\mathcal{I}}(x,y) = \bigvee_{n\geq 0} \bigvee_{z_1,\ldots,z_n\in\mathsf{Ind}} r^{\mathbf{T}}(x,z_1,\ldots,z_n,y) \text{ if the role } r \text{ is transitive, and}$$

$$r^{\mathcal{I}}(x,y) = r^{\mathbf{T}}(x,y) \vee \bigvee_{s\sqsubseteq_{\mathcal{R}} r,\ s\neq r} s^{\mathcal{I}}(x,y) \text{ otherwise.}$$

Thus, $\mathcal{I}$ correctly interprets transitive roles by transitive relations. This construction was inspired by a similar one used for crisp $\mathcal{SHI}$ in [17]. It is well-defined if $\mathcal{R}$ is acyclic (see Lemma 13). By the inv- and $\sqsubseteq_{\mathcal{R}}$-conditions, $\mathcal{I}$ satisfies $\mathcal{R}$ and inverse roles are interpreted correctly. Furthermore, one can show by induction on the role depth that for every concept C we have $C^{\mathcal{I}}(x) = C^{\mathbf{T}}(x)$ whenever the latter is defined. Together with the $\sqsubseteq_{\mathcal{T}}$-condition and the fact that $\mathcal{A} \subseteq \mathbf{T}$, this shows that $\mathcal{I}$ also satisfies $\mathcal{A}$ and $\mathcal{T}$, and thus it satisfies $\mathcal{O}$.

Let now $\mathcal{I}$ be a model of $\mathcal{O}$. We can easily construct a tableau $\mathbf{T}$ over the set $\Delta^{\mathcal{I}}$ of individuals as follows. For every concept C and $x \in \Delta^{\mathcal{I}}$, we add $\langle x : C = \ell\rangle$ to $\mathbf{T}$ if $C^{\mathcal{I}}(x) = \ell$. Similarly, for every role r and $x, y \in \Delta^{\mathcal{I}}$, we add the assertion $\langle (x, y) : r = r^{\mathcal{I}}(x, y)\rangle$ to $\mathbf{T}$. We have $\mathcal{A} \subseteq \mathbf{T}$ since $\mathcal{I}$ satisfies $\mathcal{A}$. $\mathbf{T}$ is clash-free since the values are uniquely defined by $\mathcal{I}$.

Furthermore, the semantics of L-$\mathcal{SHI}$ concepts and axioms yield completeness: consider the $\exists_+$-condition and assume that $(\exists s.C)^{\mathcal{I}}(x) = \ell$, $r^{\mathcal{I}}(x, y) = \ell_1$ with r transitive, and $r \sqsubseteq_{\mathcal{R}} s$. Since the value $\ell_2 = (\exists r.C)^{\mathcal{I}}(y)$ is defined, by monotonicity of $\otimes$ this value satisfies

$$\begin{aligned}\ell_1 \otimes \ell_2 = r^{\mathcal{I}}(x,y) \otimes (\exists r.C)^{\mathcal{I}}(y) &= \bigvee_{z\in\Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \otimes r^{\mathcal{I}}(y,z) \otimes C^{\mathcal{I}}(z) \\ \leq \bigvee_{z\in\Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,z) \otimes C^{\mathcal{I}}(z) &\leq \bigvee_{z\in\Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,z) \otimes C^{\mathcal{I}}(z) = (\exists s.C)^{\mathcal{I}}(x) = \ell.\end{aligned}$$

Similar arguments show that $\mathbf{T}$ satisfies the other completeness conditions. □

We now present a tableaux algorithm that nondeterministically expands $\mathcal{A}$ to a tree-like ABox $\widehat{\mathcal{A}}$ that represents a model of $\mathcal{O}$. It uses the conditions from Table 1 and reformulates them into *expansion rules* of the form:

> "If there is $\langle\mathsf{trigger}\rangle$ in $\widehat{\mathcal{A}}$ and there are no $\langle\mathsf{values}\rangle$ such that $\langle\mathsf{assertions}\rangle$ are in $\mathcal{A}$, then introduce $\langle\mathsf{values}\rangle$ and add $\langle\mathsf{assertions}\rangle$ to $\widehat{\mathcal{A}}$."

The rules $\exists$ and $\forall$ always introduce new individuals y that do not appear in $\widehat{\mathcal{A}}$. Initially, the ABox $\mathcal{A}$ contains the single individual a. It is expanded by the rules in a tree-like way: role connections are only created by adding new successors to existing individuals. If an individual y was created by a rule $\exists$ or $\forall$ that was applied to an assertion involving an individual x, then we say that y is a *successor* of x, and x is the *predecessor* of y; *ancestor* is the transitive closure of *predecessor*. Note that the presence of an assertion $\langle (x, y) : r = \ell \rangle$ in $\widehat{\mathcal{A}}$ does not imply that y is a successor of x—it could also be the case that this assertion was introduced by the inv-rule. We further denote by $\widehat{\mathcal{A}}_x$ the set of all concept assertions from $\widehat{\mathcal{A}}$ that involve the individual x, i.e. are of the form $\langle x : C = \ell \rangle$ for some concept C and $\ell \in L$. To ensure that the application of the rules terminates, we need to add a blocking condition. We use *anywhere blocking* [20], which is based on the idea that it suffices to examine each set $\widehat{\mathcal{A}}_x$ only once in the whole ABox $\widehat{\mathcal{A}}$.

Let $\succ$ be a total order on the individuals of $\widehat{\mathcal{A}}$ that includes the ancestor relationship, i.e. whenever y is a successor of x, then $y \succ x$. An individual y is *directly blocked* if for some other individual x in $\widehat{\mathcal{A}}$ with $y \succ x$, $\widehat{\mathcal{A}}_x$ is equal to $\widehat{\mathcal{A}}_y$ modulo the individual names; in this case, we write $\widehat{\mathcal{A}}_x \equiv \widehat{\mathcal{A}}_y$ and also say that x *blocks* y. It is *indirectly blocked* if its predecessor is either directly or indirectly blocked. A node is *blocked* if it is either directly or indirectly blocked. The rules $\exists$ and $\forall$ are applied to $\widehat{\mathcal{A}}$ only if the node x that triggers their execution is not blocked. All other rules are applied only if x is not indirectly blocked.

The total order $\succ$ avoids cycles in the blocking relation. One possibility is to simply use the order in which the individuals were created by the expansion rules. Note that the only individual a that occurs in $\mathcal{A}$, which is the root of the tree-like structure represented by $\widehat{\mathcal{A}}$, cannot be blocked since it is an ancestor of all other individuals in $\widehat{\mathcal{A}}$. With this blocking condition, we can show that the size of $\widehat{\mathcal{A}}$ is bounded exponentially in the size of $\mathcal{A}$, as in the crisp case [20].

Lemma 16. *Every application of expansion rules to $\mathcal{A}$ terminates after at most exponentially many rule applications.*

Proof. Let $\mathsf{sub}(\mathcal{O})$ denote the set of all subconcepts of concepts appearing in $\mathcal{O}$ and recall that every rule application expands $\widehat{\mathcal{A}}$ in a tree-like manner. Note that there are at most $|L||\mathsf{sub}(\mathcal{O})|$ possible concept assertions for one individual x. Thus, every node in this tree has at most $|L||\mathsf{sub}(\mathcal{O})|$ successors: one for each possible assertion with a quantified concept. Moreover, there can be at most $2^{|L||\mathsf{sub}(\mathcal{O})|}$ non-blocked nodes in $\widehat{\mathcal{A}}$ at any time, and thus, when a node becomes blocked, at most exponentially many nodes become indirectly blocked. This shows that we obtain a tree of at most exponential size before every rule application is disallowed by the blocking condition. The claim now follows from the fact that every rule application adds at least one assertion to $\widehat{\mathcal{A}}$ and cannot remove assertions from $\widehat{\mathcal{A}}$. $\square$

We say that $\widehat{\mathcal{A}}$ contains a *clash* if it contains two assertions that are equal except for their lattice value (see Definition 14). $\widehat{\mathcal{A}}$ is *complete* if it contains a clash or none of the expansion rules are applicable. The algorithm is correct in the sense that it produces a clash iff $\mathcal{O}$ is not locally consistent.

Lemma 17. *$\mathcal{O}$ is locally consistent iff some application of the expansion rules to $\mathcal{A}$ yields a complete and clash-free ABox.*

Proof. By Lemma 15, $\mathcal{O}$ is locally consistent iff it has a tableau. Assume first that $\mathbf{T}$ is a tableau for $\mathcal{O}$ over the set Ind of individuals. We show how to guide the application of the expansion rules in such a way that no clash is produced.

Observe that the initial ABox $\mathcal{A}$ is included in $\mathbf{T}$ by definition. We will ensure that the expansion rules add only assertions to $\widehat{\mathcal{A}}$ that are also in $\mathbf{T}$. Assume that, for some row of Table 1, an expansion rule is applicable, i.e. $\langle\mathsf{trigger}\rangle$ is in $\widehat{\mathcal{A}}$ and there are no $\langle\mathsf{values}\rangle$ such that $\langle\mathsf{assertions}\rangle$ are in $\widehat{\mathcal{A}}$ and the blocking condition does not apply. Since $\langle\mathsf{trigger}\rangle$ is also in the tableau $\mathbf{T}$, there must be $\langle\mathsf{values}\rangle$ such that $\langle\mathsf{assertions}\rangle$ are in $\mathbf{T}$, and thus we can add $\langle\mathsf{assertions}\rangle$ to $\widehat{\mathcal{A}}$.

Since $\mathbf{T}$ is clash-free, this process cannot create any clashes in $\widehat{\mathcal{A}}$. Lemma 16 shows that at some point $\widehat{\mathcal{A}}$ must also be complete.

Assume now that the expansion rules have produced a complete and clash-free ABox $\widehat{\mathcal{A}}$. We define a tableau $\mathbf{T}$ for $\mathcal{O}$ over the set

$$\mathsf{Ind} := \{x \in \mathsf{N_I} \mid x \text{ occurs in } \widehat{\mathcal{A}} \text{ and is not blocked}\}$$

of individuals as follows:

$$\begin{aligned}
\mathbf{T} := \{&\langle x : C = \ell\rangle \in \widehat{\mathcal{A}} \mid x \in \mathsf{Ind}\} \\
\cup \{&\langle (x,y) : r = \ell\rangle \in \widehat{\mathcal{A}} \mid x, y \in \mathsf{Ind}\} \\
\cup \{&\langle (x,y) : r = \ell\rangle \mid x, y \in \mathsf{Ind}, \langle (x,z) : r = \ell\rangle \in \widehat{\mathcal{A}}, \text{ and } y \text{ blocks } z\} \\
\cup \{&\langle (x,y) : r = \ell\rangle \mid x, y \in \mathsf{Ind}, \langle (z,y) : r = \ell\rangle \in \widehat{\mathcal{A}}, \text{ and } x \text{ blocks } z\}.
\end{aligned}$$

Thus, whenever y blocks z and z is not indirectly blocked, then all incoming role connections of z are "re-routed" back to y. Since the root a of the tree-like structure $\widehat{\mathcal{A}}$ has no predecessors, it cannot be blocked, and thus the initial ABox $\mathcal{A}$ is still contained in $\mathbf{T}$. Furthermore, since $\widehat{\mathcal{A}}$ is clash-free, $\mathbf{T}$ is also clash-free.

Assume now that $\mathbf{T}$ violates the condition specified by some row of Table 1, i.e. there is $\langle\mathsf{trigger}\rangle$ in $\mathbf{T}$, but no $\langle\mathsf{values}\rangle$ such that $\langle\mathsf{assertions}\rangle$ are in $\mathbf{T}$.

a) If $\langle\mathsf{trigger}\rangle$ involves only assertions from $\widehat{\mathcal{A}}$, then the corresponding expansion rule was applied at some point and introduced $\langle\mathsf{values}\rangle$ and $\langle\mathsf{assertions}\rangle$. If no new individual was introduced, all $\langle\mathsf{assertions}\rangle$ must also be in $\mathbf{T}$. We consider now the case of the $\exists$-rule; the $\forall$-rule can be handled similarly.

   Assume that $\langle x : \exists r.C = \ell\rangle \in \widehat{\mathcal{A}}$ and x is not blocked. Then a new individual y was introduced, together with the assertions $\langle (x,y) : r = \ell_1\rangle$ and $\langle y : C = \ell_2\rangle$, where $\ell_1 \otimes \ell_2 = \ell$. If y is not blocked, these assertions are also in $\mathbf{T}$. If y is blocked by an individual z, then the assertion $\langle (x,z) : r = \ell_2\rangle$ is in $\mathbf{T}$. Additionally, we have $\widehat{\mathcal{A}}_y \equiv \widehat{\mathcal{A}}_z$, and thus also $\langle z : C = \ell_2\rangle$ is in $\mathbf{T}$.

b) If $\langle \mathsf{trigger} \rangle$ involves a role assertion $\langle (x, y) : r = \ell_1 \rangle$ where $\langle (x, z) : r = \ell_1 \rangle \in \widehat{\mathcal{A}}$ and y blocks z, then x is not blocked and the corresponding expansion rule was applied to $\widehat{\mathcal{A}}$ with z instead of y. Consider the case of the $\exists_{\leq}$-rule. Then the assertions $\langle x : \exists r.C = \ell \rangle$ and $\langle z : C = \ell_2 \rangle$ must be in $\widehat{\mathcal{A}}$ with $\ell_1 \otimes \ell_2 \leq \ell$. Since $\widehat{\mathcal{A}}_z \equiv \widehat{\mathcal{A}}_y$, we have $\langle y : C = \ell_2 \rangle$ in $\widehat{\mathcal{A}}$ and also in $\mathbf{T}$. The rules $\exists_+$, $\forall_{\geq}$, and $\forall_+$ behave similarly. If the inv-rule was applied, then we have $\langle (z, x) : \overline{r} = \ell_1 \rangle \in \widehat{\mathcal{A}}$, and thus $\langle (y, x) : \overline{r} = \ell_1 \rangle$ is in $\mathbf{T}$. If the $\sqsubseteq_{\mathcal{R}}$-rule was applied with $r \sqsubseteq_{\mathcal{R}} s$, then $\langle (x, z) : s = \ell_2 \rangle \in \widehat{\mathcal{A}}$ with some $\ell_2 \in L$ such that $\ell_1 \leq \ell_2$. Thus, we have $\langle (x, y) : s = \ell_2 \rangle$ in $\mathbf{T}$.

c) If $\langle \mathsf{trigger} \rangle$ involves a role assertion $\langle (x, y) : r = \ell_1 \rangle$ where $\langle (z, y) : r = \ell_1 \rangle \in \widehat{\mathcal{A}}$ and x blocks z, then consider the concrete condition concerned. If it is the $\exists_{\leq}$-condition, then we have $\langle x : \exists r.C = \ell \rangle$ in $\mathbf{T}$ and also in $\widehat{\mathcal{A}}$. Since $\widehat{\mathcal{A}}_x \equiv \widehat{\mathcal{A}}_z$, this implies that $\langle z : \exists r.C = \ell \rangle$ is in $\widehat{\mathcal{A}}$. Since z must be a successor of y, z is not indirectly blocked, and thus by the $\exists_{\leq}$-rule there is $\langle y : C = \ell_2 \rangle$ in $\widehat{\mathcal{A}}$ with $\ell_1 \otimes \ell_2 \leq \ell$. The same assertion must also be present in $\mathbf{T}$ since y is not blocked. Again, the conditions $\exists_+$, $\forall_{\geq}$, and $\forall_+$ can be handled similarly. If it is the inv-condition, then since z is not indirectly blocked, we have $\langle (y, z) : \overline{r} = \ell_1 \rangle \in \widehat{\mathcal{A}}$, and thus $\langle (y, x) : \overline{r} = \ell_1 \rangle$ in $\mathbf{T}$. If it is the $\sqsubseteq_{\mathcal{R}}$-condition with $r \sqsubseteq_{\mathcal{R}} s$, then since z is not indirectly blocked, there must be a value $\ell_2 \in L$ with $\ell_1 \leq \ell_2$ such that $\langle (z, y) : s = \ell_2 \rangle$ is in $\widehat{\mathcal{A}}$, and thus $\langle (x, y) : s = \ell_2 \rangle$ is in $\mathbf{T}$. $\square$

Since the tableau rules are nondeterministic, Lemmata 16 and 17 together imply that the tableaux algorithm decides local consistency in NExpTime.

Theorem 18. *Local consistency in L-$\mathcal{SHI}$ w.r.t. witnessed models can be decided in* NExpTime.

As explained before, L-$\mathcal{SHI}$ has the n-witnessed model property for some $n \geq 1$. We have so far restricted our description to the case where $n = 1$. If $n > 1$, it does not suffice to generate only one successor for every existential and universal restriction, but one must produce n different successors to ensure that the degrees guessed for these complex concepts are indeed witnessed by the model. The only required change to the algorithm is in the rows $\exists$ and $\forall$ of Table 1, where we have to introduce n individuals $y_1, \ldots, y_n$, and $2n$ values $\ell_1^1, \ell_2^1, \ldots, \ell_1^n, \ell_2^n \in L$ that satisfy $\bigvee_{i=1}^n \ell_1^i \otimes \ell_2^i = \ell$ or $\bigwedge_{i=1}^n \ell_1^i \Rightarrow \ell_2^i = \ell$, respectively.

5 Local Completion and Other Black-Box Reductions

In the following, we assume that we have a black-box procedure that decides local consistency in a sublogic of L-$\mathcal{SHI}$. This procedure can be, e.g. the tableau-based algorithm from the previous section, or any other method for solving this decision problem. We show how to employ such a procedure to solve other reasoning problems for this sublogic.

5.1 Consistency

To reduce consistency of an arbitrary ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ to local consistency, we first make sure that the information contained in $\mathcal{A}$ is consistent "in itself", i.e. if we only consider the individuals occurring in $\mathcal{A}$. It then suffices to check a local consistency condition for each of the individuals.

Let $\mathsf{Ind}_\mathcal{A}$ denote the set of individual names occurring in $\mathcal{A}$ and $\mathsf{sub}(\mathcal{A}, \mathcal{T})$ the set of all subconcepts of concepts occurring in $\mathcal{A}$ or $\mathcal{T}$. We first guess a set $\widehat{\mathcal{A}}$ of equality assertions of the forms $\langle a : C = \ell \rangle$ and $\langle (a, b) : r = \ell \rangle$ with $a, b \in \mathsf{Ind}_\mathcal{A}$, $C \in \mathsf{sub}(\mathcal{A}, \mathcal{T})$, $r \in \mathsf{N_R}$, and $\ell \in L$. We then check whether $\widehat{\mathcal{A}}$ is clash-free and satisfies the tableau conditions listed in Table 1, except the witnessing conditions $\exists$ and $\forall$. Additionally, we impose the following condition on $\widehat{\mathcal{A}}$:

> "If there is an assertion $\langle \alpha \rhd \ell \rangle$ in $\mathcal{A}$, then there is $\ell' \in L$ such that $\ell' \rhd \ell$ and $\langle \alpha = \ell' \rangle$ is in $\widehat{\mathcal{A}}$."

We call $\widehat{\mathcal{A}}$ *locally complete* iff it is of the above form and satisfies all of the above conditions. Guessing this set and checking whether it is locally complete can be done in polynomial time in the size of $\mathcal{O}$.

Lemma 19. *An ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ is consistent iff there is a locally complete set $\widehat{\mathcal{A}}$ such that $\mathcal{O}_x = (\widehat{\mathcal{A}}_x, \mathcal{T}, \mathcal{R})$ is locally consistent for every $x \in \mathsf{Ind}_\mathcal{A}$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{O}$ and $\widehat{\mathcal{A}}$ be the set of all assertions $\langle a : C = C^\mathcal{I}(a^\mathcal{I}) \rangle$ and $\langle (a, b) : r = r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \rangle$ for $a, b \in \mathsf{Ind}_\mathcal{A}$, $r \in \mathsf{N_R}$, and $C \in \mathsf{sub}(\mathcal{A}, \mathcal{T})$. Using the same arguments as in the proof of Lemma 15, we can show that $\widehat{\mathcal{A}}$ is locally complete. Furthermore, by construction $\mathcal{I}$ satisfies $\mathcal{O}_x$ for any $x \in \mathsf{Ind}_\mathcal{A}$.

Let now $\widehat{\mathcal{A}}$ be a locally complete set for $\mathcal{O}$ and $\mathcal{O}_x$ be locally consistent for every $x \in \mathsf{Ind}_\mathcal{A}$. By Lemma 15, for each $x \in \mathsf{Ind}_\mathcal{A}$ there is a tableau $\mathbf{T}_x$ for $\mathcal{O}_x$ over the set Ind_x of individuals. We can assume that the sets Ind_x are mutually disjoint. Note that $x \in \mathsf{Ind}_x$ for every $x \in \mathsf{Ind}_\mathcal{A}$.

We now define $C^\mathbf{T}(y) = \ell$ whenever $\langle y : C = \ell \rangle \in \mathbf{T}_x$ for some $x \in \mathsf{Ind}_\mathcal{A}$. Similarly, we set $r^\mathbf{T}(y, z) = \ell$ if $\langle (y, z) : r = \ell \rangle \in \mathbf{T}_x$ for some $x \in \mathsf{Ind}_\mathcal{A}$. Note that, since $\mathbf{T}$ is clash-free and the sets Ind_x are disjoint, these values are uniquely defined. To reconnect the individuals of $\mathsf{Ind}_\mathcal{A}$, we additionally define $r^\mathbf{T}(x, y) = \ell$ whenever $\langle (x, y) : r = \ell \rangle \in \widehat{\mathcal{A}}$.

As in the proof of Lemma 15, we can now define an interpretation $\mathcal{I}$ from these values by constructing the transitive closure of $r^{\mathbf{T}}$ if r is transitive. It then holds that $C^{\mathcal{I}}(x) = C^{\mathbf{T}}(x)$ whenever the latter is defined. Since the assertions in $\widehat{\mathcal{A}}$ satisfy $\mathcal{A}$, $\mathcal{I}$ also satisfies $\mathcal{A}$ and by the $\sqsubseteq_{\mathcal{T}}$- and $\sqsubseteq_{\mathcal{R}}$-conditions, $\mathcal{I}$ satisfies $\mathcal{T}$ and $\mathcal{R}$. $\square$

Theorem 20. *If local consistency in L-$\mathcal{SHI}$ can be decided in a complexity class* C*, then consistency in L-$\mathcal{SHI}$ can be decided in any complexity class that contains both* NP *and* C.

This means that consistency in L-$\mathcal{SHI}$ is decidable in NExpTime. In [9], an automata-based algorithm was presented that can decide satisfiability and subsumption in L-$\mathcal{ALCI}$ in ExpTime. Moreover, if the TBox is acyclic, then this bound can be improved to PSpace. The algorithm can easily be adapted to decide local consistency. With the above reduction, this shows that consistency in L-$\mathcal{ALCI}$ w.r.t. general and acyclic TBoxes can be decided in ExpTime and PSpace, respectively. The same argument applies to any sublogic of L-$\mathcal{SHI}$ for which local consistency can be decided in ExpTime or PSpace.

5.2 Satisfiability, Instance Checking, and Subsumption

To decide whether a concept C is strongly ℓ-satisfiable w.r.t. $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$, we can simply check whether $(\mathcal{A} \cup \{a : C \geq \ell\}, \mathcal{T}, \mathcal{R})$ is consistent for an arbitrary individual name a. Thus, strong ℓ-satisfiability is in the same complexity class as consistency. Moreover, we can easily compute the set of *all* values $\ell \in L$ such that the ontology $(\mathcal{A} \cup \{a : C \geq \ell\}, \mathcal{T}, \mathcal{R})$ is consistent by calling the decision procedure for consistency a constant number of times, i.e. once for each $\ell \in L$. We can use this set to compute the best bound for the satisfiability of C. Formally, the *best satisfiability degree* of a concept C is the supremum of all $\ell \in L$ such that C is ℓ-satisfiable w.r.t. $\mathcal{O}$. Since we can compute the set of all elements of L satisfying this property, obtaining the best satisfiability degree requires only a supremum computation. As the lattice L is fixed, this adds a constant factor to the complexity of checking consistency.

To check ℓ-instances, we can exploit the fact that a is not an ℓ-instance of C w.r.t. $\mathcal{O}$ iff there is a model $\mathcal{I}$ of $\mathcal{O}$ and a domain element $x \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a^{\mathcal{I}}) \not\geq \ell$. This is the case iff there is a value $\ell' \not\geq \ell$ such that the ontology $(\mathcal{A} \cup \{a : C = \ell'\}, \mathcal{T}, \mathcal{R})$ is consistent. Thus, ℓ-instances can be decided by calling the decision procedure for consistency a constant number of times, namely at most once for each $\ell' \in L$ with $\ell' \not\geq \ell$. We can also compute the *best instance degree* for a and C, which is the supremum of all $\ell \in L$ such that a is an ℓ-instance of C w.r.t. $\mathcal{O}$. Let $\mathfrak{L}$ denote the set of all ℓ' such that $(\{a : C = \ell'\}, \mathcal{T}, \mathcal{R})$ is consistent. The best instance degree for a and C is the infimum of all $\ell' \in \mathfrak{L}$ since

$$\begin{aligned}\bigvee\{\ell \in L \mid a \text{ is an } \ell\text{-instance of } C\} &= \bigvee\{\ell \in L \mid \forall \ell' \ngeq \ell : \ell' \notin \mathfrak{L}\} \\ &= \bigvee\{\ell \in L \mid \forall \ell' \in \mathfrak{L} : \ell \leq \ell'\} = \bigwedge \mathfrak{L}.\end{aligned}$$

Finally, note that C is ℓ-subsumed by D iff a is an ℓ-instance of $C \to D$, where a is a new individual name. Thus, deciding ℓ-subsumption and computing the *best subsumption degree* can be done using the same approach as above.

Lemma 21. *If local consistency in L-$\mathcal{SHI}$ can be decided in a complexity class* C*, then strong satisfiability, instance checking, and subsumption in L-$\mathcal{SHI}$ can be decided in any complexity class that contains both* NP *and* C.

This shows that also strong satisfiability, instance checking, and subsumption in L-$\mathcal{SHI}$ are in NExpTime. This bound reduces to ExpTime or PSpace if we consider L-$\mathcal{ALCI}$ w.r.t. general or acyclic TBoxes, respectively [9].

6 Conclusions

We have studied fuzzy description logics with semantics based on complete residuated De Morgan lattices. We showed that even for the fairly inexpressive DL L-$\mathcal{ALC}$, strong satisfiability w.r.t. general TBoxes is undecidable when the underlying lattice is infinite. For finite lattices, decidability is regained. In fact, local consistency can be decided with a nondeterministic tableaux-based procedure in exponential time. We conjecture that this upper bound can be improved to ExpTime either by an automata-based algorithm or with the help of advanced caching techniques [14]. Other decision and computation problems can also be solved using a local consistency reasoner as a black box. In particular, this yields tight complexity bounds for deciding consistency in L-$\mathcal{ALCI}$ w.r.t. acyclic and general TBoxes–PSpace and ExpTime, respectively.

The presented tableaux algorithm has highly nondeterministic rules, and as such is unsuitable for an implementation. Most of the optimizations developed for tableaux algorithms for crisp DLs, like the use of an optimized rule-application ordering, can be transfered to our setting. However, the most important task is to reduce the search space created by the choice of lattice values in most of the rules. We plan to study these optimizations in the future.

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