Exercise 8: Datalog

Database Theory

2023-06-06

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Exercise 1

- **Exercise.** Consider the example Datalog program from the lecture:

(a) Father(alice,bob)

(b) Mother(alice,carla)

(c) Mother(evan,carla)

(d) Father(carla,david)

$$\text{Parent}(x,y) \leftarrow \text{Father}(x,y) \quad (1)$$

$$\text{Parent}(x,y) \leftarrow \text{Mother}(x,y) \quad (2)$$

$$\text{Ancestor}(x,y) \leftarrow \text{Parent}(x,y) \quad (3)$$

$$\text{Ancestor}(x,z) \leftarrow \text{Parent}(x,y) \land \text{Ancestor}(y,z) \quad (4)$$

$$\text{SameGeneration}(x,x) \leftarrow \quad (5)$$

$$\text{SameGeneration}(x,y) \leftarrow \text{Parent}(x,v) \land \text{Parent}(y,w) \land \text{SameGeneration}(v,w) \quad (6)$$

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets T_P^0, T_P^1, T_P^2, ... When is the fixed point reached?

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2. Compute the sets T_P^0, T_P^1, T_P^2, ... When is the fixed point reached?

Solution.

1.

- SameGeneration(evan, alice) — (6); $\{ x \mapsto \text{evan}, v \mapsto \text{carla}, y \mapsto \text{alice}, w \mapsto \text{carla} \}$
  - Parent(evan, carla) — (2); $\{ x \mapsto \text{evan}, y \mapsto \text{carla} \}$
    - Mother(evan, carla) — (c)
  - Parent(alice, carla) — (2); $\{ x \mapsto \text{alice}, y \mapsto \text{carla} \}$
    - Mother(alice, carla) — (b)
  - SameGeneration(carla, carla) — (5); $\{ x \mapsto \text{carla} \}$

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Solution.

2. $$T_P^0 = \emptyset$$

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2. Compute the sets T_P^0, T_P^1, T_P^2, ... When is the fixed point reached?

Solution.

2.

$$T_P^0 = \emptyset$$

$$T_P^1 = \{ \text{Father(alice,bob)}, \text{Mother(alice,carla)}, \text{Mother(evan,carla)}, \text{Father(carla,david)}, \text{SameGeneration(alice,alice)}, \text{SameGeneration(bob,bob)}, \text{SameGeneration(carla,carla)}, \text{SameGeneration(david,david)}, \text{SameGeneration(evan,evan)} \}$$

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2. Compute the sets T_P^0, T_P^1, T_P^2, ... When is the fixed point reached?

Solution.

2.

$$T_P^0 = \varnothing$$

$$T_P^1 = \{ \text{Father(alice,bob)}, \text{Mother(alice,carla)}, \text{Mother(evan,carla)}, \text{Father(carla,david)},$$
$$\text{SameGeneration(alice,alice)}, \text{SameGeneration(bob,bob)}, \text{SameGeneration(carla,carla)}, \text{SameGeneration(david,david)}, \text{SameGeneration(evan,evan)} \}$$

$$T_P^2 = T_P^1 \cup \{ \text{Parent(alice,bob)}, \text{Parent(alice,carla)}, \text{Parent(evan,carla)}, \text{Parent(carla,david)} \}$$

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$$\text{SameGeneration(alice,evan)}, \text{SameGeneration(evan,alice)}\,\}$$

$$T_P^4 = T_P^3 \cup \{\,\text{Ancestor(alice,david)}, \text{Ancestor(evan,david)}\,\} = T_P^5$$

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$$T_P^4 = T_P^3 \cup \{ \text{Ancestor(alice,david)}, \text{Ancestor(evan,david)} \} = T_P^5 = T_P^\infty$$

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $\mathrm{e}(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

1. "Which nodes in the graph are reachable from the node n?"
2. "Are all nodes of the graph reachable from the node n?"
3. "Does the graph have a directed cycle?"
4. "Does the graph have a path that is labelled by a palindrome?"
   (a palindrome is a word that reads the same forwards and backwards)
5. "Is the connected component that contains the node n 2-colourable?"
6. "Is the graph 2-colourable?"
7. "Which pairs of nodes are connected by a path with an even number of a labels?"
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Solution.

1.

$$\begin{aligned} \mathrm{Reachable}(x, x) &\leftarrow \\ \mathrm{Reachable}(x, z) &\leftarrow \mathrm{e}(x, y, v) \wedge \mathrm{Reachable}(y, z) \\ \mathrm{Ans}(x) &\leftarrow \mathrm{Reachable}(n, x) \end{aligned}$$

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Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $\mathrm{e}(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

2. Not expressible, since Datalog is *monotone*: any query that is true for some set of ground facts I is also true for every set of ground facts $J \supseteq I$, but the query is true on $I = \{\, \mathrm{e}(n, n, a) \,\}$, but not on $J = I \cup \{\, \mathrm{e}(m, m, b) \,\}$.

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $\mathrm{e}(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

3.

$$\mathrm{Reachable}(x, y) \leftarrow \mathrm{e}(x, y, v)$$
$$\mathrm{Reachable}(x, z) \leftarrow \mathrm{e}(x, y, v) \wedge \mathrm{Reachable}(y, z)$$
$$\mathrm{Ans}() \leftarrow \mathrm{Reachable}(x, x)$$

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $\mathrm{e}(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

4.

$$\mathrm{Reachable}(x, x) \leftarrow$$

$$\mathrm{Reachable}(x, y) \leftarrow \mathrm{e}(x, y, v)$$

$$\mathrm{Reachable}(x, z) \leftarrow \mathrm{e}(x, y, a), \mathrm{Reachable}(y, w), \mathrm{e}(w, z, a)$$

$$\mathrm{Reachable}(x, z) \leftarrow \mathrm{e}(x, y, b), \mathrm{Reachable}(y, w), \mathrm{e}(w, z, b)$$

$$\mathrm{Ans}() \leftarrow \mathrm{Reachable}(x, y)$$

Exercise 2

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Solution.

5. Not expressible; consider $I = \{\, \mathrm{e}(n, 1, a), \mathrm{e}(1, 2, a) \,\}$ and $J = I \cup \{\, \mathrm{e}(2, n, a) \,\}$.

Exercise 2

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Solution.

6. Not expressible; consider $I = \{\, \mathrm{e}(n, 1, a), \mathrm{e}(1, 2, a) \,\}$ and $J = I \cup \{\, \mathrm{e}(2, n, a) \,\}$.

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $\mathrm{e}(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

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Solution.

7.

$$\mathrm{Reachable}(x, y) \leftarrow \mathrm{e}(x, y, b) \qquad \mathrm{Reachable}(x, z) \leftarrow \mathrm{e}(x, y, a), \mathrm{e}(y, z, a)$$

$$\mathrm{Reachable}(x, z) \leftarrow \mathrm{e}(x, y, a), \mathrm{Reachable}(y, w), \mathrm{e}(w, z, a) \qquad \mathrm{Reachable}(x, z) \leftarrow \mathrm{Reachable}(x, y), \mathrm{Reachable}(y, z)$$

$$\mathrm{Ans}(x, y) \leftarrow \mathrm{Reachable}(x, y)$$

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9. "Is there a pair of nodes that is connected by two distinct paths?"

Solution.

8.

$$\text{Reachable}(x, x) \leftarrow \qquad \text{Reachable}(x, z) \leftarrow \text{Reachable}(x, y), \text{Reachable}(y, z)$$

$$\text{Reachable}(x, z) \leftarrow \text{e}(x, y, a), \text{Reachable}(y, w), \text{e}(w, z, b) \qquad \text{Reachable}(x, z) \leftarrow \text{e}(x, y, b), \text{Reachable}(y, w), \text{e}(w, z, a)$$

$$\text{Ans}(x, y) \leftarrow \text{Reachable}(x, y)$$

Exercise 2

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $\mathrm{e}(m, n, a)$. Moreover, assume that only constants a and b are used as labels. Can you express the following queries using Datalog?

1. "Which nodes in the graph are reachable from the node n?"
2. "Are all nodes of the graph reachable from the node n?"
3. "Does the graph have a directed cycle?"
4. "Does the graph have a path that is labelled by a palindrome?"
   (a palindrome is a word that reads the same forwards and backwards)
5. "Is the connected component that contains the node n 2-colourable?"
6. "Is the graph 2-colourable?"
7. "Which pairs of nodes are connected by a path with an even number of a labels?"
8. "Which pairs of nodes are connected by a path with the same number of a and b labels?"
9. "Is there a pair of nodes that is connected by two distinct paths?"

Solution.

9. Not expressible, since Datalog is *homomorphism-closed*; consider $I = \{\, \mathrm{e}(n, 1, a), \mathrm{e}(1, m, a), \mathrm{e}(n, 2, a), \mathrm{e}(2, m, a) \,\}$ and $J = \{\, \mathrm{e}(n, 1, a), \mathrm{e}(1, m, a) \,\}$ and the homomorphism $\varphi : I \to J = \{\, 2 \mapsto 1 \,\}$.

Exercise 3

Exercise. Consider a UCQ of the following form

$$(r_{11}(x) \wedge r_{12}(x)) \vee \ldots \vee (r_{\ell 1}(x) \wedge r_{\ell 2}(x))$$

Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on ℓ)?

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Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on ℓ)?

Solution.

$$\begin{aligned}
\mathrm{Ans}(x) &\leftarrow \mathrm{r}_{11}(x), \mathrm{r}_{12}(x) \\
\mathrm{Ans}(x) &\leftarrow \mathrm{r}_{21}(x), \mathrm{r}_{22}(x) \\
\vdots \quad & \quad \vdots \qquad \vdots \\
\mathrm{Ans}(x) &\leftarrow \mathrm{r}_{\ell 1}(x), \mathrm{r}_{\ell 2}(x)
\end{aligned}$$

Exercise 3

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$$(r_{11}(x) \wedge r_{12}(x)) \vee \ldots \vee (r_{\ell 1}(x) \wedge r_{\ell 2}(x))$$

Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on ℓ)?

Solution.

$$\begin{aligned}
\mathrm{Ans}(x) &\leftarrow \mathrm{r}_{11}(x), \mathrm{r}_{12}(x) \\
\mathrm{Ans}(x) &\leftarrow \mathrm{r}_{21}(x), \mathrm{r}_{22}(x) \\
\vdots \quad & \quad\ \vdots \qquad \vdots \\
\mathrm{Ans}(x) &\leftarrow \mathrm{r}_{\ell 1}(x), \mathrm{r}_{\ell 2}(x)
\end{aligned}$$

This solution uses ℓ rules and one additional IDB predicate.

Exercise 4

Exercise. Consider a Datalog query of the following form:

$$
\begin{array}{ccc}
A_1(x) \leftarrow r_{11}(x) & \ldots & A_\ell(x) \leftarrow r_{\ell 1}(x) \\
A_1(x) \leftarrow r_{12}(x) & \ldots & A_\ell(x) \leftarrow r_{\ell 2}(x) \\
\end{array}
$$

$$
Ans(x) \leftarrow A_1(x), \ldots, A_\ell(x)
$$

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on ℓ)?

Exercise 4

Exercise. Consider a Datalog query of the following form:

$$\begin{array}{ccc} \mathsf{A}_1(x) \leftarrow \mathsf{r}_{11}(x) & \ldots & \mathsf{A}_\ell(x) \leftarrow \mathsf{r}_{\ell 1}(x) \\ \mathsf{A}_1(x) \leftarrow \mathsf{r}_{12}(x) & \ldots & \mathsf{A}_\ell(x) \leftarrow \mathsf{r}_{\ell 2}(x) \end{array}$$

$$\mathsf{Ans}(x) \leftarrow \mathsf{A}_1(x), \ldots, \mathsf{A}_\ell(x)$$

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on ℓ)?
Solution.

Exercise 4

Exercise. Consider a Datalog query of the following form:

$$
\begin{array}{ccc}
\mathrm{A}_1(x) \leftarrow \mathrm{r}_{11}(x) & \dots & \mathrm{A}_\ell(x) \leftarrow \mathrm{r}_{\ell 1}(x) \\
\mathrm{A}_1(x) \leftarrow \mathrm{r}_{12}(x) & \dots & \mathrm{A}_\ell(x) \leftarrow \mathrm{r}_{\ell 2}(x)
\end{array}
$$

$$\mathrm{Ans}(x) \leftarrow \mathrm{A}_1(x), \dots, \mathrm{A}_\ell(x)$$

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on ℓ)?
Solution.

$$
\begin{aligned}
\varphi_{11\cdots 1}(x) &= r_{11}(x) \wedge r_{21}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\varphi_{21\cdots 1}(x) &= r_{12}(x) \wedge r_{21}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\varphi_{12\cdots 1}(x) &= r_{11}(x) \wedge r_{22}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\varphi_{22\cdots 1}(x) &= r_{12}(x) \wedge r_{22}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\vdots \quad & \qquad \vdots \qquad\quad \vdots \qquad\qquad \vdots \\
\varphi_{22\cdots 2}(x) &= r_{12}(x) \wedge r_{22}(x) \wedge \cdots \wedge r_{\ell 2}(x)
\end{aligned}
$$

$$\varphi = \bigvee_{i \in \{\, 11\cdots 1, 21\cdots 1, \dots, 22\cdots 2\,\}} \varphi_i$$

Exercise 4

Exercise. Consider a Datalog query of the following form:

$$
\begin{array}{ccc}
\mathsf{A}_1(x) \leftarrow \mathsf{r}_{11}(x) & \dots & \mathsf{A}_\ell(x) \leftarrow \mathsf{r}_{\ell 1}(x) \\
\mathsf{A}_1(x) \leftarrow \mathsf{r}_{12}(x) & \dots & \mathsf{A}_\ell(x) \leftarrow \mathsf{r}_{\ell 2}(x)
\end{array}
$$

$$
\mathsf{Ans}(x) \leftarrow \mathsf{A}_1(x), \dots, \mathsf{A}_\ell(x)
$$

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on ℓ)?
Solution.

$$
\begin{aligned}
\varphi_{11\cdots 1}(x) &= r_{11}(x) \wedge r_{21}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\varphi_{21\cdots 1}(x) &= r_{12}(x) \wedge r_{21}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\varphi_{12\cdots 1}(x) &= r_{11}(x) \wedge r_{22}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\varphi_{22\cdots 1}(x) &= r_{12}(x) \wedge r_{22}(x) \wedge \cdots \wedge r_{\ell 1}(x) \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\varphi_{22\cdots 2}(x) &= r_{12}(x) \wedge r_{22}(x) \wedge \cdots \wedge r_{\ell 2}(x)
\end{aligned}
$$

$$
\varphi = \bigvee_{i \in \{11\cdots 1, 21\cdots 1, \ldots, 22\cdots 2\}} \varphi_i
$$

This solution uses 2^ℓ CQs.

Exercise 5

Exercise. Show that T_P^∞ is the least fixed point of the T_P operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of T_P must contain every fact in T_P^∞.

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Solution.

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1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of T_P must contain every fact in T_P^∞.

Solution.

1. ▶ We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \varnothing \subseteq T_P^1 = T_P(\varnothing)$.

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1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of T_P must contain every fact in T_P^∞.

Solution.

1. - We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \varnothing \subseteq T_P^1 = T_P(\varnothing)$.
   - Assume that $T_P^i \subseteq T_P^{i+1}$ for some $i \geq 0$, and consider a ground fact $H \in T_P^{i+1} = T_P(T_P^i)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^i$. Since $T_P^i \subseteq T_P^{i+1}$, we have $B_1, \ldots, B_n \in T_P^{i+1}$, and hence $H \in T_P^{i+2} = T_P(T_P^{i+1})$.

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1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of T_P must contain every fact in T_P^∞.

Solution.

1.
   - We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \emptyset \subseteq T_P^1 = T_P(\emptyset)$.
   - Assume that $T_P^i \subseteq T_P^{i+1}$ for some $i \geq 0$, and consider a ground fact $H \in T_P^{i+1} = T_P(T_P^i)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^i$. Since $T_P^i \subseteq T_P^{i+1}$, we have $B_1, \ldots, B_n \in T_P^{i+1}$, and hence $H \in T_P^{i+2} = T_P(T_P^{i+1})$.
   - Thus, we have $T_P^i \subseteq T_P^{i+k}$ for all $i, k \geq 0$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.

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Exercise. Show that T_P^∞ is the least fixed point of the T_P operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of T_P must contain every fact in T_P^∞.

Solution.

1. - We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \varnothing \subseteq T_P^1 = T_P(\varnothing)$.
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   - Thus, we have $T_P^i \subseteq T_P^{i+k}$ for all $i, k \geq 0$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
   - Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$.

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1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
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   - Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{i_j} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{ i_1, \ldots, i_n \}$.

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   - But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.

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   - But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.
2.
   - First, note that T_P is clearly *monotone*, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.

Exercise 5

Exercise. Show that T_P^∞ is the least fixed point of the T_P operator.

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   - Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.
   - Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{i_j} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{ i_1, \ldots, i_n \}$.
   - But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.
2.
   - First, note that T_P is clearly *monotone*, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   - Consider some fixed point F of T_P. We show $T_P^i \subseteq F$ for all $i \geq 0$.

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1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
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Solution.

1.
   - We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \emptyset \subseteq T_P^1 = T_P(\emptyset)$.
   - Assume that $T_P^i \subseteq T_P^{i+1}$ for some $i \geq 0$, and consider a ground fact $H \in T_P^{i+1} = T_P(T_P^i)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^i$. Since $T_P^i \subseteq T_P^{i+1}$, we have $B_1, \ldots, B_n \in T_P^{i+1}$, and hence $H \in T_P^{i+2} = T_P(T_P^{i+1})$.
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   - Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.
   - Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{i_j} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{ i_1, \ldots, i_n \}$.
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   - First, note that T_P is clearly *monotone*, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   - Consider some fixed point F of T_P. We show $T_P^i \subseteq F$ for all $i \geq 0$.
   - Clearly, $T_P^0 = \emptyset \subseteq F$.

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1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
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1.
   - We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \emptyset \subseteq T_P^1 = T_P(\emptyset)$.
   - Assume that $T_P^i \subseteq T_P^{i+1}$ for some $i \geq 0$, and consider a ground fact $H \in T_P^{i+1} = T_P(T_P^i)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^i$. Since $T_P^i \subseteq T_P^{i+1}$, we have $B_1, \ldots, B_n \in T_P^{i+1}$, and hence $H \in T_P^{i+2} = T_P(T_P^{i+1})$.
   - Thus, we have $T_P^i \subseteq T_P^{i+k}$ for all $i, k \geq 0$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
   - Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$.
   - Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.
   - Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{i_j} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{ i_1, \ldots, i_n \}$.
   - But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.
2.
   - First, note that T_P is clearly *monotone*, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   - Consider some fixed point F of T_P. We show $T_P^i \subseteq F$ for all $i \geq 0$.
   - Clearly, $T_P^0 = \emptyset \subseteq F$.
   - Assume that $T_P^i \subseteq F$ for some $i \geq 0$. Then $T_P^{i+1} = T_P(T_P^i) \subseteq T_P(F) = F$, by monotonicity and since F is a fixed point.

Exercise 5

Exercise. Show that T_P^∞ is the least fixed point of the T_P operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of T_P must contain every fact in T_P^∞.

Solution.

1.
   - We first show that for all $i \geq 0$, we have $T_P^i \subseteq T_P^{i+1}$: Clearly $T_P^0 = \varnothing \subseteq T_P^1 = T_P(\varnothing)$.
   - Assume that $T_P^i \subseteq T_P^{i+1}$ for some $i \geq 0$, and consider a ground fact $H \in T_P^{i+1} = T_P(T_P^i)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^i$. Since $T_P^i \subseteq T_P^{i+1}$, we have $B_1, \ldots, B_n \in T_P^{i+1}$, and hence $H \in T_P^{i+2} = T_P(T_P^{i+1})$.
   - Thus, we have $T_P^i \subseteq T_P^{i+k}$ for all $i, k \geq 0$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
   - Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$.
   - Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.
   - Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{i_j} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{ i_1, \ldots, i_n \}$.
   - But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.
2.
   - First, note that T_P is clearly *monotone*, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   - Consider some fixed point F of T_P. We show $T_P^i \subseteq F$ for all $i \geq 0$.
   - Clearly, $T_P^0 = \varnothing \subseteq F$.
   - Assume that $T_P^i \subseteq F$ for some $i \geq 0$. Then $T_P^{i+1} = T_P(T_P^i) \subseteq T_P(F) = F$, by monotonicity and since F is a fixed point.
   - But then $T_P^i \subseteq F$ for all $i \geq 0$, and hence also $T_P^\infty = \bigcup_{i \geq 0} T_P^i \subseteq F$.