

Locality for Classical Logic

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Abstract In this paper we will see deductive systems for classical propositional and predicate logic in the calculus of structures. Like sequent systems, they have a cut rule which is admissible. In addition, they enjoy a top-down symmetry and some normal forms for derivations that are not available in the sequent calculus. Identity axiom, cut, weakening and also contraction can be reduced to atomic form. This leads to rules that are *local*: they do not require the inspection of expressions of unbounded size.

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1 Introduction

Inference rules that copy an unbounded quantity of information are problematic from the points of view of complexity and implementation. In the sequent calculus, an example is given by the contraction rule in Gentzen's LK [6]:

$$\frac{\Gamma \vdash \Phi, A, A}{\Gamma \vdash \Phi, A} .$$

Here, going from bottom to top in constructing a proof, a formula A of unbounded size is duplicated. Whatever mechanism performs this duplication, it has to inspect all of A , so it has to have a *global* view on A . If, for example, we had to implement contraction on a distributed system, where each processor has a limited amount of local memory, the formula A could be spread over a number of processors. In that case, no single processor has a global view on A , and we should put in place complex mechanisms to cope with the situation.

Let us call *local* those inference rules that do not require such a global view on formulae of unbounded size, and *non-local* those rules that do. Further examples of non-local rules are the promotion rule in the sequent calculus for linear logic (left, [7]) and context-sharing (or additive) rules found in various sequent systems (right, [15]):

$$\frac{\vdash A, ?B_1, \dots, ?B_n}{\vdash !A, ?B_1, \dots, ?B_n} \quad \text{and} \quad \frac{\Gamma \vdash \Phi, A \quad \Gamma \vdash \Phi, B}{\Gamma \vdash \Phi, A \wedge B} .$$

To apply the promotion rule, one has to check whether all formulae in the context are prefixed with a question mark modality: the number of formulae to check is unbounded. To apply the context-sharing $R\wedge$ rule, a context of unbounded size has to be copied.

While there are methods to solve these problems in an implementation, an interesting question is whether it is possible to approach them proof-theoretically, i.e. by avoiding non-local rules. The present work gives an affirmative answer by presenting systems for both classical propositional and first-order predicate logic in which context-sharing rules as well as contraction are replaced by local rules. For propositional logic it is even possible to obtain a system which contains local rules only (which has already been presented in [3]).

Locality is achieved by reducing the problematic rules to their atomic forms. This is not entirely new: in most sequent systems for classical logic the

identity axiom is reduced to its atomic form, i.e.

$$\overline{A \vdash A} \quad \text{is equivalently replaced by} \quad \overline{a \vdash a} \quad ,$$

where a is an atom. Contraction, however, cannot be replaced by its atomic form in known sequent systems [2]. In fact, I believe that such a system cannot be presented in the sequent calculus. To obtain local inference rules, I employ the *calculus of structures* [9, 10]. This formalism differs from the sequent calculus in two main aspects:

1. *Deepness*: inference rules apply anywhere deep inside a formula, not only at the main connective. This is sound because implication is closed under disjunction, conjunction and quantification.
2. *Symmetry*: the notion of derivation is top-down symmetric: a derivation is dualised essentially by flipping it upside-down. One example of how this symmetry is useful is the reduction of the cut rule to atomic form.

The calculus of structures was conceived by Guglielmi in an earlier unpublished version of [9]. Its original purpose was to express a logical system with a self-dual non-commutative connective resembling sequential composition in process algebras [9, 10, 11, 4]. The present work explores the ideas developed in [9] in the setting of classical logic. The calculus of structures has also been employed by Straßburger in [13] to solve the problem of the non-local behaviour of the promotion rule and in [14] to give a local system for full linear logic. In the case of classical logic it led to a cut elimination procedure similar to normalisation in natural deduction [1].

This paper is structured as follows: in Section 2 I introduce the basic notions of the proof-theoretic formalism used, the calculus of structures. Section 3 is devoted to classical propositional logic and Section 4 to predicate logic.

The section for propositional logic is structured as follows: I first present system **SKSg**: a set of inference rules for classical propositional logic, which is closed under a notion of duality. I translate derivations of a Gentzen-Schütte sequent system into this system, and vice versa. This establishes soundness and completeness with respect to classical propositional logic as well as cut elimination. In the following I obtain an equivalent system, named **SKS**, in which identity, cut, weakening and contraction are reduced to atomic form. This entails locality of the system. I go on to establish three different normal forms for derivations, by what I call ‘decomposition theorems’.

The outline of the section for predicate logic closely follows the one for propositional logic. All results for the propositional systems scale: reduction of cut, identity, weakening and contraction to atomic form, cut elimination as well as the decomposition theorems. The resulting system with atomic rules is local except for the rules that instantiate variables or check for free occurrences of a variable.

2 The Calculus of Structures

Definition 2.1. *Atoms* are denoted by a, b, \dots . The *structures* of the language KS are generated by

$$S ::= \mathbf{f} \mid \mathbf{t} \mid a \mid \underbrace{[S, \dots, S]}_{>0} \mid \underbrace{(S, \dots, S)}_{>0} \mid \bar{S} \quad ,$$

where \mathbf{f} and \mathbf{t} are the units *false* and *true*, $[S_1, \dots, S_h]$ is a *disjunction* and (S_1, \dots, S_h) is a *conjunction*. \bar{S} is the *negation* of the structure S . The negation of an atom is again an atom. Structures are denoted by S, P, Q, R, T, U, V and W . *Structure contexts*, denoted by $S\{ \}$, are structures with one occurrence of $\{ \}$, the *empty context* or *hole*, that does not appear in the scope of a negation. $S\{R\}$ denotes the structure obtained by filling the hole in $S\{ \}$ with R . We drop the curly braces when they are redundant: for example, $S[R, T]$ is short for $S\{[R, T]\}$. A structure R is a *substructure* of a structure T if there is a context $S\{ \}$ such that $S\{R\}$ is T . Structures are *equivalent* modulo the smallest equivalence relation induced by the equations shown in Figure 1. There, \vec{R}, \vec{T} and \vec{U} are finite sequences of structures, \vec{T} is non-empty. Structures are in *normal form* if negation occurs only on atoms, and extra units as well as connectives are removed using the laws for units and associativity. In general we consider structures to be in normal form and do not distinguish between two equivalent structures.

Example 2.2. The structures $\overline{[a, \mathbf{f}, b]}$ and $(\bar{a}, \mathbf{t}, \bar{b})$ are equivalent, but they are not normal; (\bar{a}, \bar{b}) is equivalent to them and normal, as well as (\bar{b}, \bar{a}) . The atom a is not a substructure of $\overline{[a, \mathbf{f}, b]}$, but \bar{a} is a substructure of (\bar{a}, \bar{b}) .

Structures are somewhere between formulae and sequents. They share with formulae their tree-like shape and with sequents the built-in, decidable equivalence modulo associativity and commutativity. The equations for negation are adopted also in one-sided sequent systems, so, apart from the equations for the units, the calculus of structures does not use new equations. However,

Associativity

$$[\vec{R}, [\vec{T}, \vec{U}]] = [\vec{R}, \vec{T}, \vec{U}]$$

$$(\vec{R}, (\vec{T}, \vec{U})) = (\vec{R}, \vec{T}, \vec{U})$$

Commutativity

$$[R, T] = [T, R]$$

$$(R, T) = (T, R)$$

Units

$$(f, f) = f \quad [f, R] = R$$

$$[t, t] = t \quad (t, R) = R$$

Negation

$$\bar{f} = t$$

$$\bar{t} = f$$

$$\overline{[R, T]} = (\bar{R}, \bar{T})$$

$$\overline{(R, T)} = [\bar{R}, \bar{T}]$$

$$\bar{\bar{R}} = R$$

Context Closure

$$\text{if } R = T \text{ then } S\{R\} = S\{T\}$$

$$\bar{\bar{R}} = \bar{T}$$

Figure 1: Syntactic equivalence of structures

from the viewpoint of the sequent calculus, it does extend the applicability of equations from the level of sequents to the level of formulae.

Definition 2.3. An *inference rule* is a scheme of the kind

$$\rho \frac{V}{U} \text{ ,}$$

where ρ is the *name* of the rule, V is its *premise* and U is its *conclusion*. If V is of the form $S\{T\}$ and U is of the form $S\{R\}$ then the inference rule is called *deep*, otherwise it is called *shallow*. In an instance of a deep inference rule

$$\pi \frac{S\{T\}}{S\{R\}} \text{ ,}$$

the structure taking the place of R is its *redex*, the structure taking the place of T is its *contractum* and the context taking the place of $S\{ \}$ is its *context*. A (*deductive*) *system* \mathcal{S} is a set of inference rules.

Most inference rules we will consider are deep. A deep inference rule can be seen as a rewrite rule with the context made explicit. For example, the rule π from the previous definition seen top-down corresponds to a rewrite rule $T \rightarrow R$. A shallow inference rule can be seen as a rewrite rule that may only be applied to the whole given term, not to arbitrary subterms.

Notation 2.4. To clarify the use of the syntactic equivalence where it is not obvious, I use the *equivalence rule*

$$= \frac{T}{R} \quad ,$$

where R and T are equivalent structures.

Definition 2.5. A *derivation* Δ in a certain deductive system is a finite sequence of instances of inference rules in the system:

$$\begin{array}{c} T \\ \pi \frac{\quad}{V} \\ \pi' \frac{\quad}{\quad} \\ \vdots \\ \rho' \frac{\quad}{U} \\ \rho \frac{\quad}{R} \end{array} .$$

A derivation can consist of just one structure. The topmost structure in a derivation is called the *premise* of the derivation, and the structure at the bottom is called its *conclusion*.

Note that the notion of derivation is top-down symmetric, contrary to the corresponding notion in the sequent calculus.

Notation 2.6. A derivation Δ whose premise is T , whose conclusion is R , and whose inference rules are in \mathcal{S} is denoted by

$$\frac{T}{\Delta \parallel_{\mathcal{S}} R} .$$

Definition 2.7. A rule ρ is *derivable* for a system \mathcal{S} if for every instance of

$$\rho \frac{T}{R} \quad \text{there is a derivation } \frac{T}{\parallel_{\mathcal{S}} R} .$$

The symmetry of derivations, where both premise and conclusion are arbitrary structures, is broken in the notion of *proof*:

Definition 2.8. A *proof* is a derivation whose premise is the unit true. A proof Π of R in system \mathcal{S} is denoted by

$$\frac{\quad}{\Pi \parallel_{\mathcal{S}} R} .$$

3 Propositional Logic

3.1 A Symmetric System

The following notion of duality is known as *contrapositive*:

Definition 3.1. The *dual* of an inference rule is obtained by exchanging premise and conclusion and replacing each connective by its De Morgan dual. For example

$$i\downarrow \frac{S\{t\}}{S[R, \bar{R}]} \quad \text{is dual to} \quad i\uparrow \frac{S(R, \bar{R})}{S\{f\}} ,$$

where the rule $i\downarrow$ is called *identity* and the rule $i\uparrow$ is called *cut*.

The rules $i\downarrow$ and $i\uparrow$ respectively correspond to the identity axiom and the cut rule in the sequent calculus, as we will see shortly.

Definition 3.2. A system of inference rules is called *symmetric* if for each of its rules it also contains the dual rule.

An example of a symmetric system is shown in Figure 2. It is called system **SKSg**, where the first **S** stands for “symmetric”, **K** stands for “klassisch” as in Gentzen’s **LK** and the second **S** says that it is a system on structures. Small letters are appended to the name of a system to denote variants. In this case, the **g** stands for “general”, meaning that rules are not restricted to atoms: they can be applied to arbitrary structures. We will see in the next section that this system is sound and complete for classical propositional logic.

The rules **s**, **w** \downarrow and **c** \downarrow are called respectively *switch*, *weakening* and *contraction*. Their dual rules carry the same name prefixed with a “co-”, so e.g. **w** \uparrow is called *co-weakening*. Rules $i\downarrow$, **w** \downarrow , **c** \downarrow are called *down-rules* and their duals are called *up-rules*. The dual of the switch rule is the switch rule itself: it is *self-dual*.

I now try to give an idea on how the familiar rules of the sequent calculus correspond to the rules of **SKSg**. For the sake of simplicity I consider the rules of the sequent calculus in isolation, i.e. not as part of a proof tree. The full correspondence is shown in Section 3.2.

The identity axiom of the sequent calculus corresponds to the identity rule $i\downarrow$:

$$\overline{\vdash A, \bar{A}} \quad \text{corresponds to} \quad i\downarrow \frac{t}{[A, \bar{A}]} .$$

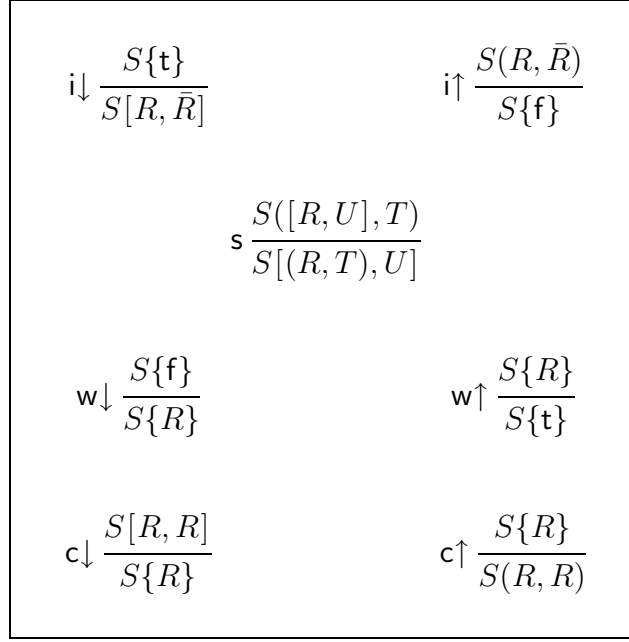


Figure 2: System SKSg

However, $i\downarrow$ can appear anywhere in a proof, not only at the top. The cut rule of the sequent calculus corresponds to the rule $i\uparrow$ followed by two instances of the switch rule:

$$\text{Cut} \frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi} \quad \text{corresponds to} \quad \begin{array}{c} s \frac{([\Phi, A], [\Psi, \bar{A}])}{[\Phi, (A, [\Psi, \bar{A}])]} \\ s \frac{[\Phi, \Psi, (A, \bar{A})]}{[\Phi, \Psi, f]} \\ i\uparrow \\ = \frac{[\Phi, \Psi]}{[\Phi, \Psi]} \end{array} .$$

The multiplicative (or context-splitting) $R\wedge$ in the sequent calculus corresponds to two instances of switch rule:

$$R\wedge \frac{\vdash \Phi, A \quad \vdash \Psi, B}{\vdash \Phi, \Psi, A \wedge B} \quad \text{corresponds to} \quad \begin{array}{c} s \frac{([\Phi, A], [\Psi, B])}{[\Phi, (A, [\Psi, B])]} \\ s \frac{[\Phi, \Psi, (A, B)]}{[\Phi, \Psi, (A, B)]} \end{array} .$$

A contraction in the sequent calculus corresponds to the $c\downarrow$ rule:

$$\text{RC} \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \quad \text{corresponds to} \quad c\downarrow \frac{[\Phi, A, A]}{[\Phi, A]} ,$$

just as the weakening in the sequent calculus corresponds to the $w\downarrow$ rule:

$$\text{RW} \frac{\vdash \Phi}{\vdash \Phi, A} \quad \text{corresponds to} \quad w\downarrow \frac{\frac{\Phi}{[\Phi, f]}}{[\Phi, A]} .$$

The $c\uparrow$ and $w\uparrow$ rules have no analogue in the sequent calculus. Their role is to ensure that our system is symmetric. They are obviously sound since they are just duals of the rules $c\downarrow$ and $w\downarrow$ which correspond to sequent calculus rules.

Derivations in a symmetric system can be dualised:

Definition 3.3. The *dual* of a derivation is obtained by turning it upside-down and replacing each rule, each connective and each atom by its dual.

For example

$$w\uparrow \frac{[(a, \bar{b}), a]}{[a, a]} \quad c\downarrow \frac{[a, a]}{a} \quad \text{is dual to} \quad c\uparrow \frac{\bar{a}}{(\bar{a}, \bar{a})} \quad w\downarrow \frac{(\bar{a}, \bar{a})}{([\bar{a}, b], \bar{a})} .$$

The notion of proof, however, is an asymmetric one: the dual of a proof is not a proof.

3.2 Correspondence to the Sequent Calculus

The sequent system that is most similar to system **SKSg** is the one-sided system **GS1p** [15], also called *Gentzen-Schütte* system. In this section we consider a version of **GS1p** with multiplicative context treatment and constants \top and \perp , and we translate its derivations to derivations in **SKSg** and vice versa. Translating from the sequent calculus to the calculus of structures is straightforward, in particular, no new cuts are introduced in the process. But to translate in the other direction we have to simulate deep inferences in the sequent calculus, which is done by using the cut rule.

$$\begin{array}{c}
\top \frac{}{\vdash \top} \qquad \text{Ax} \frac{}{\vdash A, \bar{A}} \\
\text{R}\wedge \frac{\vdash \Phi, A \quad \vdash \Psi, B}{\vdash \Phi, \Psi, A \wedge B} \qquad \text{R}\vee \frac{\vdash \Phi, A, B}{\vdash \Phi, A \vee B} \\
\text{RC} \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad \text{RW} \frac{\vdash \Phi}{\vdash \Phi, A}
\end{array}$$

Figure 3: GS1p: classical logic in Gentzen-Schütte form

One consequence of those translations is that system **SKSg** is sound and complete for classical propositional logic. Another consequence is cut elimination: one can translate a proof with cuts in **SKSg** to a proof in **GS1p + Cut**, apply cut elimination for **GS1p**, and translate back the resulting cut-free proof to obtain a cut-free proof in **SKSg**.

Definition 3.4. System **GS1p** is the set of rules shown in Figure 3. The system **GS1p + Cut** is **GS1p** together with

$$\text{Cut} \frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi} .$$

Formulae are denoted by A and B . They contain negation only on atoms and may contain the constants \top and \perp . Multisets of formulae are denoted by Φ and Ψ . The empty multiset is denoted by \emptyset . In A_1, \dots, A_h , where $h \geq 0$, a formula denotes the corresponding singleton multiset and the comma denotes multiset union. *Sequents*, denoted by Σ , are multisets of formulae.

Derivations are denoted by Δ or $\frac{\Sigma_1 \cdots \Sigma_h}{\Delta}$, where $h \geq 0$, the sequents

$\Sigma_1, \dots, \Sigma_h$ are the *premises* and Σ is the *conclusion*. Proofs, denoted by Π , are derivations where each leaf is an instance of **Ax**.

From the Sequent Calculus to the Calculus of Structures

Definition 3.5. The function $\underline{\cdot}_s$ maps formulae, multisets of formulae and sequents of **GS1p** to structures:

$$\begin{aligned}
 \underline{a}_s &= a, \\
 \underline{\top}_s &= \mathbf{t}, \\
 \underline{\perp}_s &= \mathbf{f}, \\
 \underline{A \vee B}_s &= [\underline{A}_s, \underline{B}_s], \\
 \underline{A \wedge B}_s &= (\underline{A}_s, \underline{B}_s), \\
 \underline{\emptyset}_s &= \mathbf{f}, \\
 \underline{A_1, \dots, A_{h_s}} &= [\underline{A_{1_s}}, \dots, \underline{A_{h_s}}], \quad \text{where } h > 0.
 \end{aligned}$$

In proofs, when no confusion is possible, the subscript $\underline{\cdot}_s$ may be dropped to improve readability.

In the following, we will put derivations of system **SKSg** into a context. This is possible because all inference rules of the system are deep.

Definition 3.6. Given a derivation Δ , the derivation $S\{\Delta\}$ is obtained as follows:

$$\Delta = \begin{array}{c} \pi' \frac{T}{V} \\ \pi \frac{}{} \\ \vdots \\ \rho' \frac{}{U} \\ \rho \frac{}{R} \end{array} \quad S\{\Delta\} = \begin{array}{c} \pi' \frac{S\{T\}}{S\{V\}} \\ \pi \frac{}{} \\ \vdots \\ \rho' \frac{}{S\{U\}} \\ \rho \frac{}{S\{R\}} \end{array} .$$

Theorem 3.7. For every derivation $\begin{array}{c} \Sigma_1 \cdots \Sigma_h \\ \triangle \\ \Sigma \end{array}$ in **GS1p** + Cut there exists a

derivation $\begin{array}{c} (\underline{\Sigma_{1_s}}, \dots, \underline{\Sigma_{h_s}}) \\ \parallel_{\text{SKSg} \setminus \{c\uparrow, w\uparrow\}} \\ \underline{\Sigma}_s \end{array}$ with the same number of cuts.

Proof. By structural induction on the given derivation Δ .

Base Cases

1. $\Delta = \Sigma$. Take $\underline{\Sigma}_s$.
2. $\Delta = \top \frac{}{\vdash \top}$. Take \mathbf{t} .
3. $\Delta = \text{Ax} \frac{}{\vdash A, \bar{A}}$. Take $\text{id} \frac{\mathbf{t}}{[\underline{A}_s, \bar{A}_s]}$.

Inductive Cases

In the case of the $R\wedge$ rule, we have a derivation

$$\Delta = \frac{\frac{\Sigma_1 \cdots \Sigma_k}{\vdash \Phi, A} \quad \frac{\Sigma'_1 \cdots \Sigma'_l}{\vdash \Psi, B}}{R\wedge \frac{}{\vdash \Phi, \Psi, A \wedge B}} .$$

By induction hypothesis we obtain derivations

$$\begin{array}{c} (\Sigma_1, \dots, \Sigma_k) \\ \Delta_1 \parallel \text{SKSg} \setminus \{\text{c}\uparrow, \text{w}\uparrow\} \\ [\Phi, A] \end{array} \quad \text{and} \quad \begin{array}{c} (\Sigma'_1, \dots, \Sigma'_l) \\ \Delta_2 \parallel \text{SKSg} \setminus \{\text{c}\uparrow, \text{w}\uparrow\} \\ [\Psi, B] \end{array} .$$

The derivation Δ_1 is put into the context $(\{ \}, \Sigma'_1, \dots, \Sigma'_l)$ to obtain Δ'_1 and the derivation Δ_2 is put into the context $([\Phi, A], \{ \})$ to obtain Δ'_2 :

$$\begin{array}{c} (\Sigma_1, \dots, \Sigma_k, \Sigma'_1, \dots, \Sigma'_l) \\ \Delta'_1 \parallel \text{SKSg} \setminus \{\text{c}\uparrow, \text{w}\uparrow\} \\ ([\Phi, A], \Sigma'_1, \dots, \Sigma'_l) \end{array} \quad \text{and} \quad \begin{array}{c} ([\Phi, A], \Sigma'_1, \dots, \Sigma'_l) \\ \Delta'_2 \parallel \text{SKSg} \setminus \{\text{c}\uparrow, \text{w}\uparrow\} \\ ([\Phi, A], [\Psi, B]) \end{array} .$$

The derivation in SKSg we are looking for is obtained by composing Δ'_1 and Δ'_2 and applying the switch rule twice:

$$\begin{array}{c}
(\Sigma_1, \dots, \Sigma_k, \Sigma'_1, \dots, \Sigma'_l) \\
\Delta'_1 \parallel \text{SKSg} \setminus \{c\uparrow, w\uparrow\} \\
([\Phi, A], \Sigma'_1, \dots, \Sigma'_l) \\
\Delta'_2 \parallel \text{SKSg} \setminus \{c\uparrow, w\uparrow\} \\
\frac{([\Phi, A], [\Psi, B])}{\text{S} \frac{[\Psi, ([\Phi, A], B)]}{\text{S} [\Phi, \Psi, (A, B)]}}
\end{array}$$

The other cases are similar. The only case that requires a cut in SKSg is a cut in GS1p. \square

Corollary 3.8. If a sequent Σ has a proof in GS1p + Cut then $\underline{\Sigma}_s$ has a proof in SKSg $\setminus \{c\uparrow, w\uparrow\}$.

Corollary 3.9. If a sequent Σ has a proof in GS1p then $\underline{\Sigma}_s$ has a proof in SKSg $\setminus \{i\uparrow, c\uparrow, w\uparrow\}$.

From the Calculus of Structures to the Sequent Calculus

In the following, structures are assumed to contain 1) negation only on atoms and 2) only conjunctions and disjunctions of exactly two structures. That is not a restriction because for each structure there exists an equivalent one which has these properties.

Definition 3.10. The function $\underline{\quad}_g$ maps structures of SKSg to formulae of GS1p:

$$\begin{array}{lcl}
\underline{a}_g & = & a, \\
\underline{\top}_g & = & \top, \\
\underline{\perp}_g & = & \perp, \\
\underline{[R, T]}_g & = & \underline{R}_g \vee \underline{T}_g, \\
\underline{(R, T)}_g & = & \underline{R}_g \wedge \underline{T}_g.
\end{array}$$

Lemma 3.11. For every two formulae A, B and every formula context $C\{ \}$

there exists a derivation $\frac{\vdash A, \overline{B}}{\vdash C\{A\}, \overline{C\{B\}}}$ in GS1p.

Proof. By structural induction on the context $C\{ \}$. The base case in which $C\{ \} = \{ \}$ is trivial. If $C\{ \} = C_1 \wedge C_2\{ \}$, then the derivation we are looking for is

$$\frac{\frac{\text{Ax} \frac{\overline{\vdash C_1, \bar{C}_1} \quad \overline{\vdash C_2\{A\}, C_2\{B\}}}{\text{R}\wedge} \quad \overline{\vdash C_1 \wedge C_2\{A\}, \bar{C}_1, \bar{C}_2\{B\}}}{\text{R}\vee} \quad \overline{\vdash C_1 \wedge C_2\{A\}, \bar{C}_1 \vee \bar{C}_2\{B\}}}{\vdash A, \bar{B}} \Delta$$

where Δ exists by induction hypothesis. The other case, in which $C\{ \} = C_1 \vee C_2\{ \}$, is similar. \square

Theorem 3.12. For every derivation $\frac{Q}{P} \Big\|_{\text{SKSg}}$ there exists a derivation $\frac{\vdash Q_{\text{G}}}{\vdash P_{\text{G}}}$

in $\text{GS1p} + \text{Cut}$.

Proof. We construct the sequent derivation by induction on the length of the given derivation Δ in SKSg .

Base Case

If Δ consists of just one structure P , then P and Q are the same. Take $\vdash P_{\text{G}}$.

Inductive Cases

We single out the topmost rule instance in Δ :

$$\frac{Q}{P} \Big\|_{\text{SKSg}} \Delta = \frac{\rho \frac{S\{T\}}{S\{R\}}}{P} \Big\|_{\text{SKSg}} \Delta'$$

The corresponding derivation in GS1p will be as follows:

$$\begin{array}{c}
\Pi \\
\vdash R, \bar{T} \\
\Delta_1 \\
\text{Cut} \frac{\vdash S\{R\}, \overline{S\{T\}} \quad \vdash S\{T\}}{\vdash S\{R\}} , \\
\Delta_2 \\
\vdash P
\end{array}$$

where Δ_1 exists by Lemma 3.11 and Δ_2 exists by induction hypothesis. The proof Π depends on the rule ρ . In the following we will see that the proof Π exists for all the rules of **SKSg**.

For identity and cut, i.e.

$$i\downarrow \frac{S\{t\}}{S[U, \bar{U}]} \quad \text{and} \quad i\uparrow \frac{S(U, \bar{U})}{S\{f\}} ,$$

we have the following proofs:

$$\begin{array}{c}
\text{Ax} \frac{}{\vdash U, \bar{U}} \\
\text{RV} \frac{}{\vdash U \vee \bar{U}} \\
\text{RW} \frac{}{\vdash U \vee \bar{U}, \perp}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\text{Ax} \frac{}{\vdash U, \bar{U}} \\
\text{RV} \frac{}{\vdash \bar{U} \vee U} \\
\text{RW} \frac{}{\vdash \perp, \bar{U} \vee U}
\end{array} .$$

In the case of the switch rule, i.e.

$$s \frac{S([U, V], T)}{S[(U, T), V]} ,$$

we have

$$\begin{array}{c}
\text{Ax} \frac{}{\vdash U, \bar{U}} \quad \text{Ax} \frac{}{\vdash V, \bar{V}} \\
\text{R}\wedge \frac{}{\vdash U, \bar{U} \wedge \bar{V}, V} \quad \text{Ax} \frac{}{\vdash T, \bar{T}} \\
\text{R}\wedge \frac{}{\vdash (U \wedge T), V, (\bar{U} \wedge \bar{V}), \bar{T}} \\
\text{RV}^2 \frac{}{\vdash (U \wedge T) \vee V, (\bar{U} \wedge \bar{V}) \vee \bar{T}}
\end{array} .$$

For contraction and its dual, i.e.

$$c\downarrow \frac{S[U, U]}{S\{U\}} \quad \text{and} \quad c\uparrow \frac{S\{U\}}{S(U, U)} ,$$

we have

$$\begin{array}{c} \text{Ax} \frac{}{\vdash U, \bar{U}} \quad \text{Ax} \frac{}{\vdash U, \bar{U}} \\ \text{R}\wedge \frac{}{\vdash U, U, \bar{U} \wedge \bar{U}} \\ \text{RC} \frac{}{\vdash U, \bar{U} \wedge \bar{U}} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Ax} \frac{}{\vdash U, \bar{U}} \quad \text{Ax} \frac{}{\vdash U, \bar{U}} \\ \text{R}\wedge \frac{}{\vdash U \wedge U, \bar{U}, \bar{U}} \\ \text{RC} \frac{}{\vdash U \wedge U, \bar{U}} \end{array} .$$

For weakening and its dual, i.e.

$$w\downarrow \frac{S\{f\}}{S\{U\}} \quad \text{and} \quad w\uparrow \frac{S\{U\}}{S\{t\}} ,$$

we have

$$\text{RW} \frac{\top \frac{}{\vdash \top}}{\vdash U, \top} \quad \text{and} \quad \text{RW} \frac{\top \frac{}{\vdash \top}}{\vdash \top, \bar{U}} .$$

□

Corollary 3.13. If a structure S has a proof in SKSg then $\vdash \underline{S}_{\mathbf{g}}$ has a proof in $\text{GS1p} + \text{Cut}$.

Soundness and completeness of SKSg , i.e. the fact that a structure has a proof if and only if it is valid, follows from soundness and completeness of GS1p by Corollaries 3.8 and 3.13. Moreover, a structure T implies a structure R if and only if there is a derivation from T to R , which follows from soundness and completeness and the following theorem.

Theorem 3.14.

There is a derivation $\begin{array}{c} T \\ \parallel_{\text{SKSg}} \\ R \end{array}$ if and only if there is a proof $\begin{array}{c} \parallel_{\text{SKSg}} \\ [\bar{T}, R] \end{array}$.

Proof. A proof Π can be obtained from a given derivation Δ and a derivation

Δ from a given proof Π , respectively, as follows:

$$\begin{array}{c}
\text{t} \\
\hline
i\downarrow \frac{[\bar{T}, T]}{[\bar{T}, \Delta]} \Big\|_{\text{SKSg}} \\
[\bar{T}, R]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
T \\
(T, \Pi) \Big\|_{\text{SKSg}} \\
\frac{(T, [\bar{T}, R])}{[R, (T, \bar{T})]} \\
\hline
s \\
i\uparrow \frac{[R, (T, \bar{T})]}{R}
\end{array}
.$$

□

3.3 Admissibility of the Cut and the Other Up-Rules

If one is just interested in provability, then the up-rules of the symmetric system SKSg , i.e. $i\uparrow$, $w\uparrow$ and $c\uparrow$, are superfluous. By removing them we obtain the asymmetric, cut-free system shown in Figure 4, which is called system KSg .

Definition 3.15. A rule ρ is *admissible* for a system \mathcal{S} if for every proof $\Big\|_{\mathcal{S} \cup \{\rho\}}$ there is a proof $\Big\|_{\mathcal{S}}$.

The admissibility of all the up-rules for system KSg is shown by using the translation functions from the previous section:

Theorem 3.16. The rules $i\uparrow$, $w\uparrow$ and $c\uparrow$ are admissible for system KSg .

Proof.

$$\Big\|_{\text{SKSg}} \xrightarrow{\text{Corollary 3.13}} \begin{array}{c} \text{GS1p} \\ +\text{Cut} \\ \hline \vdash \underline{S}_{\text{G}} \end{array} \xrightarrow{\text{Cut elimination for GS1p}} \begin{array}{c} \text{GS1p} \\ \hline \vdash \underline{S}_{\text{G}} \end{array} \xrightarrow{\text{Corollary 3.9}} \Big\|_{\text{KSg}}$$

□

Definition 3.17. Two systems \mathcal{S} and \mathcal{S}' are (*weakly*) *equivalent* if for every proof $\Big\|_{\mathcal{S}}$ there is a proof $\Big\|_{\mathcal{S}'}$, and vice versa.

Corollary 3.18. The systems SKSg and KSg are equivalent.

$$\boxed{
\begin{array}{cccc}
i\downarrow \frac{S\{t\}}{S[R, \bar{R}]} & s \frac{S([R, T], U)}{S[(R, U), T]} & w\downarrow \frac{S\{f\}}{S\{R\}} & c\downarrow \frac{S[R, R]}{S\{R\}}
\end{array}
}$$

Figure 4: System **KSg**

Definition 3.19. Two systems \mathcal{S} and \mathcal{S}' are *strongly equivalent* if for every derivation $\frac{T}{R} \Vdash_{\mathcal{S}}$ there is a derivation $\frac{T}{R} \Vdash_{\mathcal{S}'}$, and vice versa.

Remark 3.20. The systems **SKSg** and **KSg** are not strongly equivalent. The cut rule, for example, can not be derived in system **KSg**.

When a structure R implies a structure T then there is not necessarily a derivation from R to T in **KSg**, while there is one in **SKSg**. Therefore, I will in general use the asymmetric, cut-free system for deriving conclusions from the unit true, while I will use the symmetric system (i.e. the system with cut) for deriving conclusions from arbitrary premises.

As a result of cut elimination, sequent systems fulfill the subformula property. Our case is different, because our rules do not split the derivation according to the main connective of the active formula. However, seen bottom-up, in system **KSg** no rule introduces new atoms. It thus satisfies one main aspect of the subformula property: when given a conclusion of a rule there is only a finite number of premises to choose from. In proof search, for example, the branching of the search tree is finite.

There is also a semantic cut elimination proof for system **SKSg**, analogous to the one given in [15] for system **G3**. The given proof with cuts is thrown away, keeping only the information that its conclusion is valid, and a cut-free proof is constructed from scratch. This actually gives us more than just admissibility of the up-rules: it also yields a separation of proofs into distinct phases.

Theorem 3.21.

For every proof $\frac{}{S} \Vdash_{\mathbf{SKSg}}$ there is a proof $\frac{}{S} \Vdash_{\{i\downarrow, w\downarrow, s, c\downarrow\}}$.

Proof. Consider the rule *distribute*:

$$d\downarrow \frac{S([R, T], [R, U])}{S[R, (T, U)]} ,$$

which can be realized by a contraction and two switches:

$$\begin{array}{c} S([R, T], [R, U]) \\ \text{s} \frac{}{S[R, ([R, T], U)]} \\ \text{s} \frac{}{S[R, R, (T, U)]} \\ \text{c}\downarrow \frac{}{S[R, (T, U)]} . \end{array}$$

Build a derivation $\frac{S'}{\{d\downarrow\}}$, by going upwards from S applying $d\downarrow$ as many times as possible. Then S' will be in conjunctive normal form, i.e.

$$S' = ([a_{11}, a_{12}, \dots], [a_{21}, a_{22}, \dots], \dots, [a_{n1}, a_{n2}, \dots]) .$$

S is valid because there is a proof of it. The rule $d\downarrow$ is invertible, so S' is also valid. A conjunction is valid only if all its immediate substructures are valid. Those are disjunctions of atoms. A disjunction of atoms is valid only if it contains an atom a together with its negation \bar{a} . Thus, more specifically, S' is of the form

$$S' = ([b_1, \bar{b}_1, a_{11}, a_{12}, \dots], [b_2, \bar{b}_2, a_{21}, a_{22}, \dots], \dots, [b_n, \bar{b}_n, a_{n1}, a_{n2}, \dots]) .$$

Let $S'' = ([b_1, \bar{b}_1], [b_2, \bar{b}_2], \dots, [b_n, \bar{b}_n])$.

Obviously, there is a derivation $\frac{S''}{\{w\downarrow\}}$ and a proof $\frac{\{i\downarrow\}}{S''}$. □

3.4 Reducing Rules to Atomic Form

In the sequent calculus, the identity rule can be reduced to its atomic form. The same is true for our system, i.e.

$$i\downarrow \frac{S\{t\}}{S[R, \bar{R}]} \quad \text{is equivalently replaced by} \quad ai\downarrow \frac{S\{t\}}{S[a, \bar{a}]} ,$$

where $\mathbf{a}\downarrow$ is the *atomic identity* rule. Similarly to the sequent calculus, this is achieved by inductively replacing an instance of the general identity rule by instances on smaller structures:

$$\mathbf{i}\downarrow \frac{S\{t\}}{S[P, Q, (\bar{P}, \bar{Q})]} \quad \sim \quad \mathbf{i}\downarrow \frac{\frac{\mathbf{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]}{S([P, \bar{P}], [Q, \bar{Q}])}}{\mathbf{s} \frac{S[Q, ([P, \bar{P}], \bar{Q})]}{S[P, Q, (\bar{P}, \bar{Q})]}}{.}$$

What is new in the calculus of structures is that the cut can also be reduced to atomic form: just take the dual derivation of the one above:

$$\mathbf{i}\uparrow \frac{S(\bar{P}, \bar{Q}, [P, Q])}{S\{f\}} \quad \sim \quad \mathbf{i}\uparrow \frac{\frac{\mathbf{s} \frac{S(\bar{Q}, \bar{P}, [Q, P])}{S(\bar{Q}, [(\bar{P}, P), Q])}}{\mathbf{s} \frac{S[(\bar{P}, P), (\bar{Q}, Q)]}{S(\bar{Q}, Q)}}}{\mathbf{i}\uparrow \frac{S\{f\}}{S\{f\}}} .$$

It turns out that weakening can also be reduced to atomic form. When identity, cut and weakening are restricted to atomic form, there is only one non-local rule left in system \mathbf{KSg} : contraction. It can not be reduced to atomic form in system \mathbf{KSg} . Tiu solved this problem in when he discovered the *medial* rule [3]:

$$\mathbf{m} \frac{S[(R, U), (T, V)]}{S([R, T], [U, V])} .$$

This rule has no analogue in the sequent calculus. But it is clearly sound because we can derive it:

Proposition 3.22. The medial rule is derivable for $\{\mathbf{c}\downarrow, \mathbf{w}\downarrow\}$. Dually, the medial rule is derivable for $\{\mathbf{c}\uparrow, \mathbf{w}\uparrow\}$.

Proof. The medial rule is derivable as follows (or dually):

$$\begin{array}{c}
\frac{S[(R, U), (T, V)]}{\text{w}\downarrow \frac{S[(R, U), (T, [U, V])]}{\text{w}\downarrow \frac{S[(R, U), ([R, T], [U, V])]}{\text{w}\downarrow \frac{S[(R, [U, V]), ([R, T], [U, V])]}{\text{w}\downarrow \frac{S[(R, T], [U, V]), ([R, T], [U, V])]}{\text{c}\downarrow \frac{S[(R, T], [U, V])}}}}}}
\end{array} .$$

□

The medial rule has also been considered by Došen and Petrić as a composite arrow in the free bicartesian category, cf. the end of Section 4 in [5]. It is composed of four projections and a pairing of identities (or dually) in the same way as medial is derived using four weakenings and a contraction in the proof above.

Once we admit medial, then not only identity, cut and weakening, but also contraction is reducible to atomic form:

Theorem 3.23. The rules $i\downarrow$, $w\downarrow$ and $c\downarrow$ are derivable for $\{\mathbf{ai}\downarrow, \mathbf{s}\}$, $\{\mathbf{aw}\downarrow\}$ and $\{\mathbf{ac}\downarrow, \mathbf{m}\}$, respectively. Dually, the rules $i\uparrow$, $w\uparrow$ and $c\uparrow$ are derivable for $\{\mathbf{ai}\uparrow, \mathbf{s}\}$, $\{\mathbf{aw}\uparrow\}$ and $\{\mathbf{ac}\uparrow, \mathbf{m}\}$, respectively.

Proof. I will show derivability of the rules $\{i\downarrow, w\downarrow, c\downarrow\}$ for the respective systems. The proof of derivability of their co-rules is dual.

Given an instance of one of the following rules:

$$i\downarrow \frac{S\{\mathbf{t}\}}{S[R, \bar{R}]}, \quad w\downarrow \frac{S\{\mathbf{f}\}}{S\{R\}}, \quad c\downarrow \frac{S[R, R]}{S\{R\}},$$

construct a new derivation by structural induction on R :

1. R is an atom. Then the instance of the general rule is also an instance of its atomic form.
2. $R = [P, Q]$, where $P \neq \mathbf{f} \neq Q$. Apply the induction hypothesis respectively on

$$\begin{array}{c}
\text{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]} \\
\text{s} \frac{\text{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]}}{S([P, \bar{P}], [Q, \bar{Q}])} \\
\text{s} \frac{\text{s} \frac{\text{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]}}{S([P, \bar{P}], [Q, \bar{Q}])}}{S[P, Q, (\bar{P}, \bar{Q})]}
\end{array}
, \quad
\begin{array}{c}
= \frac{S\{f\}}{S[f, f]} \\
\text{w}\downarrow \frac{S\{f\}}{S[f, f]} \\
\text{w}\downarrow \frac{\text{w}\downarrow \frac{S\{f\}}{S[f, f]}}{S[f, Q]} \\
\text{w}\downarrow \frac{\text{w}\downarrow \frac{\text{w}\downarrow \frac{S\{f\}}{S[f, f]}}{S[f, Q]}}{S[P, Q]}
\end{array}
, \quad
\begin{array}{c}
\text{c}\downarrow \frac{S[P, P, Q, Q]}{S[P, P, Q]} \\
\text{c}\downarrow \frac{\text{c}\downarrow \frac{S[P, P, Q, Q]}{S[P, P, Q]}}{S[P, Q]}
\end{array}
.$$

3. $R = (P, Q)$, where $P \neq t \neq Q$. Apply the induction hypothesis respectively on

$$\begin{array}{c}
\text{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]} \\
\text{s} \frac{\text{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]}}{S([P, \bar{P}], [Q, \bar{Q}])} \\
\text{s} \frac{\text{s} \frac{\text{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]}}{S([P, \bar{P}], [Q, \bar{Q}])}}{S([P, \bar{P}], Q, \bar{Q})} \\
\text{s} \frac{\text{s} \frac{\text{s} \frac{\text{i}\downarrow \frac{S\{t\}}{S[Q, \bar{Q}]}}{S([P, \bar{P}], [Q, \bar{Q}])}}{S([P, \bar{P}], Q, \bar{Q})}}{S[(P, Q), \bar{P}, \bar{Q}]}
\end{array}
, \quad
\begin{array}{c}
= \frac{S\{f\}}{S(f, f)} \\
\text{w}\downarrow \frac{S\{f\}}{S(f, f)} \\
\text{w}\downarrow \frac{\text{w}\downarrow \frac{S\{f\}}{S(f, f)}}{S(f, Q)} \\
\text{w}\downarrow \frac{\text{w}\downarrow \frac{\text{w}\downarrow \frac{S\{f\}}{S(f, f)}}{S(f, Q)}}{S(P, Q)}
\end{array}
, \quad
\begin{array}{c}
\text{m} \frac{S[(P, Q), (P, Q)]}{S([P, P], [Q, Q])} \\
\text{c}\downarrow \frac{\text{m} \frac{S[(P, Q), (P, Q)]}{S([P, P], [Q, Q])}}{S([P, P], Q)} \\
\text{c}\downarrow \frac{\text{c}\downarrow \frac{\text{m} \frac{S[(P, Q), (P, Q)]}{S([P, P], [Q, Q])}}{S([P, P], Q)}}{S(P, Q)}
\end{array}
.$$

□

We now obtain the local system **SKS** from **SKSg** by restricting identity, cut, weakening and contraction to atomic form and adding medial. It is shown in Figure 5. The names of the rules are as in system **SKSg**, except that the atomic rules carry the attribute *atomic*, as for example in the name *atomic cut* for the rule $\text{ai}\uparrow$.

Theorem 3.24. System **SKS** and system **SKSg** are strongly equivalent.

Proof. Derivations in **SKSg** are translated to derivations in **SKS** by Theorem 3.23, and vice versa by Proposition 3.22. □

Thus, all results obtained for the non-local system, in particular the correspondence with the sequent calculus and admissibility of the up-rules, also hold for the local system. By removing the up-rules from system **SKS** we obtain system **KS**, shown in Figure 6.

Theorem 3.25. System **KS** and system **KSg** are strongly equivalent.

Proof. As the proof of Theorem 3.24. □

$$\begin{array}{ccc}
\text{ai}\downarrow \frac{S\{t\}}{S[a, \bar{a}]} & & \text{ai}\uparrow \frac{S(a, \bar{a})}{S\{f\}} \\
\\
\text{s} \frac{S([R, U], T)}{S[(R, T), U]} \\
\\
\text{m} \frac{S[(R, U), (T, V)]}{S([R, T], [U, V])} \\
\\
\text{aw}\downarrow \frac{S\{f\}}{S\{a\}} & & \text{aw}\uparrow \frac{S\{a\}}{S\{t\}} \\
\\
\text{ac}\downarrow \frac{S[a, a]}{S\{a\}} & & \text{ac}\uparrow \frac{S\{a\}}{S(a, a)}
\end{array}$$

Figure 5: System SKS

In the following, I will concentrate on the local system. The non-local rules, general identity, weakening, contraction and their duals $\{i\downarrow, i\uparrow, w\downarrow, w\uparrow, c\downarrow, c\uparrow\}$ do not belong to SKS. However, I will freely use them to denote a corresponding derivation in SKS according to Theorem 3.23. For example, I will use

$$\text{c}\downarrow \frac{[(a, b), (a, b)]}{(a, b)}$$

to denote either

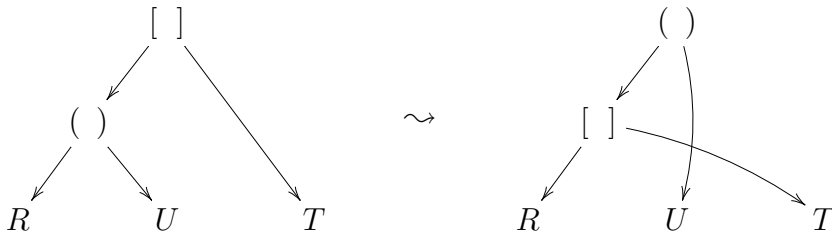
$$\begin{array}{ccc}
\text{m} \frac{[(a, b), (a, b)]}{([a, a], [b, b])} & & \text{m} \frac{[(a, b), (a, b)]}{([a, a], [b, b])} \\
\text{ac}\downarrow \frac{([a, a], [b, b])}{([a, a], b)} & \text{or} & \text{ac}\downarrow \frac{([a, a], [b, b])}{(a, [b, b])} \\
\text{ac}\downarrow \frac{([a, a], b)}{(a, b)} & & \text{ac}\downarrow \frac{(a, [b, b])}{(a, b)}
\end{array}$$

$$\begin{array}{ccc}
\text{ai} \downarrow \frac{S\{t\}}{S[a, \bar{a}]} & \text{aw} \downarrow \frac{S\{f\}}{S\{a\}} & \text{ac} \downarrow \frac{S[a, a]}{S\{a\}} \\
\text{s} \frac{S([R, T], U)}{S([R, U], T)} & \text{m} \frac{S([R, T], (U, V))}{S([R, U], [T, V])} &
\end{array}$$

Figure 6: System KS

3.5 Locality Through Atomicity

In system SKSg and also in sequent systems, there is no bound on the size of formulae that can appear as an active formula in an instance of the contraction rule. Implementing those systems for proof search thus requires either duplicating formulae of unbounded size or putting in place some complex mechanism, e.g. of sharing and copying on demand. In system SKS , no rule requires duplicating structures of unbounded size. In fact, because no rule needs to inspect structures of unbounded size, I call this system *local*. The atomic rules only need to duplicate, erase or compare atoms. The switch rule involves structures of unbounded size, namely R , T and U . But it does not require inspecting them. To see this, consider structures represented as binary trees in the obvious way. Then the switch rule can be implemented by changing the marking of two nodes and exchanging two pointers:



The same technique works for medial. The equations are local as well, including the De Morgan laws. However, since the rules in SKS introduce negation only on atoms, it is even possible to restrict negation to atoms from the beginning, as is customary in the one-sided sequent calculus, and drop the equations for negation entirely.

The concept of locality depends on the representation of structures. Rules that are local for one representation may not be local when another representation is used. For example, the switch rule is local when structures are

represented as trees, but it is not local when structures are represented as strings.

One motivation for locality is to simplify distributed implementation of an inference system. Of course, locality by itself still makes no distributed implementation. There are tasks to accomplish in an implementation of an inference system that in general require a global view on structures, for example matching a rule, i.e. finding a redex. There should also be some mechanism for backtracking. I do not see how these problems can be approached within a proof-theoretic system with properties like cut elimination. However, the application of a rule, i.e. producing the contractum from the redex, is achieved locally in system SKS. For that reason I believe that it lends itself more easily to distributed implementation than other systems.

3.6 Decomposition of Derivations

Derivations can be arranged into consecutive phases such that each phase uses only certain rules. We call this property *decomposition*. Decomposition theorems thus provide normal forms for derivations. A classic example of a decomposition theorem in the sequent calculus is proving Herbrand's Theorem by decomposing a proof tree into a bottom phase with contraction and quantifier rules and a top phase with propositional rules only. The three decomposition theorems presented here state the possibility of pushing all instances of a certain rule to the top and all instances of its dual rule to the bottom of a derivation. Except for the first, these decomposition theorems do not have analogues in the sequent calculus.

The proofs in this section use *permutation* of rules.

Definition 3.26. A rule ρ *permutes over* a rule π (or π *permutes under* ρ) if

for every derivation $\frac{T}{\pi \frac{\overline{U}}{\rho \overline{R}}}$ there is a derivation $\frac{T}{\rho \frac{\overline{V}}{\pi \overline{R}}}$ for some structure V .

Lemma 3.27. The rule $\mathbf{ac}\downarrow$ permutes under the rules $\mathbf{aw}\downarrow$, $\mathbf{ai}\downarrow$, \mathbf{s} and \mathbf{m} . Dually, the rule $\mathbf{ac}\uparrow$ permutes over the rules $\mathbf{aw}\uparrow$, $\mathbf{ai}\uparrow$, \mathbf{s} and \mathbf{m} .

Proof. Given an instance of $\mathbf{ac}\downarrow$ above an instance of a rule $\rho \in \{\mathbf{aw}\downarrow, \mathbf{ai}\downarrow, \mathbf{s}, \mathbf{m}\}$, the redex of $\mathbf{ac}\downarrow$ can be a substructure of the context of ρ . Then we permute

as follows:

$$\text{ac}\downarrow \frac{\frac{S'\{U\}}{S\{U\}}}{\rho \frac{S\{R\}}{S\{R\}}} \quad \rightsquigarrow \quad \frac{\rho \frac{S'\{U\}}{S'\{R\}}}{\text{ac}\downarrow \frac{S\{R\}}{S\{R\}}} .$$

The only other possibility occurs in case that ρ is **s** or **m**: the redex of $\text{ac}\downarrow$ can be a substructure of the contractum of ρ . Then we permute as in the following example of a switch rule, where $T\{ \}$ is a structure context:

$$\text{ac}\downarrow \frac{\frac{S([R, T[a, a]], U)}{S([R, T\{a\}], U)}}{\text{s} \frac{S[R, (T\{a\}), U]}{S[R, (T\{a\}), U]}} \quad \rightsquigarrow \quad \frac{\text{s} \frac{S([R, T[a, a]], U)}{S([R, T[a, a]], U)}}{\text{ac}\downarrow \frac{S[R, (T\{a\}), U]}{S[R, (T\{a\}), U]}} .$$

(And dually for $\text{ac}\uparrow$.) □

Lemma 3.28. The rule $\text{aw}\downarrow$ permutes under the rules $\text{ai}\downarrow$, **s** and **m**. Dually, the rule $\text{aw}\uparrow$ permutes over the rules $\text{ai}\uparrow$, **s** and **m**.

Proof. Similar to the proof of Lemma 3.27. □

We now turn to the decomposition results.

Separating Identity and Cut

Given that in system **SKS** identity is a rule, not an axiom as in the sequent calculus, a natural question to ask is whether the applications of the identity rule can be restricted to the top of a derivation. For proofs, this question is already answered positively by Theorem 3.21. It turns out that it is also true for derivations in general. Because of the duality between $\text{ai}\downarrow$ and $\text{ai}\uparrow$ we can also push the cuts to the bottom of a derivation. While this can be obtained in the sequent calculus (using cut elimination), it can not be done with a simple permutation argument as we do.

The following rules are called *super switch down* and *super switch up*:

$$\text{ss}\downarrow \frac{S\{T\{R\}\}}{S[R, T\{f\}]} \quad \text{and} \quad \text{ss}\uparrow \frac{S(R, T\{t\})}{S\{T\{R\}\}} .$$

Lemma 3.29. The rule $\text{ss}\downarrow$ is derivable for $\{\mathbf{s}\}$. Dually, the rule $\text{ss}\uparrow$ is derivable for $\{\mathbf{s}\}$.

Proof. We prove this for $\text{ss}\uparrow$ by structural induction on $T\{ \}$. The proof for $\text{ss}\downarrow$ is dual.

1. $T\{ \}$ is empty. Then premise and conclusion of the given instance of $\text{ss}\uparrow$ coincide, the rule instance can be removed.
2. $T\{ \} = [U, V\{ \}]$, where $U \neq \mathbf{f}$. Apply the induction hypothesis on

$$\text{ss}\uparrow \frac{\text{s} \frac{S(R, [U, V\{\mathbf{t}\}])}{S[U, (R, V\{\mathbf{t}\})]}}{S[U, V\{R\}]} .$$

3. $T\{ \} = (U, V\{ \})$, where $U \neq \mathbf{t}$. Apply the induction hypothesis on

$$\text{ss}\uparrow \frac{S(U, R, V\{\mathbf{t}\})}{S(U, V\{R\})} ,$$

the redex being $V\{R\}$.

□

The following rules are called *shallow* atomic identity and cut:

$$\text{ai}_s\downarrow \frac{S}{(S, [a, \bar{a}])} \quad \text{and} \quad \text{ai}_s\uparrow \frac{[S, (a, \bar{a})]}{S} .$$

Lemma 3.30. The rule $\text{ai}\downarrow$ is derivable for $\{\text{ai}_s\downarrow, \text{s}\}$. Dually, the rule $\text{ai}\uparrow$ is derivable for $\{\text{ai}_s\uparrow, \text{s}\}$.

Proof. An instance of $\text{ai}\downarrow$ can be replaced by an instance of $\text{ai}_s\downarrow$ followed by an instance of $\text{ss}\uparrow$, which is derivable for $\{\text{s}\}$. (And dually for $\text{ai}\uparrow$.) □

Theorem 3.31.

$$\text{For every derivation } \frac{T}{R} \parallel_{\text{SKS}} \text{ there is a derivation } \frac{\frac{T}{V} \parallel_{\text{SKS} \setminus \{\text{ai}\downarrow, \text{ai}\uparrow\}}}{R} \parallel_{\{\text{ai}\uparrow\}} .$$

Proof. By Lemma 3.30 we can reduce atomic identities to shallow atomic identities and the same for the cuts. It is easy to check that the rule $\text{ai}_s\downarrow$ permutes over every rule in **SKS** and the rule $\text{ai}_s\uparrow$ permutes under every rule in **SKS**. Instances of $\text{ai}_s\downarrow$ and $\text{ai}_s\uparrow$ are instances of $\text{ai}\downarrow$ and $\text{ai}\uparrow$, respectively. □

Separating Contraction

Contraction allows the repeated use of a structure in a proof by allowing us to copy it at will. It should be possible to copy everything needed in the beginning, and then go on with the proof without ever having to copy again. This intuition is made precise by the following theorem and holds for system **SKS**. There is no such result for the sequent calculus [2]. There are sequent systems for classical propositional logic that do not have an explicit contraction rule, however, since they involve context sharing, contraction is built into the logical rules and is used throughout the proof.

Theorem 3.32.

For every proof $\frac{T}{S}^{\text{KS}}$ there is a proof $\frac{S'}{S}^{\text{KS} \setminus \{\text{ac}\downarrow}}$.

Proof. Using Lemma 3.27, permute down all instances of $\text{ac}\downarrow$, starting with the bottommost. \square

This result is extended to the symmetric system as follows:

Theorem 3.33.

For every derivation $\frac{T}{R}^{\text{SKS}}$ there is a derivation $\frac{T}{U}^{\text{SKS} \setminus \{\text{ac}\downarrow, \text{ac}\uparrow}}$.

Proof. Consider the following derivations that can be obtained:

$$\begin{array}{c}
 \frac{T}{R}^{\text{SKS}} \xrightarrow{1} \frac{\text{i}\downarrow \frac{t}{[\bar{T}, T]}}{\frac{\text{SKS}}{[\bar{T}, R]}} \xrightarrow{2} \frac{\text{t}}{[\bar{T}, R]}^{\text{KS}} \xrightarrow{3} \frac{\text{S}}{\text{i}\uparrow \frac{\frac{(T, t)}{[\bar{T}, R]}^{\text{KS}}}{R}} \xrightarrow{4} \frac{\text{i}\uparrow \frac{\frac{(T, t)}{[R', (T, \bar{T}')]}}{R'}}{R}^{\text{KS} \setminus \{\text{ac}\downarrow}}
 \end{array}$$

1. Put the derivation into the context $[\bar{T}, \{ \}]$. On top of the resulting derivation, apply an $i\downarrow$ to obtain a proof.
2. Apply cut elimination.
3. Put the proof into the context $(T, \{ \})$. At the bottom of the resulting derivation, apply a switch and a cut to obtain a derivation from T to R .
4. All instances of $ac\downarrow$ are permuted down as far as possible. There are two kinds of instances: those that duplicate atoms coming from R and those that duplicate atoms coming from \bar{T} . The first kind, starting with the bottom-most instance, can be permuted down all the way to the bottom of the derivation. The second kind, also starting with the bottom-most instance, can be permuted down until they meet the cut.

Now, starting with the bottom-most $ac\downarrow$ that is above the cut, we apply the transformation

$$ac\downarrow \frac{i\uparrow \frac{S(U\{a\}, \bar{U}[\bar{a}, \bar{a}])}{S(U\{a\}, \bar{U}\{\bar{a}\})}}{S\{f\}} \quad \sim \quad ac\uparrow \frac{i\uparrow \frac{S(U\{a\}, \bar{U}[\bar{a}, \bar{a}])}{S(U(a, a), \bar{U}[\bar{a}, \bar{a}])}}{S\{f\}}$$

and permute up the resulting instance of $ac\uparrow$ all the way to the top of the derivation. This is possible because no rule in the derivation above changes T . Proceed inductively with the remaining instances of $ac\downarrow$ above the cut. The resulting derivation has the desired shape. \square

Separating Weakening

In the sequent calculus, one usually can push up to the top of the proof all the instances of weakening, or to the same effect, build weakening into the identity axiom:

$$\overline{A, \Phi \vdash A, \Psi} \quad .$$

The same *lazy* way of applying weakening can be done in system **SKS**, cf. Theorem 3.21. However, while a proof in which all weakenings occur at the top is certainly more “normal” than a proof in which weakenings are scattered all over, this is hardly an interesting normal form. In system **SKS** something more interesting can be done: applying weakening in an *eager* way.

Theorem 3.34.

For every proof $\frac{\quad}{S} \Vdash^{\text{KS}}$ there is a proof $\frac{\frac{\quad}{S'} \Vdash^{\text{KS} \setminus \{\text{aw}\downarrow}}}{\frac{\quad}{S} \Vdash^{\{\text{aw}\downarrow}}}$.

Proof. Permute down all instances of $\text{aw}\downarrow$, starting with the bottommost. This is done by using Lemma 3.28 and the following transformation:

$$\frac{\text{aw}\downarrow \frac{S[a, f]}{S[a, a]}}{\text{ac}\downarrow \frac{S[a, f]}{S\{a\}}} \rightsquigarrow = \frac{S[a, f]}{S\{a\}} .$$

□

Weakening loses information: when deducing $a \vee b$ from a , the information that a holds is lost. Given a proof with weakenings, do they lose information that we would like to keep? Can we obtain a proof of a stronger statement by removing them? The theorem above gives an affirmative answer to that question: given a proof of S , it exhibits a weakening-free proof of a structure S' , from which S trivially follows by weakenings.

Notation 3.35.

A derivation $\frac{T}{R} \Vdash^{\{\rho\}}$ of n instances of the rule ρ is denoted by $\rho^n \frac{T}{R}$.

This result extends to the symmetric case, i.e. to derivations in SKS:

Theorem 3.36.

For every derivation $\frac{T}{R} \Vdash^{\text{SKS}}$ there is a derivation $\frac{\frac{\frac{T}{V} \Vdash^{\{\text{aw}\uparrow}}}{U} \Vdash^{\text{SKS} \setminus \{\text{aw}\downarrow, \text{aw}\uparrow}}}{\frac{\quad}{R} \Vdash^{\{\text{aw}\downarrow}}}$.

Proof. There is an algorithm that produces a derivation of the desired shape. It consists of two steps: 1) pushing up all instances of $\text{aw}\uparrow$ and 2) pushing down all instances of $\text{aw}\downarrow$. Those two steps are repeated until the derivation has the desired shape. An instance of $\text{aw}\uparrow$ that is pushed up can turn into an instance of $\text{aw}\downarrow$ when meeting an instance of $\text{ai}\downarrow$, and the dual case can

also happen. However, the process terminates, since each step that does not produce the desired shape strictly decreases the combined number of instances of $\text{ai}\downarrow$ and $\text{ai}\uparrow$.

In the following, the process of pushing up instances of $\text{aw}\uparrow$ is shown, the process of pushing down instances of $\text{aw}\downarrow$ is dual.

An instance of $\text{aw}\uparrow$ is a special case of a derivation consisting of n instances of $\text{aw}\uparrow$ and is moved up as such, starting with the topmost. In addition to the cases treated in Lemma 3.28 there are the following cases:

$$\text{ac}\uparrow \frac{S'\{a\}}{S'(a, a)} \quad \text{aw}\uparrow^n \frac{S'(a, a)}{S\{t, a\}} \quad \rightsquigarrow \quad \text{aw}\uparrow^{n-1} \frac{S'\{a\}}{S\{a\}} = \frac{S'\{a\}}{S\{t, a\}}$$

$$\text{ac}\downarrow \frac{S'[a, a]}{S'\{a\}} \quad \text{aw}\uparrow^n \frac{S'\{a\}}{S\{t\}} \quad \rightsquigarrow \quad \text{aw}\uparrow^{n+1} \frac{S'[a, a]}{S\{t, t\}} = \frac{S'[a, a]}{S\{t\}}$$

$$\text{aw}\downarrow \frac{S'\{f\}}{S'\{a\}} \quad \text{aw}\uparrow^n \frac{S'\{a\}}{S\{t\}} \quad \rightsquigarrow \quad \text{aw}\uparrow^{n-1} \frac{S'\{f\}}{S\{f\}} = \frac{S'\{f\}}{S\{f\}} = \frac{S'([t, t], f)}{S([t, t], f)} \stackrel{s}{=} \frac{S'\{f\}}{S\{t, (t, f)\}} = \frac{S'\{f\}}{S\{t\}}$$

$$\text{ai}\downarrow \frac{S'\{t\}}{S'[a, \bar{a}]} \quad \text{aw}\uparrow^n \frac{S'\{t\}}{S\{t, \bar{a}\}} \quad \rightsquigarrow \quad \text{aw}\uparrow^{n-1} \frac{S'\{t\}}{S\{t\}} = \frac{S'\{t\}}{S\{t\}} = \frac{S'\{t\}}{S\{t, f\}} \stackrel{aw\downarrow}{=} \frac{S'\{t\}}{S\{t, \bar{a}\}}$$

□

Separating all Atomic Rules

Decomposition results can be applied consecutively. Here, all rules that deal with atoms, namely $\text{ai}\downarrow$, $\text{ac}\downarrow$, $\text{aw}\downarrow$ and their duals, are separated from the rules that deal with the connectives, namely s and m :

Theorem 3.37.

For every derivation $\frac{T}{R} \Vdash_{\text{KSq}}$ there is a derivation $\frac{\frac{\frac{\frac{\frac{\frac{\frac{T}{\Vdash_{\{\text{ac}\uparrow}\}}{T_1}}{\Vdash_{\{\text{aw}\uparrow}\}}{T_2}}{\Vdash_{\{\text{ai}\downarrow}\}}{T_3}}{\Vdash_{\{\text{s,m}\}}}{R_3}}{\Vdash_{\{\text{ai}\uparrow}\}}{R_2}}{\Vdash_{\{\text{aw}\downarrow}\}}{R_1}}{\Vdash_{\{\text{ac}\downarrow}\}}{R}}$.

Proof. We first push contractions to the outside, using Theorem 3.33. In the contraction-free part of the obtained derivation, we push weakening to the outside, using the procedure from the proof of Theorem 3.36, which does not introduce new instances of contraction. In the contraction- and weakening-free part of the resulting derivation we then separate out identity and cut by applying the procedure from the proof of Theorem 3.31, which neither introduces new contractions nor weakenings. \square

4 Predicate Logic

Definition 4.1. *Variables* are denoted by x and y . *Terms* are defined as usual in first-order predicate logic. *Atoms*, denoted by a, b , etc., are expressions of the form $p(t_1, \dots, t_n)$, where p is a *predicate symbol* of *arity* n and t_1, \dots, t_n are terms. The negation of an atom is again an atom. The *structures* of the language KSq are generated by the following grammar, which is the one for the propositional case extended by existential and universal quantifier:

$$S ::= f \mid \mathbf{t} \mid a \mid \underbrace{[S, \dots, S]}_{>0} \mid \underbrace{(S, \dots, S)}_{>0} \mid \bar{S} \mid \exists xS \mid \forall xS \quad .$$

Definition 4.2. Structures are *equivalent* modulo the smallest equivalence re-

lation induced by the equations in Fig. 1 extended by the following equations:

$$\begin{array}{ll}
\mathbf{Variable\ Renaming} & \forall xR = \forall yR[x/y] \quad \text{if } y \text{ is not free in } R \\
& \exists xR = \exists yR[x/y] \\
\mathbf{Vacuous\ Quantifier} & \forall yR = \exists yR = R \quad \text{if } y \text{ is not free in } R \\
\mathbf{Negation} & \overline{\exists x\bar{R}} = \forall x\bar{R} \\
& \overline{\forall x\bar{R}} = \exists x\bar{R}
\end{array}$$

Definition 4.3. The notions of *structure context* and *substructure* are defined in the same way as in the propositional case. A structure of language \mathbf{KSq} is in *normal form* if negation occurs only on atoms, and extra units, connectives and quantifiers are removed using the laws for units, associativity and vacuous quantifier.

As in the propositional case, we in general consider structures to be in normal form and do not distinguish between equivalent structures.

4.1 A Symmetric System

The rules of system \mathbf{SKSgq} , a symmetric system for predicate logic, are shown in Figure 7. The first and last column show the rules that deal with quantifiers, in the middle there are the rules for the propositional fragment. The rules $\mathbf{u}\downarrow$ and $\mathbf{u}\uparrow$ were obtained by Guglielmi. They follow a scheme or recipe [8], which also yields the switch rule and ensures atomicity of cut and identity not only for classical logic but also for several other logics. The $\mathbf{u}\downarrow$ rule corresponds to the $\mathbf{R}\forall$ rule in $\mathbf{GS1}$, shown in Figure 8. Because of the equational theory, we can equivalently replace it by

$$\mathbf{u}\downarrow \frac{S\{\forall x[R, T]\}}{S[\forall xR, T]} \quad \text{if } x \text{ is not free in } T.$$

In the sequent calculus, going up, the $\mathbf{R}\forall$ rule removes a universal quantifier from a formula to allow other rules to access this formula. In system \mathbf{SKSgq} , inference rules apply deep inside formulae, so there is no need to remove the quantifier: it can be moved out of the way using the rule $\mathbf{u}\downarrow$ and it vanishes once the proof is complete because of the equation $\forall xt = t$. As a result, the premise of the $\mathbf{u}\downarrow$ rule implies its conclusion, which is not true for the $\mathbf{R}\forall$ rule

$$\begin{array}{ccc}
i\downarrow \frac{S\{t\}}{S[R, \bar{R}]} & & i\uparrow \frac{S(R, \bar{R})}{S\{f\}} \\
\\
u\downarrow \frac{S\{\forall x[R, T]\}}{S[\forall xR, \exists xT]} & s \frac{S([R, U], T)}{S[(R, T), U]} & u\uparrow \frac{S(\exists xR, \forall xT)}{S\{\exists x(R, T)\}} \\
\\
w\downarrow \frac{S\{f\}}{S\{R\}} & & w\uparrow \frac{S\{R\}}{S\{t\}} \\
\\
c\downarrow \frac{S[R, R]}{S\{R\}} & & c\uparrow \frac{S\{R\}}{S(R, R)} \\
\\
n\downarrow \frac{S\{R[x/t]\}}{S\{\exists xR\}} & & n\uparrow \frac{S\{\forall xR\}}{S\{R[x/t]\}}
\end{array}$$

Figure 7: System SKSgq

of the sequent calculus. The $R\forall$ rule is the only rule in **GS1** with such bad behaviour. In all the rules that I presented in the calculus of structures the premise implies the conclusion.

The rule $n\downarrow$ corresponds to $R\exists$. As usual, the substitution operation requires t to be free for x in R : quantifiers in R do not capture variables in t . The term t is not required to be free for x in $S\{R\}$: quantifiers in S may capture variables in t .

4.2 Correspondence to the Sequent Calculus

We extend the translations between **SKSg** and **GS1p** to translations between **SKSgq** and **GS1**. System **GS1** is system **GS1p** extended by the rules shown in Figure 8.

$$\begin{array}{c}
\text{R}\exists \frac{\vdash \Phi, A[x/t]}{\vdash \Phi, \exists x A} \qquad \text{R}\forall \frac{\vdash \Phi, A[x/y]}{\vdash \Phi, \forall x A} \\
\text{Proviso: } y \text{ is not free in the conclusion of R}\forall.
\end{array}$$

Figure 8: Quantifier rules of GS1

The functions $\underline{\quad}_s$ and $\underline{\quad}_g$ are extended in the obvious way:

$$\begin{array}{ccc}
\underline{\exists x A}_s = \exists x \underline{A}_s & \text{and} & \underline{\exists x S}_g = \exists x \underline{S}_g \\
\underline{\forall x A}_s = \forall x \underline{A}_s & & \underline{\forall x S}_g = \forall x \underline{S}_g
\end{array}$$

From the Sequent Calculus to the Calculus of Structures

Theorem 4.4.

For every derivation $\frac{\Sigma_1 \cdots \Sigma_h}{\Sigma}$ in GS1 + Cut, in which the free variables in

the premises that are introduced by R \forall instances are x_1, \dots, x_n , there exists

a derivation $\frac{\forall x_1 \dots \forall x_n (\underline{\Sigma}_{1_s}, \dots, \underline{\Sigma}_{h_s})}{\underline{\Sigma}_s} \parallel \text{SKSgq} \setminus \{w\uparrow, c\uparrow, u\uparrow, n\uparrow\}$ with the same number of cuts.

Proof. The proof is an extension of the proof of Theorem 3.7. There are two more inductive cases, one for R \exists , which is easily translated into an n \downarrow , and one for R \forall , which is shown here:

$$\text{R}\forall \frac{\frac{\Sigma_1 \cdots \Sigma_{h'}}{\vdash \Phi, A[x/y]}}{\vdash \Phi, \forall x A}$$

By induction hypothesis we have the derivation

$$\frac{\forall x_1 \dots \forall x_{n'} (\underline{\Sigma}_{1_s}, \dots, \underline{\Sigma}_{h'_s}) \quad \Delta \left\| \text{SKSgq} \setminus \{w\uparrow, c\uparrow, u\uparrow, n\uparrow\} \right.}{[\underline{\Phi}_s, \underline{A}[x/y]_s]} \quad ,$$

from which we build

$$\frac{\forall y \forall x_1 \dots \forall x_{n'} (\underline{\Sigma}_{1_s}, \dots, \underline{\Sigma}_{h'_s}) \quad \forall y \{ \Delta \} \left\| \text{SKSgq} \setminus \{w\uparrow, c\uparrow, u\uparrow, n\uparrow\} \right.}{\text{u}\downarrow \frac{\forall y [\underline{\Phi}_s, \underline{A}[x/y]_s]}{[\exists y \underline{\Phi}_s, \forall y \underline{A}[x/y]_s]} \quad ,}{=} \frac{[\underline{\Phi}_s, \forall y \underline{A}[x/y]_s]}{[\underline{\Phi}_s, \forall x \underline{A}_s]}$$

where in the lower instance of the equivalence rule y is not free in $\forall x \underline{A}_s$ and in the upper instance of the equivalence rule y is not free in $\underline{\Phi}_s$: both due to the proviso of the $R\forall$ rule. \square

Corollary 4.5. If a sequent Σ has a proof in $\text{GS1} + \text{Cut}$ then $\underline{\Sigma}_s$ has a proof in $\text{SKSgq} \setminus \{w\uparrow, c\uparrow, u\uparrow, n\uparrow\}$.

Corollary 4.6. If a sequent Σ has a proof in GS1 then $\underline{\Sigma}_s$ has a proof in $\text{SKSgq} \setminus \{i\uparrow, w\uparrow, c\uparrow, u\uparrow, n\uparrow\}$.

From the Calculus of Structures to the Sequent Calculus

Lemma 4.7. For every two formulae A, B and every formula context $C\{ \}$

there exists a derivation $\frac{\vdash A, \bar{B}}{\vdash C\{A\}, \overline{C\{B\}}}$ in GS1 .

Proof. There are two cases needed in addition to the proof of Lemma 3.11: $C\{ \} = \exists x C'\{ \}$ and $C\{ \} = \forall x C'\{ \}$. The first case is shown here, the

second is similar:

$$\frac{\frac{\frac{\vdash A, \bar{B}}{\Delta}}{\text{R}\exists} \frac{\vdash C'\{A\}, \overline{C'\{B\}}}{\vdash \exists x C'\{A\}, \overline{C'\{B\}}} }{\text{R}\forall} \frac{\vdash \exists x C'\{A\}, \forall x \overline{C'\{B\}}}{\vdash \exists x C'\{A\}, \forall x \overline{C'\{B\}}},$$

where Δ exists by induction hypothesis. □

Theorem 4.8. For every derivation $\begin{array}{c} Q \\ \parallel_{\text{SKS}_{\text{gq}}} \\ P \end{array}$ there exists a derivation

$$\frac{\vdash \underline{Q}_{\text{G}}}{\vdash \underline{P}_{\text{G}}}$$

in $\text{GS1} + \text{Cut}$.

Proof. The proof is an extension of the proof of Theorem 3.12. The base cases are the same, in the inductive cases the existence of Δ_1 follows from Lemma 4.7. Corresponding to the rules for quantifiers, there are four additional inductive cases. For the rules

$$\text{u}\downarrow \frac{S\{\forall x[R, T]\}}{S[\forall x R, \exists x T]} \quad \text{and} \quad \text{u}\uparrow \frac{S\{\exists x R, \forall x T\}}{S\{\exists x(R, T)\}}$$

we have the proofs

$$\frac{\frac{\text{Ax} \frac{\vdash R, \bar{R}}{\vdash R, \bar{R}} \quad \text{Ax} \frac{\vdash T, \bar{T}}{\vdash T, \bar{T}}}{\text{R}\wedge} \frac{\vdash R, T, \bar{R} \wedge \bar{T}}{\vdash R, \exists x T, \bar{R} \wedge \bar{T}}}{\text{R}\exists} \frac{\vdash R, \exists x T, \exists x(\bar{R} \wedge \bar{T})}{\vdash R, \exists x T, \exists x(\bar{R} \wedge \bar{T})}}{\text{R}\forall} \frac{\vdash \forall x R, \exists x T, \exists x(\bar{R} \wedge \bar{T})}{\vdash \forall x R \vee \exists x T, \exists x(\bar{R} \wedge \bar{T})} \quad \text{and} \quad \frac{\frac{\text{Ax} \frac{\vdash R, \bar{R}}{\vdash R, \bar{R}} \quad \text{Ax} \frac{\vdash T, \bar{T}}{\vdash T, \bar{T}}}{\text{R}\wedge} \frac{\vdash R \wedge T, \bar{R}, \bar{T}}{\vdash R \wedge T, \bar{R}, \exists x \bar{T}}}{\text{R}\exists} \frac{\vdash R \wedge T, \bar{R}, \exists x \bar{T}}{\vdash \exists x(R \wedge T), \bar{R}, \exists x \bar{T}}}{\text{R}\forall} \frac{\vdash \exists x(R \wedge T), \forall x \bar{R}, \exists x \bar{T}}{\vdash \exists x(R \wedge T), \forall x \bar{R} \vee \exists x \bar{T}},$$

and for the rules

$$\text{n}\downarrow \frac{S\{R[x/t]\}}{S\{\exists x R\}} \quad \text{and} \quad \text{n}\uparrow \frac{S\{\forall x R\}}{S\{R[x/t]\}}$$

we have the proofs

$$\frac{\text{Ax} \frac{\vdash R[x/t], \overline{R[x/t]}}{\vdash \exists x R, \overline{R[x/t]}}}{\text{R}\exists} \quad \text{and} \quad \frac{\text{Ax} \frac{\vdash R[x/t], \bar{R}[x/t]}{\vdash R[x/t], \exists x \bar{R}}}{\text{R}\exists}.$$

□

Corollary 4.9. If a structure S has a proof in SKSgq then $\vdash \underline{S}_c$ has a proof in GS1 .

Soundness and completeness of SKSgq , i.e. the fact that a structure has a proof in SKSgq if and only if it is valid, follows from soundness and completeness of GS1 by Corollaries 4.5 and 4.9. Moreover, a structure T implies a structure R if and only if there is a derivation from T to R , which follows from soundness and completeness and the following theorem.

Theorem 4.10.

There is a derivation $\frac{T}{R} \Big\|_{\text{SKSgq}}$ if and only if there is a proof $\frac{}{[\bar{T}, R]} \Big\|_{\text{SKSgq}}$.

Proof. (same as the proof of Theorem 3.14.) A proof Π can be obtained from a given derivation Δ and a derivation Δ from a given proof Π , respectively, as follows:

$$\begin{array}{c} \text{i}\downarrow \frac{\text{t}}{[\bar{T}, T]} \\ [\bar{T}, \Delta] \Big\|_{\text{SKSgq}} \\ [\bar{T}, R] \end{array} \quad \text{and} \quad \begin{array}{c} \frac{T}{(T, \Pi) \Big\|_{\text{SKSgq}}} \\ \text{s} \frac{(T, [\bar{T}, R])}{[R, (T, \bar{T})]} \\ \text{i}\uparrow \frac{}{R} \end{array} .$$

□

Note that the above is not true for system GS1 , because the premise of the $R\forall$ rule does not imply its conclusion. The proof above does not work for the sequent calculus because adding to the context of a derivation can violate the proviso of the $R\forall$ rule.

4.3 Admissibility of the Cut and the Other Up-Rules

Just like in the propositional case, the up-rules of the symmetric system are admissible. By removing them from SKSgq we obtain the asymmetric, cut-free system shown in Figure 9, which is called system KSgq .

Theorem 4.11. The rules $\text{i}\uparrow$, $\text{w}\uparrow$, $\text{c}\uparrow$, $\text{u}\uparrow$ and $\text{n}\uparrow$ are admissible for system KSgq .

$$\begin{array}{ccc}
i\downarrow \frac{S\{t\}}{S[R, \bar{R}]} & w\downarrow \frac{S\{f\}}{S\{R\}} & c\downarrow \frac{S[R, R]}{S\{R\}} \\
s \frac{S([R, T], U)}{S[(R, U), T]} & u\downarrow \frac{S\{\forall x[R, T]\}}{S[\forall xR, \exists xT]} & n\downarrow \frac{S\{R[x/t]\}}{S\{\exists xR\}}
\end{array}$$

Figure 9: System KSgq

Proof.

$$\begin{array}{ccccc}
\mathbb{I}_{S}^{\text{SKSgq}} & \xrightarrow{\text{Corollary 4.8}} & \begin{array}{c} \text{GS1} \\ +\text{Cut} \\ \hline \vdash \underline{S}_G \end{array} & \xrightarrow{\text{Cut elimination for GS1}} & \begin{array}{c} \text{GS1} \\ \hline \vdash \underline{S}_G \end{array} & \xrightarrow{\text{Corollary 4.6}} & \mathbb{I}_{S}^{\text{KSgq}} \\
& & & & & & \square
\end{array}$$

Corollary 4.12. The systems SKSgq and KSgq are equivalent.

4.4 Reducing Rules to Atomic Form

Consider the following local rules:

$$\begin{array}{cc}
l_1\downarrow \frac{S[\exists xR, \exists xT]}{S\{\exists x[R, T]\}} & l_1\uparrow \frac{S\{\forall x(R, T)\}}{S(\forall xR, \forall xT)} \\
l_2\downarrow \frac{S[\forall xR, \forall xT]}{S\{\forall x[R, T]\}} & l_2\uparrow \frac{S\{\exists x(R, T)\}}{S(\exists xR, \exists xT)} \quad .
\end{array}$$

Like medial, they have no analogues in the sequent calculus. In system SKSgq, and similarly in the sequent calculus, the corresponding inferences are made using contraction and weakening:

Proposition 4.13. The rules $\{l_1\downarrow, l_2\downarrow\}$ are derivable for $\{c\downarrow, w\downarrow\}$. Dually, the rules $\{l_1\uparrow, l_2\uparrow\}$ are derivable for $\{c\uparrow, w\uparrow\}$.

Proof. We show the case for $l_1\downarrow$, the other cases are similar or dual:

$$\begin{array}{c} \text{w}\downarrow \frac{S[\exists xR, \exists xT]}{S[\exists xR, \exists x[R, T]]} \\ \text{w}\downarrow \frac{S[\exists x[R, T], \exists x[R, T]]}{S\{\exists x[R, T]\}} \\ \text{c}\downarrow \frac{S\{\exists x[R, T]\}}{S\{\exists x[R, T]\}} \end{array} .$$

□

Using medial and the rules $\{l_1\downarrow, l_2\downarrow, l_1\uparrow, l_2\uparrow\}$ we can reduce identity, cut and weakening to atomic form, similarly to the propositional case.

Theorem 4.14. The rules $i\downarrow$, $w\downarrow$ and $c\downarrow$ are derivable for $\{ai\downarrow, s, u\downarrow\}$, $\{aw\downarrow\}$ and $\{ac\downarrow, m, l_1\downarrow, l_2\downarrow\}$, respectively. Dually, the rules $i\uparrow$, $w\uparrow$ and $c\uparrow$ are derivable for $\{ai\uparrow, s, u\uparrow\}$, $\{aw\uparrow\}$ and $\{ac\uparrow, m, l_1\uparrow, l_2\uparrow\}$, respectively.

Proof. The proof is an extension of the proof of Theorem 3.23 by the inductive cases for the quantifiers. Given an instance of one of the following rules:

$$i\downarrow \frac{S\{t\}}{S[R, \bar{R}]} \quad , \quad w\downarrow \frac{S\{f\}}{S\{R\}} \quad , \quad c\downarrow \frac{S[R, R]}{S\{R\}} \quad ,$$

construct a new derivation by structural induction on R :

1. $R = \exists xT$, where x occurs free in T . Apply the induction hypothesis respectively on

$$\begin{array}{c} \frac{S\{t\}}{S\{\forall xt\}} \\ i\downarrow \frac{S\{\forall x[T, \bar{T}]\}}{S[\exists xT, \forall x\bar{T}]} \\ u\downarrow \end{array} \quad , \quad \frac{S\{f\}}{S\{\exists xf\}} \\ w\downarrow \frac{S\{\exists xT\}}{S\{\exists xT\}} \quad , \quad \frac{S[\exists xT, \exists xT]}{S\{\exists x[T, T]\}} \\ l_1\downarrow \frac{S\{\exists xT\}}{S\{\exists xT\}} \\ c\downarrow .$$

2. $R = \forall xT$, where x occurs free in T . Apply the induction hypothesis respectively on

$$\begin{array}{c} \frac{S\{t\}}{S\{\forall xt\}} \\ i\downarrow \frac{S\{\forall x[T, \bar{T}]\}}{S[\forall xT, \exists x\bar{T}]} \\ u\downarrow \end{array} \quad , \quad \frac{S\{f\}}{S\{\forall xf\}} \\ w\downarrow \frac{S\{\forall xT\}}{S\{\forall xT\}} \quad , \quad \frac{S[\forall xT, \forall xT]}{S\{\forall x[T, T]\}} \\ l_2\downarrow \frac{S\{\forall xT\}}{S\{\forall xT\}} \\ c\downarrow .$$

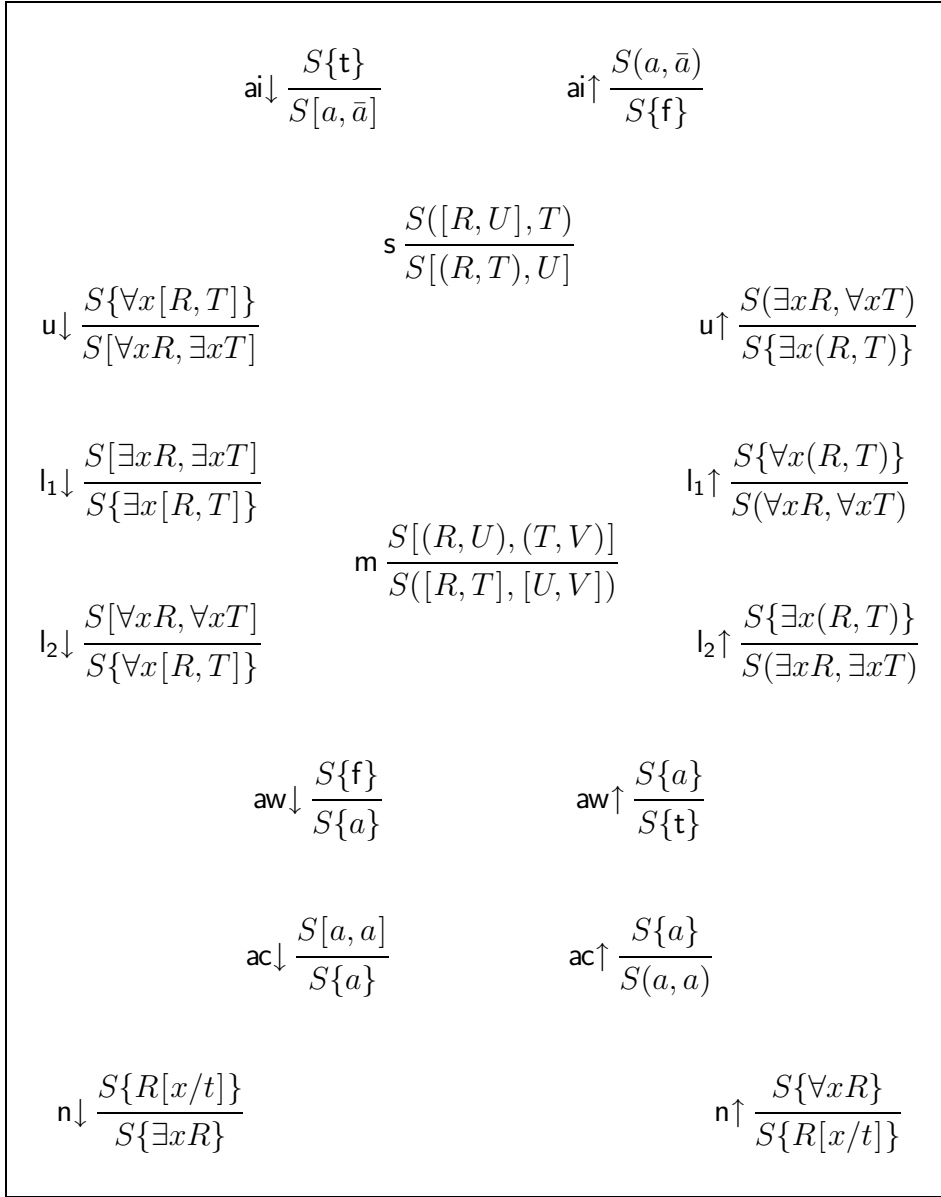


Figure 10: System SKSq

□

We now obtain system **SKSq** from **SKSgq** by restricting identity, cut, weakening and contraction to atomic form and adding the rules $\{\text{m}, \text{l}_1\downarrow, \text{l}_2\downarrow, \text{l}_1\uparrow, \text{l}_2\uparrow\}$. It is shown in Figure 10.

$\text{ai}\downarrow \frac{S\{t\}}{S[a, \bar{a}]}$		$\text{aw}\downarrow \frac{S\{f\}}{S\{a\}}$		$\text{ac}\downarrow \frac{S[a, a]}{S\{a\}}$	
$\text{s} \frac{S([R, T], U)}{S[(R, U), T]}$			$\text{m} \frac{S[(R, T), (U, V)]}{S[(R, U), [T, V]]}$		
$\text{u}\downarrow \frac{S\{\forall x[R, T]\}}{S[\forall xR, \exists xT]}$	$\text{n}\downarrow \frac{S\{R[x/t]\}}{S\{\exists xR\}}$	$\text{l}_1\downarrow \frac{S[\exists xR, \exists xT]}{S\{\exists x[R, T]\}}$	$\text{l}_2\downarrow \frac{S[\forall xR, \forall xT]}{S\{\forall x[R, T]\}}$		

Figure 11: System **KSq**

As in all the systems considered, the up-rules, i.e. $\{\text{n}\uparrow, \text{u}\uparrow, \text{l}_1\uparrow, \text{l}_2\uparrow\}$ are admissible. Hence, system **KSq**, shown in Figure 11, is complete.

Theorem 4.15. System **SKSq** and system **SKSgq** are strongly equivalent. Also, system **KSq** and system **KSgq** are strongly equivalent.

Proof. Derivations in **SKSgq** are translated to derivations in **SKSq** by Theorem 4.14, and vice versa by Proposition 4.13. The same holds for **KSgq** and **KSq**. \square

Thus, all results obtained for system **SKSgq** also hold for system **SKSq**. As in the propositional case, I will freely use general identity, cut, weakening and contraction to denote a corresponding derivation in **SKSq** according to Theorem 4.14.

4.5 Locality Through Atomicity

As we have seen in the previous section, the technique of reducing contraction to atomic form to obtain locality also works in the case of predicate logic: the non-local rule $\text{c}\downarrow$ is equivalently replaced by local ones, namely $\{\text{ac}\downarrow, \text{m}, \text{l}_1\downarrow, \text{l}_2\downarrow\}$.

However, there are other sources of non-locality in system **SKSq**. One is the condition on the quantifier equations:

$$\forall yR = \exists yR = R \quad \text{where } y \text{ is not free in } R.$$

To add or remove a quantifier, a structure of unbounded size has to be checked for occurrences of the variable y .

Another is the $n\downarrow$ rule, in which a term t of unbounded size has to be copied into an unbounded number of occurrences of x in R . It is non-local for two distinct reasons: 1) the unbounded size of t and 2) the unbounded number of occurrences of x in R . The unboundedness of term t can be dealt with, since $n\downarrow$ can be derived and thus replaced by the following two rules:

$$n_1\downarrow \frac{S\{\exists y_1 \dots \exists y_n R[x/f(y_1, \dots, y_n)]\}}{S\{\exists x R\}} \quad \text{and} \quad n_2\downarrow \frac{S\{R\}}{S\{\exists x R\}} ,$$

where f is a function symbol of arity n . Still, rule $n_1\downarrow$ is not local because of the unbounded number of occurrences of x in R .

Is it possible to obtain a local system for first-order predicate logic? I do not know how to do it without adding new symbols to the language of predicate logic. But it is conceivable to obtain a local system by introducing substitution operators together with rules that explicitly handle the instantiation of variables piece by piece. The question is whether this can be done without losing the good properties, especially cut elimination and simplicity.

4.6 Decomposition of Derivations

In the following I show how all decomposition results for the propositional system from Section 3.6 extend to predicate logic in a straightforward way.

As in the propositional case, atomic identity and cut can be reduced to their shallow versions using the super switch rules. In the predicate case the rules shallow atomic identity and shallow atomic are as follows:

$$ai_s\downarrow \frac{S}{(S, \forall[a, \bar{a}])} \quad \text{and} \quad ai_s\uparrow \frac{[S, \exists(a, \bar{a})]}{S} ,$$

where \forall and \exists denote sequences of quantifiers that universally close $[a, \bar{a}]$ and existentially close (a, \bar{a}) , respectively.

The super switch rules for predicate logic,

$$ss\downarrow \frac{S\{T\{R\}\}}{S[R, T\{f\}]} \quad \text{and} \quad ss\uparrow \frac{S(R, T\{t\})}{S\{T\{R\}\}} ,$$

carry a proviso: quantifiers in T do not capture variables in R . This is not a restriction because bound variables can always be renamed such that the proviso is fulfilled.

Lemma 4.16. The rule $\text{ss}\downarrow$ is derivable for $\{\text{s}, \text{n}\downarrow, \text{u}\downarrow\}$. Dually, the rule $\text{ss}\uparrow$ is derivable for $\{\text{s}, \text{n}\uparrow, \text{u}\uparrow\}$.

Proof. The proof is an extension of the proof of Lemma 3.29. I show the two cases that have to be considered in addition to the proof in the propositional case:

1. $T\{ \} = \forall xU\{ \}$, where x occurs freely in U . Apply the induction hypothesis on

$$\begin{aligned} &= \frac{S(R, \forall xU\{t\})}{S\{\forall x(R, \forall xU\{t\})\}} \\ \text{n}\uparrow &\frac{S\{\forall x(R, U\{t\})\}}{S\{\forall xU\{R\}\}} \\ \text{ss}\uparrow & \end{aligned} .$$

2. $T\{ \} = \exists xU\{ \}$, where x occurs freely in U . Apply the induction hypothesis on

$$\begin{aligned} &= \frac{S(R, \exists xU\{t\})}{S(\forall xR, \exists xU\{t\})} \\ \text{u}\uparrow &\frac{S\{\exists x(R, U\{t\})\}}{S\{\exists xU\{R\}\}} \\ \text{ss}\uparrow & \end{aligned} .$$

□

Lemma 4.17. The rule $\text{ai}\downarrow$ is derivable for $\{\text{ai}_s\downarrow, \text{s}, \text{n}\uparrow, \text{u}\uparrow\}$. Dually, the rule $\text{ai}\uparrow$ is derivable for $\{\text{ai}_s\uparrow, \text{s}, \text{n}\downarrow, \text{u}\downarrow\}$.

Proof.

$$\text{An instance of } \text{ai}\downarrow \frac{S\{t\}}{S[a, \bar{a}]} \text{ is replaced by } \begin{array}{c} \text{ai}\downarrow \frac{S\{t\}}{(S\{t\}, \forall[a, \bar{a}])} \\ \text{ss}\uparrow \frac{S\{\forall[a, \bar{a}]\}}{S[a, \bar{a}]} \\ \text{n}\uparrow^n \end{array} .$$

(And dually for $\text{ai}\uparrow$.)

□

Theorem 4.18 (Decomposition in Predicate Logic). All theorems of section 3.6 also hold in the case of predicate logic, i.e. with SKS replaced by SKSq and KS replaced by KSq . In Theorem 3.37, $\{\text{s}, \text{m}\}$ has to be extended by the quantifier rules, i.e. $\{\text{u}\downarrow, \text{u}\uparrow, \text{l}_1\downarrow, \text{l}_1\uparrow, \text{l}_2\downarrow, \text{l}_2\uparrow, \text{n}\downarrow, \text{n}\uparrow\}$.

Proof. Identity and cut are separated as in the propositional case, using Lemma 4.17 instead of Lemma 3.30.

Contraction is separated as in the propositional case, using the proof of Theorem 3.33. The only difference is in step four, where instances of $\text{ac}\downarrow$ have to be permuted under instances of rules from $\text{KSq} \setminus \text{KS}$. None of those rules except for $\text{n}\downarrow$ changes atoms, so $\text{ac}\downarrow$ trivially permutes under those instances. It also easily permutes under instances of $\text{n}\downarrow$:

$$\text{ac}\downarrow \frac{S\{R[a, a][x/t]\}}{S\{R\{a\}[x/t]\}} \quad \text{n}\downarrow \frac{S\{\exists x R\{a\}\}}{S\{\exists x R\{a\}\}} \quad \rightsquigarrow \quad \text{n}\downarrow \frac{S\{R[a, a][x/t]\}}{S\{\exists x R[a, a]\}} \quad \text{ac}\downarrow \frac{S\{\exists x R\{a\}\}}{S\{\exists x R\{a\}\}} .$$

Weakening is separated as in the propositional case. When moved over $\text{n}\downarrow$ and $\text{n}\uparrow$, derivations of weakenings will contain weakenings on different atoms (with the same predicate symbol but differently instantiated):

$$\text{n}\downarrow \frac{S'\{R'\{a\}[x/t]\}}{S'\{\exists x R'\{a\}\}} \quad \text{aw}\uparrow^n \frac{S'\{\exists x R'\{a\}\}}{S\{\exists x R\{t\}\}} \quad \rightsquigarrow \quad \text{aw}\uparrow^n \frac{S'\{R'\{a\}[x/t]\}}{S\{R\{t\}[x/t]\}} \quad \text{n}\downarrow \frac{S\{R\{t\}[x/t]\}}{S\{\exists x R\{t\}\}}$$

□

5 Conclusions

We have seen deductive systems for classical propositional and predicate logic in the calculus of structures. They are sound and complete, and the cut rule is admissible. In contrast to sequent systems, their rules apply *deep* inside formulae, and derivations enjoy a top-down *symmetry* which allows to dualise them.

Those features allow to reduce the cut, weakening and contraction to atomic form, which is not possible in the sequent calculus. This leads to *local* rules, i.e. rules that do not require the inspection of expressions of unbounded size. For propositional logic, I presented system SKS , which is local, i.e. contains only local rules. For predicate logic I presented system SKSq which is local except for the treatment of variables.

The freedom in applying inference rules in the calculus of structures allows permutations that can not be observed in the sequent calculus. This leads to more normal forms for derivations, as shown in the decomposition theorems.

Normal forms for derivations are an interesting area for future work. The decomposition theorems given here barely scratch the surface of what seems to be achievable. For example, consider the following two conjectures:

Conjecture 5.1 (Interpolation).

$$\text{For every derivation } \begin{array}{c} T \\ \parallel_{\text{SKS}} \\ R \end{array} \text{ there is a derivation } \begin{array}{c} T \\ \parallel_{\text{SKS} \setminus \{\text{ai}\downarrow, \text{aw}\downarrow\}} \\ P \\ \parallel_{\text{SKS} \setminus \{\text{ai}\uparrow, \text{aw}\uparrow\}} \\ R \end{array} .$$

Here, a derivation is separated into two phases: the top one, with rules that do not introduce new atoms going down, and the bottom one, with rules that do not introduce new atoms going up. Consequently, the structure P contains only atoms that occur in both T and R and is thus an *interpolant*. This form of interpolation can be seen as the symmetric closure of cut elimination: not only are cuts pushed up, but also their duals, identities, are pushed down. Cut elimination is an immediate corollary of this property: if T is equivalent to the unit true then also P is equivalent to true, and in the bottom part of the derivation there are no cuts. I will show elsewhere a semantic proof of interpolation in the propositional case. A syntactic proof that scales to the predicate case would be desirable.

Another decomposition theorem, that has been proved for two other systems [10] in the calculus of structures and led to cut elimination, is the separation of the *core* and the *non-core* fragment. So far, all the systems in the calculus of structures allow for an easy reduction of both cut and identity to atomic form by means of rules that can be obtained in a uniform way. Those rules are called the *core* fragment. In **SKS**, the core consists of one single rule: the switch. The core of **SKSq**, in addition to the switch rule, also contains the rules $\text{u}\downarrow$ and $\text{u}\uparrow$. All rules that are not in the core and are not identity or cut are called *non-core*. The problem is separating switch and medial.

Conjecture 5.2 (Separation core – non-core).

$$\text{For every derivation } \begin{array}{c} T \\ \parallel_{\text{SKS}} \\ R \end{array} \text{ there is a derivation } \begin{array}{c} T \\ \parallel_{\text{non-core}} \\ T' \\ \parallel_{\{\text{ai}\downarrow\}} \\ T'' \\ \parallel_{\text{core}} \\ R'' \\ \parallel_{\{\text{ai}\uparrow\}} \\ R' \\ \parallel_{\text{non-core}} \\ R \end{array} .$$

A cut elimination procedure that is based on permuting up instances of the cut would be easy to obtain, could we rely on this conjecture. Then all the problematic rules that could stand in the way of the cut can be moved either below all the cuts or to the top of the proof, rendering them trivial, since their premise is *true*. Cut elimination is thus an easy consequence of such a decomposition theorem.

The proof of the separation of contraction (Theorem 3.33) relies on the admissibility of the cut. It should be provable directly, i.e. without using cut admissibility, just by very natural permutations. The difficulty is in proving termination of the process of bouncing contractions up and down between cuts and identities, as happens in [13].

The above mentioned freedom in applying inference rules is a mixed blessing. Compared to the sequent calculus, it implies a greater non-determinism in proof search. It will be interesting to see whether it is possible to restrict this non-determinism by finding a suitable notion of *uniform proofs* [12]. Another interesting question for future research is whether there is a local system for intuitionistic logic.

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Web Site

Information about the calculus of structures is available from the following URL:

<http://www.wv.inf.tu-dresden.de/~guglielm/Research/> .

References

- [1] Kai Brünnler. Atomic cut elimination for classical logic. Technical Report WV-02-11, Dresden University of Technology, 2002.
- [2] Kai Brünnler. Two restrictions on contraction. Technical Report WV-02-04, Dresden University of Technology, 2002.

- [3] Kai Brännler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.
- [4] Paola Bruscoli. A purely logical account of sequentiality in proof search. In Peter J. Stuckey, editor, *Logic Programming, 18th International Conference*, volume 2401 of *Lecture Notes in Artificial Intelligence*, pages 302–316. Springer-Verlag, 2002.
- [5] Kosta Došen and Zoran Petrić. Bicartesian coherence. *Studia Logica*, 71(3):331–353, 2002.
- [6] Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland, Amsterdam, 1969.
- [7] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
- [8] Alessio Guglielmi. Recipe. Manuscript. Available on the web at <http://www.wv.inf.tu-dresden.de/~guglielm/Research/Notes/AG2.pdf>, 2002.
- [9] Alessio Guglielmi. A system of interaction and structure. Technical Report WV-02-10, 2002. Available at <http://www.wv.inf.tu-dresden.de/~guglielm/Research/Gug/Gug.pdf>.
- [10] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In L. Fribourg, editor, *CSL 2001*, volume 2142 of *Lecture Notes in Computer Science*, pages 54–68. Springer-Verlag, 2001.
- [11] Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 231–246. Springer-Verlag, 2002.
- [12] Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51:125–157, 1991.
- [13] Lutz Straßburger. MELL in the calculus of structures. Technical Report WV-2001-03, Dresden University of Technology, 2001. Available at <http://www.ki.inf.tu-dresden.de/~lutz/els.pdf>.

- [14] Lutz Straßburger. A local system for linear logic. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 388–402. Springer-Verlag, 2002.
- [15] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.