Algorithms for Reasoning in Very Expressive Description Logics under Infinitely Valued Gödel Semantics

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Abstract

Fuzzy description logics (FDLs) are knowledge representation formalisms capable of dealing with imprecise knowledge by allowing intermediate membership degrees in the interpretation of concepts and roles. One option for dealing with these intermediate degrees is to use the so-called Gödel semantics, under which conjunction is interpreted by the minimum of the degrees of the conjuncts. Despite its apparent simplicity, developing reasoning techniques for expressive FDLs under this semantics is a hard task.

In this paper, we introduce two new algorithms for reasoning in very expressive FDLs under Gödel semantics. They combine the ideas of a previous automata-based algorithm for Gödel FDLs with the known crispification and tableau approaches for FDL reasoning. The results are the two first practical algorithms capable of reasoning in infinitely valued FDLs supporting general concept inclusions.

Keywords: Fuzzy Description Logics, Gödel Fuzzy Logic, Mathematical Fuzzy Logic, Tableau Algorithm, Fuzzy OWL 2

1. Introduction

Description logics (DLs) [1] are a well-known family of knowledge representation formalisms that have been successfully used for modeling many real-world domains. Their basic building blocks are *individuals*, representing elements of the application domain, *concepts* expressing unary predicates over the domain, and *roles* encoding binary relations. Complex statements are built using different concept constructors. For example, the DL axioms $\mathsf{Father}(\text{bob})$ and $\mathsf{Father} \sqsubseteq \exists \mathsf{hasChild}.\top$ express that the individual Bob belongs to the concept Father, and that every father must have a child, i.e. an individual that is in the $\mathsf{hasChild}$-relation with him. There is an obvious trade-off between the expressivity of a DL and the complexity of reasoning tasks. Hence, many different DLs have been proposed in the literature. They range from $\mathcal{ALC}$—the smallest propositionally closed DL—and its sub-logics, to the very expressive $\mathcal{SROIQ}$—one of the largest decidable DLs, and the logic underlying the Web Ontology Language OWL 2 [2].

One of the known limitations of classical logic is its inability to handle imprecise concepts for which a clear-cut characterization is impossible [3, 4]. To cover this gap, the semantics of DLs, which originally is based on first-order logic, has been extended following the ideas of mathematical fuzzy logic [3, 5]. The resulting *fuzzy description logics* (FDLs) allow intermediate truth degrees—usually rational numbers between 0 (false) and 1 (true)—to be used while defining and reasoning with imprecise knowledge [6]. To interpret the knowledge that uses these intermediate degrees, the logical connectives need to be extended accordingly. In general, many possible such extensions can be considered. Hence, every classical DL gives rise to a family of FDLs, whose members differ on the interpretation of the connectives they use. Unfortunately, it has been shown that for most of these extensions, reasoning is undecidable, even if the underlying DL

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is relatively inexpressive [7]. In fact, the only decidable expressive FDLs are those based on the Gödel semantics [8], and the related Zadeh semantics [9]. For the rest of this paper, we focus on fuzzy DLs that use Gödel semantics. These are denoted by the prefix "G-".

Already in the classical case, developing a reasoning algorithm for the very expressive DL $\mathcal{SROIQ}$ is far from trivial. In fact, one needs to handle all possible interactions between the many constructors available. For instance, it is known that the combination nominals and number restrictions can be problematic. This difficulty is accentuated when the Gödel semantics are considered, since this logic does not have the *finitely valued model property* [10]. This means that there are consistent ontologies whose models must use infinitely many truth degrees. Indeed, this is one of the reasons why the *crispification approach* as described in [11–13]—in which concepts are partitioned according to the degrees of their elements—is only correct under finitely valued semantics.

The study of reasoning algorithms for expressive Gödel FDLs that can handle infinitely many truth degrees started in [10, 14]. The main contribution of that work was the development of automata-based methods for testing the existence of (potentially infinite) models of a fuzzy ontology. These methods were used to show that the loss of the finitely valued model property does not affect the complexity of reasoning in G-$\mathcal{ALC}$. Rather than trying to find a model directly, this algorithm produces an abstract representation of a large class of models. In this representation, the actual degrees of truth used in a model are abstracted to consider only the order among them. As an added benefit, considering only the order between concepts allows for a more flexible representation of the domain knowledge. For instance, one can express that an individual is *taller* than another, i.e. $\mathsf{Tall}(\mathrm{bob}) > \mathsf{Tall}(\mathrm{chris})$, without having to specify explicit degrees of tallness for them.

In this paper, we present two algorithms that exploit the same idea of considering the order between degrees, rather than the degrees explicitly. The first algorithm is an extension of the crispification approach for finitely valued FDLs, which translates a fuzzy ontology into a classical ontology by using concepts that simulate the order between the relevant truth degrees.[1] Although it yields good theoretical results such as tight complexity bounds for reasoning, this approach is restricted to sublogics of G-$\mathcal{SROIQ}$ that have the forest-model property [16], and there is no obvious way to extend them to the full expressivity of G-$\mathcal{SROIQ}$. To overcome these limitations, we develop a novel combination of the classical tableau algorithm for $\mathcal{SROIQ}$ with the order-based abstraction from [10]. It inherits the pay-as-you-go behavior from the classical tableau algorithms, and is the first reasoning algorithm that can handle the full expressivity of fuzzy $\mathcal{SROIQ}$ under Gödel semantics.

The paper is structured as follows. We start by recalling the basic notions of the Gödel semantics for fuzzy logic and fuzzy description logics that are relevant for the rest of the paper. In Section 3 we provide the main intuitions behind our algorithms with the help of a detailed example, and Section 4 describes an automata-based method for handling complex role inclusions that is needed for the two algorithms. We then present in detail the two proposed algorithms for reasoning in G-$\mathcal{SROIQ}$: Section 5 provides a reduction to classical DLs, followed by a new tableau algorithm in Section 6. We finish the paper with a description of related work and some concluding remarks. To improve readability, detailed proofs of our results are deferred to Appendix A.

2. Preliminaries

We start by recalling some of the basic notions of Gödel fuzzy logic and fuzzy description logics [3, 6, 17] that will be used throughout the paper.

2.1. Gödel Fuzzy Logic and Order Structures

The two basic operators of Gödel fuzzy logic are conjunction and implication, interpreted by the *Gödel t-norm* and *residuum*, respectively. The Gödel t-norm of two fuzzy degrees[2] $x, y \in [0, 1]$ is defined as the

[1] This algorithm has already been presented in the short conference paper [15].

[2] For the scope of this paper, we limit the possible degrees to be rational numbers only.

minimum $\min\{x, y\}$. Its residuum $\Rightarrow$ is the operation uniquely defined by the equivalence $\min\{x, y\} \leqslant z$ iff $y \leqslant (x \Rightarrow z)$ for all $x, y, z \in [0, 1]$. Equivalently, this operation can be computed as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise.} \end{cases}$$

Additionally, we consider both the *residual negation* $x \mapsto (x \Rightarrow 0)$ and the *involutive negation* $x \mapsto (1 - x)$ in this paper. Note that min is monotone in both arguments, and hence preserves arbitrary infima and suprema, while $\Rightarrow$ is monotone in the second argument and antitone in the first argument. The following property will be useful throughout this paper [3].

Proposition 2.1. *For all values $x, x_1, \dots, x_n \in [0, 1]$, we have*

$$\big((x_1 \wedge \dots \wedge x_n) \Rightarrow x\big) = \big(x_1 \Rightarrow \dots (x_n \Rightarrow x) \dots\big).$$

An *order structure* S is a finite set containing at least the numbers 0, 0.5, and 1, together with an involutive unary operation $\mathsf{inv}\colon S \to S$ such that $\mathsf{inv}(x) = 1 - x$ for all numbers $x \in S \cap [0, 1]$. A *total preorder* (on S) is a transitive and total binary relation $\preccurlyeq_* \subseteq S \times S$. Notice that any such relation is necessarily also reflexive. For $\alpha, \beta \in S$, we write $\alpha \simeq_* \beta$ if $\alpha \preccurlyeq_* \beta$ and $\beta \preccurlyeq_* \alpha$, and we write $\alpha \prec_* \beta$ if it is not the case that $\beta \preccurlyeq_* \alpha$ (but then $\alpha \preccurlyeq_* \beta$ holds since $\preccurlyeq_*$ is total). We emphasise here that $\simeq_*$ is an equivalence relation on S. For a relation symbol $\bowtie \in \{<, \leqslant, =, \geqslant, >\}$, we denote by $\bowtie_*$ the corresponding relation induced by $\preccurlyeq_*$; that is, $\prec_*$, $\preccurlyeq_*$, $\simeq_*$, $\succcurlyeq_*$, or $\succ_*$, respectively. The set $\mathsf{order}(S)$ contains exactly those total preorders $\preccurlyeq_*$ over S which

- have 0 and 1 as least and greatest element, respectively,
- coincide with the order of the rational numbers on $S \cap [0, 1]$, i.e. for all $\alpha, \beta \in S \cap [0, 1]$ it holds that $\alpha \leqslant \beta$ iff $\alpha \preccurlyeq_* \beta$, and
- satisfy $\alpha \preccurlyeq_* \beta$ iff $\mathsf{inv}(\beta) \preccurlyeq_* \mathsf{inv}(\alpha)$ for all $\alpha, \beta \in S$.

Given a preorder $\preccurlyeq_* \in \mathsf{order}(S)$, the following functions extend the operators of Gödel fuzzy logic from the elements in $S \cap [0, 1]$ to the whole order structure S:

$$\min\nolimits_*\{\alpha, \beta\} := \begin{cases} \alpha & \text{if } \alpha \preccurlyeq_* \beta \\ \beta & \text{otherwise,} \end{cases}$$

$$\alpha \Rightarrow_* \beta := \begin{cases} 1 & \text{if } \alpha \preccurlyeq_* \beta \\ \beta & \text{otherwise.} \end{cases}$$

An *order assertion* (over S) is an expression of the form $\alpha \bowtie \beta$, where $\bowtie \in \{<, \leqslant, =, \geqslant, >\}$ and $\alpha, \beta \in S$; the elements α, β are not necessarily numbers from $[0, 1]$, but arbitrary elements of S. We do not distinguish between $\alpha \trianglelefteq \beta$ and $\beta \trianglerighteq \alpha$, where $\trianglelefteq$ is either $<$, $\leqslant$, or $=$, and $\trianglerighteq$ is $>$, $\geqslant$, or $=$, respectively. An *order formula* is a Boolean combination of order assertions. An element $\preccurlyeq_* \in \mathsf{order}(S)$ *satisfies* (or is a *model* of)

- the order assertion $\alpha \bowtie \beta$ iff $\alpha \bowtie_* \beta$ holds;
- an order formula if there is a satisfying Boolean valuation of all its order assertions such that $\preccurlyeq_*$ satisfies all order assertions evaluated to true, and does not satisfy any order assertions evaluated to false;
- a set of order assertions if it satisfies all assertions contained in it.

A set of order assertions Φ is *satisfiable* if it has a model, and it *entails* an order assertion ϕ if all models of Φ are also models of ϕ.

In the following, we describe a procedure for deciding satisfiability of a set of order assertions Φ. We construct the labeled directed graph $G_\Phi = (S, E_\Phi)$ with labels from the set $\{\leqslant, <\}$, whose nodes are the elements of the order structure S, and the labeled edges

$$\begin{aligned}
E_\Phi := \{&(\alpha, <, \beta) \mid \alpha < \beta \in S \text{ or } \mathsf{inv}(\alpha) > \mathsf{inv}(\beta) \in S\} \cup \\
\{&(\alpha, \leqslant, \beta) \mid \alpha \leqslant \beta \in S \text{ or } \alpha = \beta \in S \text{ or } \mathsf{inv}(\alpha) \geqslant \mathsf{inv}(\beta) \in S \text{ or } \mathsf{inv}(\alpha) = \mathsf{inv}(\beta) \in S\} \cup \\
\{&(\alpha, <, \beta) \mid \alpha, \beta \in S \cap [0,1],\ \alpha < \beta,\ \text{there is no } \gamma \in S \cap [0,1] \text{ with } \alpha < \gamma < \beta\} \cup \\
\{&(0, \leqslant, \alpha),\ (\alpha, \leqslant, 1) \mid \alpha \in S \setminus \{0,1\}\}
\end{aligned}$$

encode Φ and the properties of $\mathsf{order}(S)$. A cycle involving an edge labeled with $<$ is called a *$<$-cycle*.

Lemma 2.2. *A set Φ of order assertions over an order structure S is satisfiable iff G_Φ has no $<$-cycle.*

Furthermore, Φ entails $\alpha \leqslant \beta$ iff $\Phi \cup \{\beta > \alpha\}$ is unsatisfiable (and similarly for $<$), and Φ entails $\alpha = \beta$ iff it entails both $\alpha \leqslant \beta$ and $\alpha \geqslant \beta$. Hence, satisfiability and entailment of order assertions can be decided in polynomial time.

For convenience, we sometimes use a generalized form of order assertions, like $\alpha \geqslant \min\{\beta, \gamma\}$ or $\alpha = (\beta \Rightarrow \gamma)$, where min and $\Rightarrow$ are interpreted using the operators $\min_*$ and $\Rightarrow_*$, respectively, introduced above.

2.2. G-$\mathcal{SROIQ}$

We now introduce the very expressive fuzzy description logic G-$\mathcal{SROIQ}$. As is common for fuzzy DLs, we add the involutive negation to the "pure" Gödel logic that includes the residual negation. The involutive negation will be handled by our reasoning algorithms through appropriate (encodings of) order structures, whose definition already accounts for the behavior of the involutive negation (through the operator inv). The semantics of the residual negation will be handled through constructions or rules dedicated to the implication constructor (see Sections 5 and 6).

Let $\mathsf{N_I}$, $\mathsf{N_C}$, and $\mathsf{N_R}$ be three mutually disjoint sets of *individual names*, *concept names*, and *role names*, respectively. The set of *roles* is $\mathsf{N_R^-} := \mathsf{N_R} \cup \{r^- \mid r \in \mathsf{N_R}\}$, where the elements of the form r^- are called *inverse roles*. Since we need to make several syntactic restrictions based on which roles appear in which role axioms, we first consider the notion of role hierarchies.

2.2.1. Regular Role Hierarchies

A *role hierarchy* $\mathcal{R}_h$ is a finite set of *(complex) role inclusions* of the form $w \sqsubseteq r \geqslant p$, where r is a role name, $w \in (\mathsf{N_R^-})^+$ is a non-empty *role chain*, and $p \in (0,1]$. Such a role inclusion is called *simple* if $w \in \mathsf{N_R^-}$. We extend the notation $\cdot^-$ to inverse roles and role chains as usual, by setting $(r^-)^- := r$ and $(r_1 \ldots r_n)^- := r_n^- \ldots r_1^-$.

We recall now the regularity condition from classical DLs [18, 19]. Let $\lessdot$ be a strict partial order on $\mathsf{N_R^-}$ such that $r \lessdot s$ iff $r^- \lessdot s$. A role inclusion $w \sqsubseteq r \geqslant p$ is *$\lessdot$-regular* if

- w is of the form rr or r^-, or
- w is of the form $r_1 \ldots r_n$, $rr_1 \ldots r_n$, or $r_1 \ldots r_n r$, and for all $1 \leqslant i \leqslant n$ it holds that $r_i \lessdot r$.

A role hierarchy $\mathcal{R}_h$ is *regular* if there is a strict partial order $\lessdot$ as above such that each role inclusion in $\mathcal{R}_h$ is $\lessdot$-regular. The set of *non-simple* role names (w.r.t. $\mathcal{R}_h$) is the smallest set satisfying the following condition: If $w \sqsubseteq r \geqslant p \in \mathcal{R}_h$ is not simple or w is of the form s or s^- for a non-simple role s, then r is non-simple. All other role names are *simple*. An inverse role r^- is *simple* (*non-simple*) if r is simple (non-simple). For the rest of this paper, let $\mathcal{R}_h$ be a regular role hierarchy.

Table 1: Syntax and semantics of G-$\mathcal{SROIQ}$

Concept	Syntax (C)	Semantics ($C^{\mathcal{I}}(d)$)
concept name	$A \in \mathsf{N_C}$	$A^{\mathcal{I}}(d) \in [0,1]$
truth constant	$\overline{p}$	p
conjunction	$C \sqcap D$	$\min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\}$
implication	$C \to D$	$C^{\mathcal{I}}(d) \Rightarrow D^{\mathcal{I}}(d)$
negation	$\neg C$	$1 - C^{\mathcal{I}}(d)$
existential restriction	$\exists r.C$	$\sup_{e \in \Delta^{\mathcal{I}}} \min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\}$
value restriction	$\forall r.C$	$\inf_{e \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(d,e) \Rightarrow C^{\mathcal{I}}(e)$
nominal	$\{a\}$	$\begin{cases} 1 & \text{if } d = a^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$
at-least restriction	$\geqslant n\, s.C$	$\sup_{\substack{e_1,\dots,e_n \in \Delta^{\mathcal{I}} \\ \text{pairwise different}}} \min_{i=1}^{n} \min\{s^{\mathcal{I}}(d,e_i), C^{\mathcal{I}}(e_i)\}$
local reflexivity	$\exists s.\mathsf{Self}$	$s^{\mathcal{I}}(d,d)$
Role (chain)	**Syntax (r)**	**Semantics ($r^{\mathcal{I}}(d,e)$)**
role name	$r \in \mathsf{N_R}$	$r^{\mathcal{I}}(d,e) \in [0,1]$
inverse role	r^-	$r^{\mathcal{I}}(e,d)$
role chain	$r_1 \dots r_n \in (\mathsf{N_R^-})^+$	$\sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}}} \min\{r_1^{\mathcal{I}}(d,d_1), \dots, r_n^{\mathcal{I}}(d_{n-1},e)\}$

2.2.2. Concepts

G-$\mathcal{SROIQ}$ *concepts* [15] are built using the constructors listed in the upper part of Table 1, where C, D denote concepts, $p \in [0,1]$, $n \in \mathbb{N}$, $A \in \mathsf{N_C}$, $a \in \mathsf{N_I}$, $r \in \mathsf{N_R^-}$, and $s \in \mathsf{N_R^-}$ is a simple role. The restriction to simple roles in at-least restrictions is necessary to ensure decidability, already in the classical case [20]. We also use the common DL constructors $\top := \overline{1}$ (top concept), $\bot := \overline{0}$ (bottom concept), $C \sqcup D := \neg(\neg C \sqcap \neg D)$ (disjunction), and $\leqslant n\, s.C := \neg(\geqslant(n+1)\, s.C)$ (at-most restriction).

The semantics of G-$\mathcal{SROIQ}$ is based on G-*interpretations*, which are pairs of the form $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set, called the *domain*, and $\cdot^{\mathcal{I}}$ is the *interpretation function* which assign to each individual name $a \in \mathsf{N_I}$ an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, to each concept name $A \in \mathsf{N_C}$ a fuzzy set $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$, and to each role name $r \in \mathsf{N_R}$ a fuzzy binary relation $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. This G-interpretation is extended to complex concepts, roles, and role chains as defined in the last column of Table 1, for all $d, e \in \Delta^{\mathcal{I}}$.

Remark 2.3. In contrast to classical DLs, existential and value restrictions are not equally expressive; i.e. in general it does not hold that $(\forall r.C)^{\mathcal{I}}(x) = (\neg\exists r.\neg C)^{\mathcal{I}}(x)$. However, they can still be viewed as dual in the sense that we can obtain one from the other by inverting the order on $[0,1]$ and replacing $\Rightarrow$ by min, or vice versa. This also means that the fuzzy semantics of existential restrictions exhibits a behavior that is classically ascribed to value restrictions: the supremum used to define $(\exists r.C)^{\mathcal{I}}(d)$ implicitly imposes an upper bound on the value of $\min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\}$, for *all* domain elements e. Dually, value restrictions also require the existence of a particular element through the witnessing conditions, as described in Section 2.2.3. This property of fuzzy role restrictions is important for the definition of the algorithms in Sections 5 and 6. A similar behavior can be observed in at-least restrictions: the semantics of $(\geqslant n\, r.C)^{\mathcal{I}}(d)$ puts an *at-most restriction* ($\leqslant n-1$) on the number of nodes e for which $\min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\}$ can exceed this value.

In some previous work on fuzzy extensions of $\mathcal{SROIQ}$ [12], fuzzy at-most restrictions are defined using the residual negation; that is $\leqslant n\, s.C := (\geqslant(n+1)\, s.C) \to \bot$. This has the effect that the value of $\leqslant n\, r.C$

under Gödel semantics is always either 0 or 1 (see the formal semantics below). However, this discrepancy in definitions is not an issue: as we will show, our algorithms in Sections 5 and 6 can handle both the involutive and the residual negation, and hence also this alternative notion of at-most restrictions. More precisely, the semantics of $\neg$ can be expressed by the operator inv of an appropriate order structure. In contrast, the residual negation $C \to \bot$ is treated by dedicated rules for the implication and bottom constructors.

The use of truth constants $\overline{p}$ for $p \in [0,1]$ is not standard in FDLs, but it allows us to simulate e.g. *fuzzy nominals* [21] of the form $\{p_1/a_1, \dots, p_n/a_n\}$ with $p_i \in [0,1]$ and $a_i \in \mathsf{N_I}$, $1 \leqslant i \leqslant n$, using the concept $(\{a_1\} \sqcap \overline{p_1}) \sqcup \dots \sqcup (\{a_n\} \sqcap \overline{p_n})$.

2.2.3. Witnessed Interpretations

As it is usual in fuzzy DLs, we restrict reasoning to *witnessed* G-interpretations [22]. Intuitively, witnessed interpretations require that the suprema and infima in the semantics are in fact maxima and minima, respectively. In other words, the degrees of these constructors are witnessed by an element in the domain. Formally, a G-interpretation $\mathcal{I}$ is *witnessed* if, for every $d \in \Delta^{\mathcal{I}}$, $n \geqslant 0$, $r \in \mathsf{N_R^-}$, simple $s \in \mathsf{N_R^-}$, and concept C, there are $e, e', e_1, \dots, e_n \in \Delta^{\mathcal{I}}$ such that $e_1, \dots, e_n$ are pairwise different,

$$\begin{aligned}(\exists r.C)^{\mathcal{I}}(d) &= \min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\},\\ (\forall r.C)^{\mathcal{I}}(d) &= r^{\mathcal{I}}(d,e') \Rightarrow C^{\mathcal{I}}(e'), \qquad \text{and}\\ ({\geqslant} n\, s.C)^{\mathcal{I}}(d) &= \min_{i=1}^{n} \min\{s^{\mathcal{I}}(d,e_i), C^{\mathcal{I}}(e_i)\}.\end{aligned}$$

2.2.4. Ontologies

As we have seen already with the role inclusions, the axioms of G-$\mathcal{SROIQ}$ extend classical axioms by allowing to state a degree in $(0,1]$ to which the axioms hold. A *(classical) assertion* is either a *concept assertion* of the form $C(a)$ or a *role assertion* of the form $r(a,b)$ for $a, b \in \mathsf{N_I}$, a concept C, and a role r. A *(fuzzy) assertion* is of the form $\alpha \bowtie p$ or $\alpha \bowtie \beta$, where α, β are classical assertions, $\bowtie \in \{<, \leqslant, =, \geqslant, >\}$, and $p \in [0,1]$. An *ABox* is a finite set of fuzzy assertions and *individual (in)equality assertions* of the form $a \approx b$ ($a \not\approx b$) for $a, b \in \mathsf{N_I}$. A *TBox* is a finite set of *general concept inclusions (GCIs)* of the form $C \sqsubseteq D \geqslant p$ for concepts C, D and $p \in (0,1]$. An *RBox* $\mathcal{R} = \mathcal{R}_h \cup \mathcal{R}_a$ consists of the role hierarchy $\mathcal{R}_h$ and a finite set $\mathcal{R}_a$ of *disjoint role axioms* $\mathsf{dis}(s_1, s_2) \geqslant p$, where s_1, s_2 are simple roles and $p \in (0,1]$. An *ontology* $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ consists of an ABox $\mathcal{A}$, a TBox $\mathcal{T}$, and an RBox $\mathcal{R}$. For an ontology $\mathcal{O}$, we denote by $\mathsf{rol}(\mathcal{O})$ the set of all roles occurring in $\mathcal{O}$, together with their inverses, and by $\mathsf{ind}(\mathcal{O})$ the set of all individual names appearing in $\mathcal{O}$.

A G-interpretation $\mathcal{I}$ *satisfies* (or is a *model* of)

- the fuzzy assertion $\alpha \bowtie \beta$ if $\alpha^{\mathcal{I}} \bowtie \beta^{\mathcal{I}}$, where $(C(a))^{\mathcal{I}} := C^{\mathcal{I}}(a^{\mathcal{I}})$, $(r(a,b))^{\mathcal{I}} := r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$, and $p^{\mathcal{I}} := p$ for all $p \in [0,1]$;
- the (in)equality assertion $a \approx b$ ($a \not\approx b$) if $a^{\mathcal{I}} = b^{\mathcal{I}}$ ($a^{\mathcal{I}} \neq b^{\mathcal{I}}$);
- the GCI $C \sqsubseteq D \geqslant p$ iff $C^{\mathcal{I}}(d) \Rightarrow D^{\mathcal{I}}(d) \geqslant p$ for all $d \in \Delta^{\mathcal{I}}$;
- the role inclusion $r_1 \dots r_n \sqsubseteq r \geqslant p$ iff $(r_1 \dots r_n)^{\mathcal{I}}(d,e) \Rightarrow r^{\mathcal{I}}(d,e) \geqslant p$ for all $d, e \in \Delta^{\mathcal{I}}$;
- the disjoint role axiom $\mathsf{dis}(s_1, s_2) \geqslant p$ iff $\min\{s_1^{\mathcal{I}}(d,e), s_2^{\mathcal{I}}(d,e)\} \leqslant 1 - p$ for all $d, e \in \Delta^{\mathcal{I}}$;
- an ontology if it satisfies all its axioms.

Using the axioms previously introduced, it is possible to simulate other expressions that are commonly considered to be part of (fuzzy) $\mathcal{SROIQ}$ [12, 23] as follows, where r is a role and s is a simple role:

- transitivity axioms $\mathsf{tra}(r) \geqslant p$ by $rr \sqsubseteq r \geqslant p$;
- symmetry axioms $\mathsf{sym}(r) \geqslant p$ by $r^- \sqsubseteq r \geqslant p$;

- asymmetry axioms $\mathsf{asy}(s) \geqslant p$ by $\mathsf{dis}(s, s^-) \geqslant p$;
- reflexivity axioms $\mathsf{ref}(r) \geqslant p$ by $\top \sqsubseteq \exists s.\mathsf{Self} \geqslant 1$ and $s \sqsubseteq r \geqslant p$, where s is a fresh (and hence simple) role name;
- irreflexivity axioms $\mathsf{irr}(s) \geqslant p$ by $\exists s.\mathsf{Self} \sqsubseteq \neg\overline{p} \geqslant 1$;
- negated role assertions $\neg r(a, b) \geqslant p$ by $r(a, b) \leqslant 1 - p$; and
- the universal role r_u by $\mathsf{tra}(r_u) \geqslant 1$, $\mathsf{sym}(r_u) \geqslant 1$, $\mathsf{ref}(r_u) \geqslant 1$, $r \sqsubseteq r_u \geqslant 1$ for all $r \in \mathsf{rol}(\mathcal{O})$, and $r_u(a, b) = 1$ for all $a, b \in \mathsf{ind}(\mathcal{O})$, as it is commonly done in the literature [16, 23]. There are also other possibilities to simulate the universal role, e.g. using nominals instead of symmetry and reflexivity.

A G-$\mathcal{SROIQ}$ ontology is *consistent* if it has a witnessed G-model. Other common reasoning problems for FDLs, such as concept satisfiability and subsumption can be reduced to consistency in linear time [14]. For instance, the subsumption between C and D to degree p w.r.t. a TBox $\mathcal{T}$ and an RBox $\mathcal{R}$ is equivalent to the inconsistency of $(\{(C \to D)(a) < p\}, \mathcal{T}, \mathcal{R})$, where a is a fresh individual name. Likewise, the satisfiability of C to degree p w.r.t. $\mathcal{T}$ and $\mathcal{R}$ is equivalent to the consistency of $(\{C(a) \geqslant p\}, \mathcal{T}, \mathcal{R})$. One can even show that the *best* satisfiability and subsumption degrees are always values occurring in the input ontology, and can be computed using linearly many consistency tests [14]. Hence, we can restrict the following considerations to the problem of deciding consistency of ontologies.

2.2.5. Sublogics of G-$\mathcal{SROIQ}$

The letter $\mathcal{I}$ in G-$\mathcal{SROIQ}$ denotes the presence of inverse roles. If such roles are not allowed (and hence we also cannot express (a)symmetry axioms), the resulting logic is written G-$\mathcal{SROQ}$. Likewise, the name G-$\mathcal{SRIQ}$ indicates the absence of nominals, and G-$\mathcal{SROI}$ that of at-least and at-most restrictions. Replacing the letter $\mathcal{R}$ with $\mathcal{H}$ indicates that RBoxes are restricted to simple role inclusions (i.e. $r \sqsubseteq s \geqslant p$), ABoxes are restricted to fuzzy assertions, and local reflexivity is not allowed; however, the letter $\mathcal{S}$ indicates that transitivity axioms are still allowed. Hence, in G-$\mathcal{SHOIQ}$ we can use role inclusions of the forms $r \sqsubseteq s \geqslant p$ and $rr \sqsubseteq r \geqslant p$. Disallowing axioms of the first type removes the letter $\mathcal{H}$, while the absence of transitivity axioms is denoted by replacing $\mathcal{S}$ with $\mathcal{ALC}$.

Classical DLs are obtained from the above definitions by restricting the set of truth values to 0 and 1, and hence removing the prefix "G-" from the name. The semantics of a classical concept C is then viewed as a set $C^\mathcal{I} \subseteq \Delta^\mathcal{I}$ instead of the characteristic function $C^\mathcal{I} \colon \Delta^\mathcal{I} \to \{0, 1\}$, and likewise for roles. In this setting, all axioms are restricted to be of the form $\alpha \geqslant 1$, and this is abbreviated to α, e.g. $C \sqsubseteq D$ instead of $C \sqsubseteq D \geqslant 1$. We also use $C \equiv D$ as short-hand for the two axioms $C \sqsubseteq D$ and $D \sqsubseteq C$. Furthermore, the implication constructor $C \to D$, although usually not included in classical DLs, can be expressed via $\neg C \sqcup D$.

In this paper, we present two algorithms for deciding consistency in (sublogics of) G-$\mathcal{SROIQ}$. Before we describe them in all details, we illustrate the main ideas on an example, involving only a small subset of the constructors and axioms of G-$\mathcal{SROIQ}$.

3. A First Example

The main idea for our two algorithms is that, instead of explicitly defining the degrees of all concepts and roles for all domain elements, we only represent the *order* between different values. For example, to satisfy the semantics of the implication $\to$, i.e. $(C \to D)^\mathcal{I}(x) = C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x)$, it suffices to consider the two cases

- $(C \to D)^\mathcal{I}(x) = 1$ and $C^\mathcal{I}(x) \leqslant D^\mathcal{I}(x)$; or
- $(C \to D)^\mathcal{I}(x) = D^\mathcal{I}(x)$ and $C^\mathcal{I}(x) > D^\mathcal{I}(x)$.

In both cases, it is irrelevant what the actual values of $C^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(x)$ are, as long as they satisfy a certain order relationship. We exploit this property of the Gödel operators in the following constructions, by using order structures and order assertions to represent the semantics of concepts. This idea has also been previously used for other reasoning problems based on the Gödel semantics [24].

Before we describe the algorithms in detail, in this section we consider the consistency problem for the small example ontology $\mathcal{O} := (\mathcal{A}, \mathcal{T}, \emptyset)$, where[3]

$$\mathcal{A} := \{(\exists r.A)(a) \geqslant p_A,\ (\exists r.B)(a) \geqslant p_B,\ ({\leqslant}1\, r.C)(a) \geqslant p_C\},$$
$$\mathcal{T} := \{A \sqsubseteq C \geqslant 1,\ B \sqsubseteq C \geqslant 1\},$$

and p_A, p_B, p_C are arbitrary values, to illustrate the main ideas behind our methods. The formal details of these approaches will be presented in Sections 4–6. For the nonce, our goal is to make the intuitions behind these procedures clear to the reader.

3.1. Reduction to Classical DLs

Our first algorithm is based on a reduction of the fuzzy ontology $\mathcal{O}$ to a classical ontology $\mathsf{red}(\mathcal{O})$. We use special concept names to express order assertions over a specific order structure $\mathcal{U}$. This order structure contains all values occurring in $\mathcal{O}$, all relevant subconcepts and roles, e.g. $\exists r.A_1$ and r, relevant assertions over known individuals, such as $({\leqslant}1\, r.C)(a)$, and special role assertions of the form $r(*, a)$, as explained below. For example, the concept $\boxed{C > (\exists r.A)(a)}$ expresses that the value of C at the current domain element should exceed the value of $\exists r.A$ at a. We call them *order concepts*, and, to improve readability, always denote them with a surrounding box. This approach can be seen as an extension of previous algorithms for reasoning in fuzzy DLs based on reductions to classical DLs [12, 13, 21], which use *cut-concepts* of the form $\boxed{A \geqslant p}$, but are applicable only when the fuzzy semantics is based on finitely many values.

To achieve the correct behavior, our reduction explicitly specifies the semantics of the order structure and the concept constructors. For example, we use the classical axioms $\top \sqsubseteq \boxed{\alpha \leqslant \beta} \sqcup \boxed{\beta \leqslant \alpha}$, for all $\alpha, \beta \in \mathcal{U}$, to express that $\leqslant$ should be a total relation. The assertions in our ABox $\mathcal{A}$ are translated into classical assertions, e.g. $\boxed{(\exists r.A)(a) \geqslant p_A}(a)$. To ensure that $(\exists r.A)(a)$ actually represents the value of the existential restriction $\exists r.A$ at the individual a, we use the additional assertion $\boxed{(\exists r.A)(a) = (\exists r.A)}(a)$. The GCIs from our example ontology have the straightforward translations

$$\top \sqsubseteq \boxed{A \Rightarrow C \geqslant 1} \text{ and } \top \sqsubseteq \boxed{B \Rightarrow C \geqslant 1},$$

which require that they are satisfied in every element of the domain.

In the reduction, domain elements are connected via only one special role, denoted by $\mathfrak{r}$. This role is used to transfer information between domain elements. The goal is that, except for the named individuals, the role $\mathfrak{r}$ generates a forest-shaped structure in the classical interpretation; hence this approach is restricted to logics having the forest-model property, i.e. $\mathcal{SRIQ}$, $\mathcal{SROQ}$, and $\mathcal{SROI}$ [16].

Information about the named individuals is transferred to all $\mathfrak{r}$-connected domain elements using GCIs like $\boxed{(\exists r.A)(a) \geqslant (\exists r.B)(a)} \sqsubseteq \forall\mathfrak{r}.\boxed{(\exists r.A)(a) \geqslant (\exists r.B)(a)}$ and $\boxed{p_A \geqslant (\exists r.B)(a)} \sqsubseteq \forall\mathfrak{r}.\boxed{p_A \geqslant (\exists r.B)(a)}$, i.e. whenever a domain element x "knows" something about the behavior of a, then all $\mathfrak{r}$-successors of x share that knowledge. This is not strictly necessary for the current example, but in general it is needed to ensure the correct treatment of nominals: if an arbitrary domain element refers back to a named individual, e.g. via $B \sqsubseteq \exists r.(\{a\} \sqcap A)$, then we need to specify the relative order of $A(a)$ and B. This behavior can only be modeled consistently if all (connected) domain elements share all relevant (order) information about the named elements, in particular in relation to fixed values like p_A.

Special elements of $\mathcal{U}$ of the form $\langle C\rangle_\uparrow$ are used to refer to the value of a concept C at the parent node in the tree. These elements are restricted by axioms like $\boxed{(\exists r.B) \leqslant C} \sqsubseteq \forall\mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle\exists r.B\rangle_\uparrow \leqslant \langle C\rangle_\uparrow}\big)$, which express that order relations between concepts of the parent are known by all child nodes, i.e. $\mathfrak{r}$-successors. The

[3] Recall that ${\leqslant}1\, r.C$ is an abbreviation for $\neg{\geqslant}2\, r.C$.

special concept name AN is used to distinguish anonymous domain elements from those that are designated by an individual name (and are hence not part of the forest).

In our example, to generate a witness for the existential restriction $\exists r.A$ at a, we introduce the axiom

$$\top \sqsubseteq \exists \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \exists r.A\rangle_{\uparrow} \leqslant \min\{r, A\}}\big) \sqcup \big(\exists \mathfrak{r}.\{a\} \sqcap \boxed{(\exists r.A) \leqslant \min\{r(*,a), A(a)\}}\big). \tag{1}$$

That is, either a has an anonymous (AN) $\mathfrak{r}$-successor at which the value of $\exists r.A$ at the parent node ($\langle \exists r.A\rangle_{\uparrow}$), in this case a, is bounded by the minimum between the r-connection to the parent node (r) and the value of A at the current node (A); or there is an $\mathfrak{r}$-successor that satisfies $\{a\}$, i.e. a itself, and the value of $\exists r.A$ at a is bounded by the minimum between the value of the role connection from the current node (represented by $*$) to a and the value of A at a ($A(a)$). In general, the second part has to consider all named domain elements as possible successors; in our example we have only a.

On the other hand, all $\mathfrak{r}$-successors of a have to be restricted to not exceed the value of $\exists r.A$ (cf. Remark 2.3) using the similar axioms

$$\top \sqsubseteq \forall \mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle \exists r.A\rangle_{\uparrow} \geqslant \min\{r, A\}}\big) \text{ and } \exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\exists r.A) \geqslant \min\{r(*,a), A(a)\}}. \tag{2}$$

Analogous axioms are introduced to express the semantics of $\exists r.B$.

For the number restriction $\leqslant 1\, r.C = \neg{\geqslant}2\, r.C$, we first create witnesses as for the existential restrictions above:[4]

$$\top \sqsubseteq {\geqslant}2\, \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle {\geqslant}2\, r.C\rangle_{\uparrow} \leqslant \min\{r, C\}}\big) \sqcup \big({\geqslant}1\, \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle {\geqslant}2\, r.C\rangle_{\uparrow} \leqslant \min\{r, C\}}\big) \sqcap \boxed{({\geqslant}2\, r.C) \leqslant \min\{r(*,a), C(a)\}}\big)$$

That is, either there exist two anonymous witnesses for the value of ${\geqslant}2\, r.C$, or one anonymous witness and a serves as another witness. In general, the reduction needs to consider all possible (exponentially many) combinations of named and unnamed domain elements as witnesses for number restrictions; in this example there are only 2 cases. Dually, there can be at most one r-successor that exceeds the value given by ${\geqslant}2\, r.C$ at a, which is encoded in the assertion

$${\leqslant}1\, \mathfrak{r}.\big(\big(\mathsf{AN} \sqcap \boxed{\langle {\geqslant}2\, r.C\rangle_{\uparrow} < \min\{r, C\}}\big) \sqcup \big(\neg\mathsf{AN} \sqcap \boxed{({\geqslant}2\, r.C)(a) < \min\{r(a,*), C\}}\big)\big)(a).$$

All the axioms listed above are collected into a classical ontology $\mathsf{red}(\mathcal{O})$, and any classical model of this ontology obtained by a classical reasoner can be used to construct a G-model of $\mathcal{O}$. Hence, while this reduction incurs an exponential blowup in the size of the ontology, it enables us to use existing optimized reasoners to decide consistency of G-$\mathcal{SROIQ}$ ontologies.

3.2. The Tableau Algorithm

Our second algorithm explicitly creates a G-model of $\mathcal{O}$ by introducing new domain elements, called *nodes*. It uses an order structure that is similar to the one used for the reduction described above. The main difference is that the order structure now contains concept and role assertions of the form $B(x)$ and $r(x, y)$, where x and y are nodes. In this way, we can express the semantics directly using order assertions, e.g. $(\exists r.A)(x) \geqslant \min\{r(x, y), A(y)\}$ for all nodes x and y. However, the latter expression is not fully determined: that is, we do not know whether $(\exists r.A)(x) \geqslant r(x, y)$, or $(\exists r.A)(x) \geqslant A(y)$ holds. In our tableau algorithm, we resolve this nondeterminism by considering only *atomic* order assertions, i.e. without using the abbreviations min and $\Rightarrow$. In order to guarantee that these sets can be used to construct a G-model of $\mathcal{O}$, we need to ensure that they remain satisfiable.

In our example, the tableau algorithm is initialized with one node a representing the individual of the same name, and the order assertions from $\mathcal{A}$, where the at-most assertion is equivalent to an upper bound on the corresponding at-least-restriction: $({\geqslant}2\, r.C)(a) \leqslant 1 - p_C$. Afterwards, (nondeterministic) tableau rules are applied exhaustively to create new nodes and order assertions; we only present a few selected

[4]The negation $\neg$ will be handled by the involution inv of the order structure, and hence does not have to be explicitly represented in the reduction.

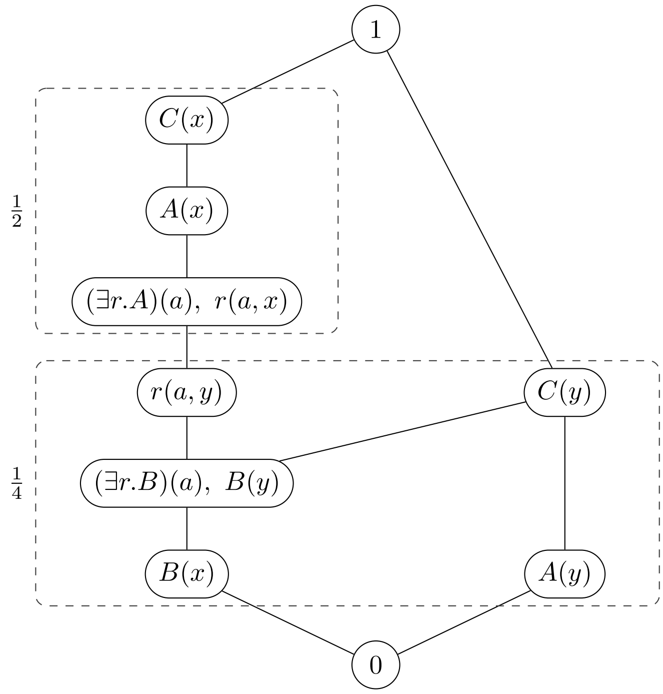

Figure 1: Order diagram of the preorder induced by the order assertions produced in the example by the first applications of tableau rules. The dashed boxes indicate a value assignment that satisfies $\mathcal{O}$ in the case that $p_A = \frac{1}{2}$ and $p_C = p_B = \frac{1}{4}$.

nondeterministic choices here. Similar to classical tableau algorithms, first we create two witnesses x and y for the existential restrictions $\exists r.A$ and $\exists r.B$, respectively, at a. For example, we need to ensure that $(\exists r.A)(a) = \min\{r(a,x), A(x)\}$ is satisfied. One possibility is to introduce the order assertions

$$(\exists r.A)(a) = r(a,x) \text{ and } (\exists r.A)(a) \leqslant A(x),$$

expressing that the above minimum is realized by the value of the role connection from a to x. Likewise, for y we assert that

$$(\exists r.B)(a) \leqslant r(a,y) \text{ and } (\exists r.B)(a) = B(y).$$

Moreover, the supremum-based semantics of existential restrictions imposes an upper bound on all other r-successors. Hence, we also assert that

$$(\exists r.B)(a) \geqslant B(x) \text{ and } (\exists r.A)(a) \geqslant r(a,y).$$

In the next step, the GCIs are applied to all nodes; we ignore a here since it is not relevant for this example. For the node x, we know already that

$$B(x) \leqslant (\exists r.B)(a) \leqslant r(a,y) \leqslant (\exists r.A)(a) \leqslant A(x),$$

and hence it suffices to assert in addition that $A(x) \leqslant C(x)$, which then implies that also $B(x) \leqslant C(x)$ holds. For y, we introduce the order assertions

$$A(y) \leqslant C(y) \text{ and } B(y) \leqslant C(y).$$

The resulting set of assertions entails the preorder depicted in Figure 1, where we ignore p_A, p_B, p_C, and all irrelevant elements of the order structure. Note that, although we consider as models only total preorders, the assertions themselves need not define a single total order over all elements of the order structure.

Now we deal with the number restriction $(\leqslant 1\, r.C)(a)$. In the following, we ignore the required witnesses, as they are not essential for the example. As in the classical tableau algorithm, we use a tableau rule that forces each r-successor of a to choose whether it wants to contribute to the number restriction or not. In the classical setting, this means choosing whether to satisfy C or not. An r-successor satisfying C contributes to a number restriction $\leqslant 1\, r.C$ in the sense that it reduces the total number of other r-successors that can also satisfy C (in this case to 0); in contrast, r-successors satisfying $\neg C$ do not contribute to the number restriction since they are irrelevant to its satisfaction. However, it is important that we know which of the nodes contribute to the number restriction, and which do not.

With the Gödel semantics, this corresponds to checking whether $\neg(\leqslant 1\, r.C)(a) < \min\{r(a,x), C(x)\}$ holds for a node x. If this inequality holds for at least 2 nodes, then the supremum in the semantics of the at-least restriction $(\geqslant 2\, r.C) = \neg(\leqslant 1\, r.C)$ is violated. We analyze several possibilities:

- If $(\geqslant 2\, r.C)(a) \geqslant r(a,x) = \min\{r(a,x), C(x)\}$ and $(\geqslant 2\, r.C)(a) < \min\{r(a,y), C(y)\}$, then

  $$r(a,y) \leqslant (\exists r.A)(a) = r(a,x) \leqslant (\geqslant 2\, r.C)(a) < \min\{r(a,y), C(y)\} \leqslant r(a,y).$$

  In this case, the resulting set of order assertions is not satisfiable anymore.

- If $(\geqslant 2\, r.C)(a) < r(a,x)$ and $(\geqslant 2\, r.C)(a) \geqslant \min\{r(a,y), C(y)\}$, then it depends on the values p_A, p_B, and p_C whether we can build a G-model. If $(\geqslant 2\, r.C)(a) \leqslant 1-p_C < p_B \leqslant (\exists r.B)(a) \leqslant (\geqslant 2\, r.C)(a)$, then this is obviously not possible. On the other hand, supposing that $p_A = \frac{1}{2}$ and $p_C = p_B = \frac{1}{4}$, we can construct a G-model by assigning the value $\frac{1}{4}$ to $A(y), B(x), B(y), r(a,y), C(y)$ and $\frac{1}{2}$ to $r(a,x), A(x), C(x)$. This means that $(\exists r.A)(a)$ evaluates to $\frac{1}{2}$, $(\exists r.B)(a)$ to $\frac{1}{4}$, and $(\leqslant 1\, r.C)(a)$ to $\frac{1}{4}$, and hence $\mathcal{O}$ is satisfied (see Figure 1).

- If $(\geqslant 2\, r.C)(a) < r(a,x)$ and $(\geqslant 2\, r.C)(a) < \min\{r(a,y), C(y)\}$, then the at-least restriction is violated. Thus, we have to apply another rule to *merge* the node y into x (or vice versa), which essentially amounts to discarding the node y and replacing all occurrences of y in the order assertions by x. Hence, almost all relevant elements of the order structure become equivalent, the only exception being $(\geqslant 2\, r.C)(a)$, which must be strictly smaller than all other elements. A possible resulting G-model could simply assign 1 to $A(x), B(x), C(x), r(a,x)$, which would result in $(\geqslant 2\, r.C)(a)$ being evaluated to 0. Again, all axioms of $\mathcal{O}$ are satisfied.

4. Weighted Automata Recognizing Complex Role Inclusions

Before we can finally describe the algorithms in detail, we first need to lift a method of dealing with complex role inclusions from classical $\mathcal{SROIQ}$ to the Gödel semantics. Let $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R}_h \cup \mathcal{R}_a)$ be a G-$\mathcal{SROIQ}$ ontology. We extend the idea from [18] of using finite automata to characterize all role chains that imply a given role w.r.t. $\mathcal{R}_h$. In our setting, we need to use a certain kind of weighted automata [25], which compute a weight for any given input word.

Definition 4.1 (WFA)**.** A *weighted finite automaton (WFA)* is a tuple $\mathbf{A} = (Q, \Sigma, q_{\mathsf{ini}}, \mathsf{wt}, q_{\mathsf{fin}})$, consisting of a non-empty set Q of *states*, a non-empty *input alphabet* Σ, an *initial state* $q_{\mathsf{ini}} \in Q$, a *transition weight function* $\mathsf{wt}\colon Q \times (\Sigma \cup \{\varepsilon\}) \times Q \to [0,1]$, and a *final state* $q_{\mathsf{fin}} \in Q$. Given an input word $w \in \Sigma^*$, a *run* of $\mathbf{A}$ on w is a non-empty sequence of pairs $\mathsf{r} = (w_i, q_i)_{0 \leqslant i \leqslant m}$ such that $(w_0, q_0) = (w, q_{\mathsf{ini}})$, $(w_m, q_m) = (\varepsilon, q_{\mathsf{fin}})$, and for each i, $1 \leqslant i \leqslant m$, it holds that $w_{i-1} = x_i w_i$ for some $x_i \in \Sigma \cup \{\varepsilon\}$. The *weight* of such a run is $\mathsf{wt}(\mathsf{r}) := \min_{i=1}^{m} \mathsf{wt}(q_{i-1}, x_i, q_i)$. The *behavior* of $\mathbf{A}$ on w is defined as $(\|\mathbf{A}\|, w) := \sup_{\mathsf{r} \text{ run of } \mathbf{A} \text{ on } w} \mathsf{wt}(\mathsf{r})$.

We abbreviate by $q \xrightarrow{x,p} q' \in \mathbf{A}$ the fact that $\mathsf{wt}(q, x, q') = p$. Further, for a state q of $\mathbf{A}$, we denote by $\mathbf{A}^q$ the automaton resulting from $\mathbf{A}$ by making q the initial state. The following connection is a direct consequence of the definition of the behavior of a WFA.

Proposition 4.2. *Let $\mathbf{A}$ be a WFA, $q \xrightarrow{x,p} q' \in \mathbf{A}$, and $w \in \Sigma^*$. Then we have*

$$(\|\mathbf{A}^q\|, xw) \geqslant \min\{p, (\|\mathbf{A}^{q'}\|, w)\}.$$

In the following, we consider WFA over the input alphabet $\mathsf{rol}(\mathcal{O})$. The *mirrored copy* $\mathbf{A}^-$ of the WFA $\mathbf{A}$ over $\mathsf{rol}(\mathcal{O})$ is constructed by exchanging initial and final states, and replacing each transition $q \xrightarrow{x,p} q'$ by $q' \xrightarrow{x^-,p} q$, where $\varepsilon^- := \varepsilon$.

Proposition 4.3. *Let $\mathbf{A}$ be a WFA, $\mathbf{A}'$ be a mirrored copy of $\mathbf{A}$, and $w \in \mathsf{rol}(\mathcal{O})^*$. Then we have*

$$(\|\mathbf{A}\|, w) = (\|\mathbf{A}'\|, w^-).$$

Following [18], we now construct, for each role r, a WFA $\mathbf{A}_r$ that recognizes all role chains that "imply" r w.r.t. $\mathcal{R}_h$ (with associated degrees). This construction proceeds in several steps. The first automaton, denoted $\mathbf{A}_r^0$, contains the initial state i_r, the final state f_r, and the transition $i_r \xrightarrow{r,1} f_r$, as well as the following states and transitions for each $w \sqsubseteq r \geqslant p \in \mathcal{R}$:

- if $w = rr$, then $f_r \xrightarrow{\varepsilon,p} i_r$;
- if $w = r_1 \dots r_n$ with $r_1 \neq r \neq r_n$, then $i_r \xrightarrow{r_1,1} q_w^1 \xrightarrow{r_2,1} \dots \xrightarrow{r_n,1} q_w^n \xrightarrow{\varepsilon,p} f_r$;
- if $w = rr_1 \dots r_n$, then $f_r \xrightarrow{r_1,1} q_w^1 \xrightarrow{r_2,1} \dots \xrightarrow{r_n,1} q_w^n \xrightarrow{\varepsilon,p} f_r$; and
- if $w = r_1 \dots r_n r$, then $i_r \xrightarrow{r_1,1} q_w^1 \xrightarrow{r_2,1} \dots \xrightarrow{r_n,1} q_w^n \xrightarrow{\varepsilon,p} i_r$,

where all states q_w^i are distinct. Here and in the following, all transitions that are not explicitly mentioned have weight 0.

The next WFA $\mathbf{A}_r^1$ is the same as $\mathbf{A}_r^0$ if there is no role inclusion of the form $r^- \sqsubseteq r \geqslant p \in \mathcal{R}$; otherwise, $\mathbf{A}_r^1$ is the disjoint union of $\mathbf{A}_r^0$ and a mirrored copy of $\mathbf{A}_r^0$, where i_r is the only initial state, f_r is the only final state, and the following transitions are added for the copy f_r' of f_r and the copy i_r' of i_r: $i_r \xrightarrow{\varepsilon,p} f_r'$, $f_r' \xrightarrow{\varepsilon,p} i_r$, $f_r \xrightarrow{\varepsilon,p} i_r'$, and $i_r' \xrightarrow{\varepsilon,p} f_r$.

Finally, we define the WFA $\mathbf{A}_r$ by induction on the regular order $\lessdot$ as follows:

- If r is minimal w.r.t. $\lessdot$, then $\mathbf{A}_r := \mathbf{A}_r^1$.
- Otherwise, $\mathbf{A}_r$ is the disjoint union of $\mathbf{A}_r^1$ with a copy $\mathbf{A}_s^{1'}$ of $\mathbf{A}_s^1$ for each transition $q \xrightarrow{s,1} q'$ in $\mathbf{A}_r^1$ with $s \neq r$, and we add ε-transitions with weight 1 from q to the initial state of $\mathbf{A}_s^{1'}$ and from the final state of $\mathbf{A}_s^{1'}$ to q'.[5]
- The automaton $\mathbf{A}_{r^-}$ is a mirrored copy of $\mathbf{A}_r$.

The difference to the construction in [18] is only the inclusion of the appropriate weights for each considered role inclusion. As shown in [18], the size of each $\mathbf{A}_r$ is bounded exponentially in the length of the longest chain $r_1 \lessdot \dots \lessdot r_n$ for which there are role inclusions $u_i r_{i-1} v_i \sqsubseteq r_i \geqslant p_i \in \mathcal{R}_h$ for all i, $2 \leqslant i \leqslant n$.

We now present the promised characterization of the role inclusions in $\mathcal{R}_h$ in terms of the behavior of the automata $\mathbf{A}_r$. Intuitively, the degree to which the interpretation of w must be included in the interpretation of r is determined by the behavior of $\|\mathbf{A}_r\|$ on w.

Lemma 4.4. *A G-interpretation $\mathcal{I}$ satisfies $\mathcal{R}_h$ iff for every $r \in \mathsf{rol}(\mathcal{O})$, every $w \in \mathsf{rol}(\mathcal{O})^+$, and all $d, e \in \Delta^\mathcal{I}$, it holds that*

$$w^\mathcal{I}(d,e) \Rightarrow r^\mathcal{I}(d,e) \geqslant (\|\mathbf{A}_r\|, w).$$

For the final observation of this section, we define the relation $\sqsubseteq_p$ as the "transitive closure" of the simple role inclusions in $\mathcal{R}_h$: we set $r \sqsubseteq_p s$ iff p is the supremum of the values $\min\{p_1, \dots, p_n\}$ over all sequences $r \sqsubseteq r_1 \geqslant p_1, \dots, r_{n-1} \sqsubseteq s \geqslant p_n$ in $\mathcal{R}_h$. Note that $r \sqsubseteq_1 r$ holds because of the empty sequence.

[5] Note that all transitions labeled with roles have weight 0 or 1, and the only roles occurring in $\mathbf{A}_r^1$ (except r itself) are smaller than r w.r.t. $\lessdot$.

Proposition 4.5. *For a simple role r and $w \in \mathsf{rol}(\mathcal{O})^*$, we have*

$$(\|\mathbf{A}_r\|, w) = \begin{cases} p & \text{if } w = s \in \mathsf{rol}(\mathcal{O}) \text{ and } s \sqsubseteq_p r, \\ 0 & \text{otherwise.} \end{cases}$$

In the following algorithms, we use a new kind concepts of the form $\exists \mathbf{A}_r^q.C$ and $\forall \mathbf{A}_r^q.C$ that allow us to ignore the complex role inclusions for checking the satisfaction of $\exists r.C$ and $\forall r.C$, but instead we have to follow the transitions of $\mathbf{A}_r$. For any WFA $\mathbf{A}$, the semantics of $\forall \mathbf{A}.C$ is defined in analogy to ordinary value restrictions as follows (see also Lemma 4.4):

$$(\forall \mathbf{A}.C)^{\mathcal{I}}(d) := \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e),$$

where $\varepsilon^{\mathcal{I}}(d,e) := 1$ if $d = e$, and $\varepsilon^{\mathcal{I}}(d,e) := 0$ otherwise. For $\exists \mathbf{A}.C$, we similarly define

$$(\exists \mathbf{A}.C)^{\mathcal{I}}(d) := \sup_{w \in \mathsf{rol}(\mathcal{O})^*} \sup_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}\|, w), w^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\}.$$

5. A Reduction to Classical $\mathcal{SROIQ}$

We now describe a method for reasoning in Gödel FDLs based on transforming a given fuzzy ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R}_h \cup \mathcal{R}_a)$ into a classical ontology $\mathsf{red}(\mathcal{O})$. As mentioned before, it is only applicable to G-$\mathcal{SRIQ}$, G-$\mathcal{SROQ}$, and G-$\mathcal{SROI}$. This reduction always uses nominals, even if they are not present in the ontology $\mathcal{O}$, but it does not require complex role constructors or axioms. Hence, $\mathsf{red}(\mathcal{O})$ will be formulated in $\mathcal{ALCOQ}$. However, if number restrictions are not present in $\mathcal{O}$, then $\mathsf{red}(\mathcal{O})$ is an $\mathcal{ALCO}$ ontology. The main case distinctions we need to make in the following description are whether the following (groups of) constructors or axioms are present in the input ontology or not:

- inverse roles,
- number restrictions,
- nominals,
- disjointness axioms or local reflexivity.

Parts of the reduction that are contingent on the presence of these constructors are labeled with (I), (Q), (O), and (R), respectively. Additionally, we require that any roles occurring in a concept (axiom) labeled by (R) are *simple*; otherwise, the whole concept (axiom) is omitted. If no inverse roles occur, then we modify the set $\mathsf{rol}(\mathcal{O})$ to also not contain any inverse roles.

As a first pre-processing step, we eliminate role assertions $r(a,b)$ from the ABox by replacing them with equivalent concept assertions using nominals: $(\exists r.\{b\})(a)$. We denote by $\mathsf{val}(\mathcal{O})$ the smallest set containing the following values:

- the constants 0, 0.5, and 1;
- all elements of $[0,1]$ appearing in $\mathcal{O}$, either in axioms or as truth constants; and
- for each $p \in \mathsf{val}(\mathcal{O})$, its involutive negation $1 - p$.

The size of this set is linear in the size of $\mathcal{O}$.

We further define the set $\mathsf{sub}(C)$ of (extended) subconcepts of a concept C, which contains at least C and $\neg C$, as well as the following concepts, which are defined recursively:

$$\begin{cases} \mathsf{sub}(D) & \text{if } C = \neg D \text{ or } C = \geqslant n\, r.D, \\ \mathsf{sub}(D) \cup \mathsf{sub}(E) & \text{if } C \in \{D \sqcap E, D \to E\}, \\ \mathsf{sub}(D) \cup \{\forall \mathbf{A}_r^q.D, \neg\forall \mathbf{A}_r^q.D \mid q \text{ is a state of } \mathbf{A}_r\} & \text{if } C = \forall r.D, \\ \mathsf{sub}(D) \cup \{\exists \mathbf{A}_r^q.D, \neg\exists \mathbf{A}_r^q.D \mid q \text{ is a state of } \mathbf{A}_r\} & \text{if } C = \exists r.D, \\ \emptyset & \text{otherwise.} \end{cases}$$

To simplify the descriptions, we do not distinguish between $\neg\neg C$ and C. The set $\mathsf{sub}(\mathcal{O})$ of all relevant subconcepts of $\mathcal{O}$ is defined as

$$\begin{aligned}\mathsf{sub}(\mathcal{O}) := \bigcup\{&\mathsf{sub}(C),\ \mathsf{sub}(D) \mid C \sqsubseteq D \geqslant p,\ C(a) \bowtie D(b),\ \text{or } C(a) \bowtie p \text{ occurs in } \mathcal{O}\} \cup \\ &\{\{a\} \mid a \in \mathsf{ind}(\mathcal{O})\} \cup && (\mathsf{O}) \\ &\{\exists r.\mathsf{Self} \mid r \in \mathsf{rol}(\mathcal{O}),\ r \text{ is simple}\} && (\mathsf{R})\end{aligned}$$

The size of $\mathsf{sub}(\mathcal{O})$ is exponential in the size of the role hierarchy (due to the use of the automata $\mathbf{A}_r$ in the definition); since we eliminate complex role inclusions in the reduction, this blowup cannot be avoided in general [26]. However, if all roles are simple, then the size of these automata is polynomial in the size of $\mathcal{R}_h$.

In our reduction, we do not explicitly represent all role connections, but only a "skeleton" of connections that are necessary to satisfy the witnessing conditions for role restrictions. The restrictions for all implied role connections are then handled by the concepts $\forall\mathbf{A}_r.C$ and $\exists\mathbf{A}_r.C$ by simulating the transitions of $\mathbf{A}_r$; each transition corresponds to a role connection to a new domain element. We do not need to introduce concepts of the form $\geqslant n\, \mathbf{A}_r.C$ since all roles in at-least restrictions must be simple, i.e. there can be no role chains of length > 1 that imply them (at least not with a degree that is strictly greater than 0).

A restriction of our reduction is that we consider only *(quasi-)forest-shaped* models of $\mathcal{O}$ [16]. In such a model, the domain elements identified by individual names serve as the roots of several tree-shaped structures. The roots themselves may be arbitrarily interconnected by roles. Due to nominals, there may also be role connections from any domain element back to the roots. Although complex role inclusions can imply role connections between arbitrary domain elements, the underlying tree-shaped "skeleton" is what is important for reasoning (for details, see [16] and our correctness proof in the appendix). This dependence on forest-shaped models is the reason why our reduction works only for G-$\mathcal{SROI}$, G-$\mathcal{SROQ}$, and G-$\mathcal{SRIQ}$. Notice that even classical $\mathcal{ALCOIQ}$ does not have the forest model property [27].

We now formally define the order structure $\mathcal{U}$ introduced in the example of Section 3.1:

$$\begin{aligned}\mathcal{U}_{\mathcal{A}} &:= \mathsf{val}(\mathcal{O}) \cup \{C(a) \mid a \in \mathsf{ind}(\mathcal{O}),\ C \in \mathsf{sub}(\mathcal{O})\} \cup \{s(a,b) \mid a,b \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O}),\ s \in \{r, \neg r\}\} \\ \mathcal{U} &:= \mathcal{U}_{\mathcal{A}} \cup \mathsf{sub}(\mathcal{O}) \cup \mathsf{sub}_{\uparrow}(\mathcal{O}) \cup \{s,\ s(a,*),\ s(*,a) \mid a \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O}),\ s \in \{r, \neg r\}\},\end{aligned}$$

where $\mathsf{sub}_{\uparrow}(\mathcal{O}) := \{\langle C\rangle_{\uparrow} \mid C \in \mathsf{sub}(\mathcal{O})\}$ and the function inv is defined using negation; that is, $\mathsf{inv}(C) := \neg C$, $\mathsf{inv}(C(a)) := (\neg C)(a)$, $\mathsf{inv}(r(a,*)) := (\neg r)(a,*)$, etc.

Total preorders on assertions in $\mathcal{U}_{\mathcal{A}}$ are used to describe the behavior of the named root elements in the forest-shaped model. For each domain element of $\mathcal{I}$, total preorders on the elements of $\mathsf{sub}(\mathcal{O})$ describe the degrees of all relevant concepts in a similar way. The elements of $\mathsf{sub}_{\uparrow}(\mathcal{O})$ are used to refer back to degrees of concepts at the unique predecessor element in the tree-shaped parts of the interpretation. For convenience, we also define $\langle\alpha\rangle_{\uparrow} := \alpha$ for all $\alpha \in \mathcal{U}_{\mathcal{A}}$ since the elements of $\mathcal{U}_{\mathcal{A}}$ are *global*, i.e. their values do not depend on the current domain element. The elements $r \in \mathsf{rol}(\mathcal{O})$ represent the values of the role connections from the predecessor. The special assertions $r(*,a)$ and $r(a,*)$ are used to describe role connections between the current domain element (represented by $*$) and the named elements in the roots.

5.1. The Reduction

In order to describe total preorders over $\mathcal{U}$ in a classical $\mathcal{ALCOQ}$ ontology, we use special concept names of the form $\boxed{\alpha \leqslant \beta}$ for $\alpha, \beta \in \mathcal{U}$, which we call *order concepts*. This differs from previous reductions for finitely valued FDLs [11, 12, 28] in that we not only consider *cut concepts* like $\boxed{q \leqslant \alpha}$ with $q \in \mathsf{val}(\mathcal{O})$, but also relationships between different concepts.[6] We can express arbitrary order assertions over $\mathcal{U}$ through the abbreviations $\boxed{\alpha \geqslant \beta} := \boxed{\beta \leqslant \alpha}$, $\boxed{\alpha < \beta} := \neg\boxed{\alpha \geqslant \beta}$, $\boxed{\alpha > \beta} := \neg\boxed{\alpha \leqslant \beta}$, and $\boxed{\alpha = \beta} := \boxed{\alpha \leqslant \beta} \sqcap \boxed{\alpha \geqslant \beta}$. For brevity, we further extend this notation as follows:

- $\boxed{\alpha \geqslant \min\{\beta,\gamma\}} := \boxed{\alpha \geqslant \beta} \sqcup \boxed{\alpha \geqslant \gamma}$,

[6] For the rest of this paper, the expressions $\boxed{\alpha \leqslant \beta}$ always denote DL concept names.

- $\boxed{\alpha \leqslant \min\{\beta,\gamma\}} := \boxed{\alpha \leqslant \beta} \sqcap \boxed{\alpha \leqslant \gamma}$,
- $\boxed{\alpha \geqslant \beta \Rightarrow \gamma} := (\boxed{\beta \leqslant \gamma} \to \boxed{\alpha \geqslant 1}) \sqcap (\boxed{\beta > \gamma} \to \boxed{\alpha \geqslant \gamma})$,
- $\boxed{\alpha \leqslant \beta \Rightarrow \gamma} := \boxed{\beta \leqslant \gamma} \sqcup \boxed{\alpha \leqslant \gamma}$,

and analogously define $\boxed{\alpha \bowtie \beta \Rightarrow \gamma}$ and $\boxed{\alpha \bowtie \min\{\beta,\gamma\}}$, with $\bowtie \in \{<,=,>\}$. This can be straightforwardly extended to even more complex expressions using $\Rightarrow$ and min.

In our reduction, we additionally use the special concept name AN to identify the *anonymous* domain elements, i.e. those which are not of the form $b^{\mathcal{I}}$ for any $b \in \mathsf{ind}(\mathcal{O})$. The reduction uses only one role name $\mathfrak{r}$. The reduced ontology $\mathsf{red}(\mathcal{O})$ is divided into several parts, called $\mathsf{red}(\mathcal{U})$, $\mathsf{red}(\mathcal{A})$, $\mathsf{red}(\mathsf{AN})$, $\mathsf{red}(\uparrow)$, $\mathsf{red}(\mathcal{R})$, $\mathsf{red}(\mathcal{T})$, and $\mathsf{red}(C)$ for all $C \in \mathsf{sub}(\mathcal{O})$, which we describe in the following. Before giving the full details, we want to emphasize that $\mathsf{red}(\mathcal{O})$ is formulated in $\mathcal{ALCOQ}$, whenever $\mathcal{O}$ is in G-$\mathcal{SRIQ}$ or G-$\mathcal{SROQ}$, and in $\mathcal{ALCO}$ if $\mathcal{O}$ is a G-$\mathcal{SROI}$ ontology. This is due to the fact that we always use nominals to distinguish the named from the anonymous part of the forest-shaped model, and the inverse of the (unique) role $\mathfrak{r}$ is not needed in the reduction.

The first part of $\mathsf{red}(\mathcal{O})$ is

$$\begin{aligned}\mathsf{red}(\mathcal{U}) := {} & \{\boxed{\alpha \leqslant \beta} \sqcap \boxed{\beta \leqslant \gamma} \sqsubseteq \boxed{\alpha \leqslant \gamma} \mid \alpha,\beta,\gamma \in \mathcal{U}\} \cup \\ & \{\top \sqsubseteq \boxed{\alpha \leqslant \beta} \sqcup \boxed{\beta \leqslant \alpha} \mid \alpha,\beta \in \mathcal{U}\} \cup \\ & \{\top \sqsubseteq \boxed{0 \leqslant \alpha} \sqcap \boxed{\alpha \leqslant 1} \mid \alpha \in \mathcal{U}\} \cup \\ & \{\top \sqsubseteq \boxed{\alpha \lhd \beta} \mid \alpha,\beta \in \mathsf{val}(\mathcal{O}),\ \alpha \lhd \beta,\ \lhd \in \{<,\leqslant\}\} \cup \\ & \{\boxed{\alpha \leqslant \beta} \sqsubseteq \boxed{\mathsf{inv}(\beta) \leqslant \mathsf{inv}(\alpha)} \mid \alpha,\beta \in \mathcal{U}\}.\end{aligned}$$

In this order, these axioms ensure that the relation formalized by "$\leqslant$" is transitive on all elements of $\mathcal{U}$, it is total on $\mathcal{U}$, it has 0 and 1 as least and greatest elements, respectively, it reflects the natural order on $\mathsf{val}(\mathcal{O})$, and inv is antitone w.r.t. $\leqslant$. In short, $\leqslant$ represents an element of $\mathsf{order}(\mathcal{U})$.

To describe the behavior of all named elements, we use the following axioms:

$$\begin{aligned}\mathsf{red}(\mathcal{A}) := {} & \{\boxed{\alpha \bowtie \beta}(c) \mid \alpha \bowtie \beta \in \mathcal{A}\} \cup \\ & \{a \approx b \in \mathcal{A}\} \cup \{a \not\approx b \in \mathcal{A}\} \cup \\ & \{\mathfrak{r}(a,b) \mid a,b \in \mathsf{ind}(\mathcal{O})\} \cup \\ & \{\boxed{\alpha \lhd \beta} \sqsubseteq \forall \mathfrak{r}.\boxed{\alpha \lhd \beta} \mid \alpha,\beta \in \mathcal{U}_{\mathcal{A}},\ \lhd \in \{<,\leqslant\}\} \cup \\ & \{\boxed{C(a) = C}(a) \mid a \in \mathsf{ind}(\mathcal{O}),\ C \in \mathsf{sub}(\mathcal{O})\} \cup \\ & \{\boxed{r(a,b) = r(a,*)}(b) \mid a,b \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O})\} \cup \\ & \{\boxed{r(a,b) = r(*,b)}(a) \mid a,b \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O})\} \cup \\ & \{\top \sqsubseteq \boxed{r(a,a) = (\exists r.\mathsf{Self})(a)} \mid a \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O})\} \cup && \text{(R)} \\ & \{\top \sqsubseteq \boxed{r(a,b) = r^-(b,a)} \mid a,b \in \mathsf{ind}(\mathcal{O}) \cup \{*\},\ r \in \mathsf{rol}(\mathcal{O})\}, && \text{(I)}\end{aligned}$$

where c is an arbitrary individual name. The first four lines are responsible for enforcing that the ABox is satisfied and that information about the behavior of the named individuals is available throughout the whole model. The remaining axioms describe various equivalences for named individuals, e.g. that $r(a,b)$ and $r(*,b)$ should have the same value when evaluated at a.

Example 5.1. The fuzzy assertion $(\exists r.A)(a) \geqslant p_A$ of Section 3 is encoded by first asserting that it holds at the individual designated by c (see the first line of $\mathsf{red}(\mathcal{A})$):

$$\boxed{(\exists r.A)(a) \geqslant p_A}(c).$$

Since all individual names are connected by $\mathfrak{r}$ (third line), this statement is transferred to a by the axioms in the fourth line; note that both $(\exists r.A)(a)$ and p_A are elements of $\mathcal{U}_{\mathcal{A}}$. Hence, $\mathsf{red}(\mathcal{A})$ enforces that

$\boxed{(\exists r.A)(a) \geqslant p_A}(a)$ holds as well. Since we have $\boxed{(\exists r.A)(a) = \exists r.A}(a)$ by the fifth line, together with $\mathsf{red}(\mathcal{U})$ we arrive at $\boxed{\exists r.A \geqslant p_A}(a)$, which intuitively corresponds directly to the original assertion.

The axioms of $\mathsf{red}(\mathcal{A})$ ensure that this information about the order relationship between $\exists r.A$ and p_A at a is also available to all other domain elements connected via $\mathfrak{r}$. This is important if these domain elements refer back to a via the nominal $\{a\}$.

The next axiom defines the concept AN of all *anonymous* elements:

$$\mathsf{red}(\mathsf{AN}) := \Big\{\mathsf{AN} \equiv \neg \bigsqcup_{a \in \mathsf{ind}(\mathcal{O})} \{a\}\Big\}.$$

In other words, AN denotes the complement of the set of all named individuals.

Next, we need to ensure that the order of a node in a tree-shaped part of the model is known at each of its successors via the elements of $\mathsf{sub}_\uparrow(\mathcal{O})$. This is guaranteed through the axioms

$$\mathsf{red}(\uparrow) := \{\boxed{\alpha \lhd \beta} \sqsubseteq \forall \mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle\alpha\rangle_\uparrow \lhd \langle\beta\rangle_\uparrow}\big) \mid \alpha, \beta \in \mathcal{U}_\mathcal{A} \cup \mathsf{sub}(\mathcal{O}),\ \lhd \in \{<, \leqslant\}\}.$$

We now come to the reduction of the RBox:

$$\mathsf{red}(\mathcal{R}) := \bigcup_{\substack{r \sqsubseteq s \geqslant p \in \mathcal{R}_h \\ r,s \text{ simple}}} \mathsf{red}(r \sqsubseteq s \geqslant p) \cup \bigcup_{\mathsf{dis}(r,s) \geqslant p \in \mathcal{R}_a} \mathsf{red}(\mathsf{dis}(r,s) \geqslant p),$$

where

$$\begin{aligned}
\mathsf{red}(r \sqsubseteq s \geqslant p) := {} & \{\top \sqsubseteq \boxed{r(a,b) \Rightarrow s(a,b) \geqslant p} \mid a, b \in \mathsf{ind}(\mathcal{O}) \cup \{*\}\} \cup \\
& \{\top \sqsubseteq \boxed{r \Rightarrow s \geqslant p}, \\
& \ \ \top \sqsubseteq \boxed{r^- \Rightarrow s^- \geqslant p}, && \text{(I)} \\
& \ \ \top \sqsubseteq \boxed{\exists r.\mathsf{Self} \Rightarrow \exists s.\mathsf{Self} \geqslant p}\} && \text{(R)} \\
\mathsf{red}(\mathsf{dis}(r,s) \geqslant p) := {} & \{\top \sqsubseteq \boxed{\min\{r(a,b), s(a,b)\} \leqslant 1-p} \mid a, b \in \mathsf{ind}(\mathcal{O}) \cup \{*\}\} \cup \\
& \{\top \sqsubseteq \boxed{\min\{r,s\} \leqslant 1-p}, \\
& \ \ \top \sqsubseteq \boxed{\min\{r^-, s^-\} \leqslant 1-p}, && \text{(I)} \\
& \ \ \top \sqsubseteq \boxed{\min\{\exists r.\mathsf{Self}, \exists s.\mathsf{Self}\} \leqslant 1-p}\} && \text{(R)}
\end{aligned}$$

These axioms ensure that the various elements of $\mathcal{U}$ that represent the values of role connections, such as $r(a,b)$, $\exists r.\mathsf{Self}$, and r, respect the axioms in $\mathcal{R}$. Although the simple role inclusions $r \sqsubseteq s \geqslant p$ are already expressed in the automata $\mathbf{A}_r$, we include them also in $\mathsf{red}(\mathcal{R})$. The reason for this is that the reduction of at-least restrictions below does not use these automata since only simple roles can occur in them.

The GCIs in $\mathcal{T}$ are translated in a straightforward manner, ensuring that all domain elements satisfy the necessary order relation between the concepts:

$$\mathsf{red}(\mathcal{T}) := \{\top \sqsubseteq \boxed{C \Rightarrow D \geqslant p} \mid C \sqsubseteq D \geqslant p \in \mathcal{T}\}$$

We now describe the reductions of the concepts. Intuitively, the axioms in $\mathsf{red}(C)$ describe the semantics of C in terms of its order relationships to other elements of $\mathcal{U}$. Note that the semantics of the involutive negation $\neg C = \mathsf{inv}(C)$ is already handled by the operator inv (see the definition of $\mathsf{red}(\mathcal{U})$ above):

$$\begin{aligned}
\mathsf{red}(\{a\}) &:= \{\{a\} \sqsubseteq \boxed{1 \leqslant \{a\}},\ \neg\{a\} \sqsubseteq \boxed{\{a\} \leqslant 0}\} && \text{(O)} \\
\mathsf{red}(\overline{p}) &:= \{\top \sqsubseteq \boxed{\overline{p} = p}\} \\
\mathsf{red}(\exists r.\mathsf{Self}) &:= \{\top \sqsubseteq \boxed{\exists r.\mathsf{Self} = \exists r^-.\mathsf{Self}}\} && \text{(I,R)} \\
\mathsf{red}(\neg C) &:= \emptyset \\
\mathsf{red}(C \sqcap D) &:= \{\top \sqsubseteq \boxed{C \sqcap D = \min\{C, D\}}\} \\
\mathsf{red}(C \to D) &:= \{\top \sqsubseteq \boxed{C \to D = C \Rightarrow D}\}
\end{aligned}$$

The constructions needed for handling role restrictions are more involved. In particular, in the case of value and existential restrictions we have to deal with non-simple roles, for which we employ the automata $\mathbf{A}_r$ from Section 4.

$$
\begin{aligned}
\mathsf{red}(\exists r.C) := \{\top \sqsubseteq{} & \boxed{(\exists r.C) \geqslant (\exists \mathbf{A}_r.C)}, \\
& \mathsf{AN} \sqsubseteq \exists \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \exists r.C\rangle_{\uparrow} \leqslant \min\{r, C\}}\big) \sqcup \\
& \qquad \boxed{(\exists r.C) \leqslant \min\{r^-, \langle C\rangle_{\uparrow}\}} \sqcup && \text{(I)} \\
& \qquad \boxed{(\exists r.C) \leqslant \min\{\exists r.\mathsf{Self}, C\}} \sqcup && \text{(R)} \\
& \qquad \bigsqcup_{a \in \mathsf{ind}(\mathcal{O})} \big(\exists \mathfrak{r}.\{a\} \sqcap \boxed{(\exists r.C) \leqslant \min\{r(*, a), C(a)\}}\big)\} \cup && \text{(O)} \\
& \{\exists \mathfrak{r}.\big(\big(\mathsf{AN} \sqcap \boxed{\langle \exists r.C\rangle_{\uparrow} \leqslant \min\{r, C\}}\big) \sqcup \big(\neg \mathsf{AN} \sqcap \boxed{(\exists r.C)(a) \leqslant \min\{r(a, *), C\}}\big)\big)(a) \mid a \in \mathsf{ind}(\mathcal{O})\}
\end{aligned}
$$

The second axiom of $\mathsf{red}(\exists r.C)$ ensures the existence of a witness for $\exists r.C$ at each anonymous domain element. This roughly corresponds to axiom (1) from Section 3, where we did not need the parts indicated by (I) and (O); a slight difference is that we used this axiom for the named domain element a, and hence the precondition AN was missing.

Example 5.2. Assume for instance that the preorder represented by the order concepts at some anonymous domain element d satisfies $0 < \exists r.C < 0.5$. The first possibility is that there is an anonymous element e that is connected to d via $\mathfrak{r}$, and hence by $\mathsf{red}(\uparrow)$ we know that e satisfies $0 < \langle \exists r.C\rangle_{\uparrow} < 0.5$. The axiom further requires that $\langle \exists r.C\rangle_{\uparrow} \leqslant \min\{r, C\}$, which implies that $0 < \langle \exists r.C\rangle_{\uparrow} \leqslant C$ and $0 < \langle \exists r.C\rangle_{\uparrow} \leqslant r$. We will see below that the reduction of $\exists \mathbf{A}_r.C$ further ensures that $\langle \exists r.C\rangle_{\uparrow} \geqslant \min\{r, C\}$, and thus we obtain $\langle \exists r.C\rangle_{\uparrow} = \min\{r, C\}$. Hence, e can be seen as an abstract representation of the witness of $\exists r.C$ at d; the precise values of C at e and of the r-connection between d and e (represented by the element r) is irrelevant, as long as their minimum is equal to the value of $\exists r.C$ at d.

The other disjuncts of this axiom deal with the possibilities that (i) d itself, (ii) its predecessor, or (iii) a named domain element acts as the witness for the existential restriction in a similar way. The assertions in the last line of $\mathsf{red}(\exists r.C)$ deal with the case of a named domain element, in which case the options (i) and (ii) are already covered by (iii).

Together with the first axiom of $\mathsf{red}(\exists r.C)$, the following axioms ensure that no other r-successor of d violates the upper bound on $\min\{r, C\}$ given by $\exists r.C$ at d:

$$
\begin{aligned}
\mathsf{red}(\exists \mathbf{A}^q.C) &:= \{\top \sqsubseteq \boxed{(\exists \mathbf{A}^q.C) \geqslant C} \mid q \text{ is final}\} \cup \bigcup_{q \xrightarrow{x,p} q' \in \mathbf{A}} \mathsf{red}_{x,p,q'}(\exists \mathbf{A}^q.C) \\
\mathsf{red}_{\varepsilon,p,q'}(\exists \mathbf{A}^q.C) &:= \{\top \sqsubseteq \boxed{(\exists \mathbf{A}^q.C) \geqslant \min\{p, \exists \mathbf{A}^{q'}.C\}}\} \\
\mathsf{red}_{s,p,q'}(\exists \mathbf{A}^q.C) &:= \{\top \sqsubseteq \forall \mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle \exists \mathbf{A}^q.C\rangle_{\uparrow} \geqslant \min\{p, s, \exists \mathbf{A}^{q'}.C\}}\big), \\
& \qquad \mathsf{AN} \sqsubseteq \boxed{(\exists \mathbf{A}^q.C) \geqslant \min\{p, s^-, \langle \exists \mathbf{A}^{q'}.C\rangle_{\uparrow}\}}, && \text{(I)} \\
& \qquad \top \sqsubseteq \boxed{(\exists \mathbf{A}^q.C) \geqslant \min\{p, \exists s.\mathsf{Self}, \exists \mathbf{A}^{q'}.C\}}\} \cup && \text{(R)} \\
& \quad \{\exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\exists \mathbf{A}^q.C) \geqslant \min\{p, s(*, a), (\exists \mathbf{A}^{q'}.C)(a)\}}, \\
& \qquad \exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\exists \mathbf{A}^q.C)(a) \geqslant \min\{p, s^-(*, a), \exists \mathbf{A}^{q'}.C\}} \mid a \in \mathsf{ind}(\mathcal{O})\} && \text{(I,O)}
\end{aligned}
$$

Example 5.3. The connection between this construction and the axioms (2) in Section 3 is less obvious, because there we did not consider automata to simulate role inclusions. Since the example ontology has no role inclusions, the automaton $\mathbf{A}_r$ only contains the initial state i_r, the final state f_r, and the transition

$i_r \xrightarrow{r,1} f_r$. Hence, we have $\mathbf{A}_r = \mathbf{A}_r^{i_r}$, and we obtain

$$\begin{aligned}
\mathsf{red}(\exists \mathbf{A}_r^{i_r}.A) &= \mathsf{red}_{r,1,f_r}(\exists \mathbf{A}_r^{i_r}.A) \\
&= \{\top \sqsubseteq \forall \mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle \exists \mathbf{A}_r^{i_r}.A\rangle_\uparrow \geqslant \min\{1, r, \exists \mathbf{A}_r^{f_r}.A\}}\big), \\
&\quad\ \exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\exists \mathbf{A}_r^{i_r}.A) \geqslant \min\{1, r(*,a), (\exists \mathbf{A}_r^{f_r}.A)(a)\}}\}, \\
\mathsf{red}(\exists \mathbf{A}_r^{f_r}.A) &= \{\top \sqsubseteq \boxed{(\exists \mathbf{A}_r^{f_r}.A) \geqslant A}\}, \text{ and} \\
\mathsf{red}(\exists r.A) &\supseteq \{\top \sqsubseteq \boxed{(\exists r.A) \geqslant (\exists \mathbf{A}_r^{i_r}.A)}\}.
\end{aligned}$$

Now $\mathsf{red}(\mathcal{U})$, $\mathsf{red}(\mathcal{A})$, and $\mathsf{red}(\uparrow)$ tell us that in the first two axioms we can essentially replace $\langle \exists \mathbf{A}_r^{i_r}.A\rangle_\uparrow$ by $\langle \exists r.A\rangle_\uparrow$, $\exists \mathbf{A}_r^{f_r}.A$ by A, $\exists \mathbf{A}_r^{i_r}.A$ by $\exists r.A$, and $(\exists \mathbf{A}_r^{f_r}.A)(a)$ by $A(a)$, which yields the two axioms of (2). Again, the other axioms in $\mathsf{red}_{r,1,f_r}(\exists \mathbf{A}_r^{i_r}.A)$ deal with the other possible kinds of successors (inverse and self), which were not relevant for Section 3.

To illustrate the treatment of complex role inclusions, not shown in the previous example, we return to Example 5.2.

Example 5.4. Recall Example 5.2. If $r^\mathcal{I}(d, e_1) = 0.6$ and $r^\mathcal{I}(e_1, e_2) = 0.5$ for two other (anonymous) domain elements e_1, e_2, and we further have the role inclusion $rr \sqsubseteq r \geqslant 0.7$, then we know that $r^\mathcal{I}(d, e_2)$ is at least 0.5. Although this r-connection is not explicitly represented in the forest structure, concepts of the form $\exists \mathbf{A}_r^q.C$ are appropriately transferred from d via e_1 to e_2 in order to ensure that the value of C at e_2 satisfies $0.5 > (\exists r.C)^\mathcal{I}(d) \geqslant \min\{r^\mathcal{I}(d, e_2), C^\mathcal{I}(e_2)\}$. In this example, since we know only that $r^\mathcal{I}(d, e_2) \geqslant 0.5$, it must be the case that $C^\mathcal{I}(e_2) \leqslant (\exists r.C)^\mathcal{I}(d) < 0.5$.

The reduction for value restrictions is dual to the one for existential restrictions; intuitively, it is obtained from $\mathsf{red}(\exists r.C)$ by exchanging $\geqslant$ with $\leqslant$ and min with $\Rightarrow$:

$$\begin{aligned}
\mathsf{red}(\forall r.C) := \{&\top \sqsubseteq \boxed{(\forall r.C) \leqslant (\forall \mathbf{A}_r.C)}, \\
&\mathsf{AN} \sqsubseteq \exists \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}\big) \sqcup \\
&\quad \boxed{(\forall r.C) \geqslant r^- \Rightarrow \langle C\rangle_\uparrow} \sqcup && \text{(I)} \\
&\quad \boxed{(\forall r.C) \geqslant (\exists r.\mathsf{Self}) \Rightarrow C} \sqcup && \text{(R)} \\
&\quad \bigsqcup_{a \in \mathsf{ind}(\mathcal{O})} \big(\exists \mathfrak{r}.\{a\} \sqcap \boxed{(\forall r.C) \geqslant r(*,a) \Rightarrow C(a)}\big)\} \cup && \text{(O)} \\
&\{\exists \mathfrak{r}.\big(\big(\mathsf{AN} \sqcap \boxed{\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}\big) \sqcup \big(\neg \mathsf{AN} \sqcap \boxed{(\forall r.C)(a) \geqslant r(a,*) \Rightarrow C}\big)\big)(a) \mid a \in \mathsf{ind}(\mathcal{O})\},
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{red}(\forall \mathbf{A}^q.C) &:= \{\top \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant C} \mid q \text{ is final}\} \cup \bigcup_{q \xrightarrow{x,p} q' \in \mathbf{A}} \mathsf{red}_{x,p,q'}(\forall \mathbf{A}^q.C) \\
\mathsf{red}_{\varepsilon,p,q'}(\forall \mathbf{A}^q.C) &:= \{\top \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant p \Rightarrow (\forall \mathbf{A}^{q'}.C)}\} \\
\mathsf{red}_{s,p,q'}(\forall \mathbf{A}^q.C) &:= \{\top \sqsubseteq \forall \mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle \forall \mathbf{A}^q.C\rangle_\uparrow \leqslant \min\{p, s\} \Rightarrow (\forall \mathbf{A}^{q'}.C)}\big), \\
&\quad\ \mathsf{AN} \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, s^-\} \Rightarrow \langle \forall \mathbf{A}^{q'}.C\rangle_\uparrow}, && \text{(I)} \\
&\quad\ \top \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, \exists s.\mathsf{Self}\} \Rightarrow (\forall \mathbf{A}^{q'}.C)}\} \cup && \text{(R)} \\
&\quad \{\exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, s(*,a)\} \Rightarrow (\forall \mathbf{A}^{q'}.C)(a)}, \\
&\quad\ \exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\forall \mathbf{A}^q.C)(a) \leqslant \min\{p, s^-(*,a)\} \Rightarrow (\forall \mathbf{A}^{q'}.C)} \mid a \in \mathsf{ind}(\mathcal{O})\}. && \text{(I,O)}
\end{aligned}$$

We now define final component of $\mathsf{red}(\mathcal{O})$ for handling number restrictions, namely $\mathsf{red}({\geqslant} n\, r.C)$, as

$$\Big\{\mathsf{AN} \sqsubseteq \bigsqcup_{z_i=0}^{1} \bigsqcup_{z_s=0}^{1} \bigsqcup_{m=0}^{n-z_i-z_s} \bigsqcup_{\substack{S\subseteq\mathsf{ind}(\mathcal{O})\\ |S|=n-m-z_i-z_s}} \bigsqcap \mathsf{red}_{z_i,z_s,m,S,\leqslant}({\geqslant} n\, r.C),$$

$$\mathsf{AN} \sqsubseteq \neg \bigsqcup_{z_i=0}^{1} \bigsqcup_{z_s=0}^{1} \bigsqcup_{m=0}^{n-z_i-z_s} \bigsqcup_{\substack{S\subseteq\mathsf{ind}(\mathcal{O})\\ |S|=n-m-z_i-z_s}} \bigsqcap \mathsf{red}_{z_i,z_s,m,S,<}({\geqslant} n\, r.C)\Big\} \cup$$

$$\Big\{{\geqslant} n\, \mathfrak{r}.\big((\mathsf{AN} \sqcap \boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow \leqslant \min\{r, C\}}) \sqcup (\neg\mathsf{AN} \sqcap \boxed{({\geqslant} n\, r.C)(a) \leqslant \min\{r(a,*), C\}})\big)(a),$$

$$\neg{\geqslant} n\, \mathfrak{r}.\big((\mathsf{AN} \sqcap \boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow < \min\{r, C\}}) \sqcup (\neg\mathsf{AN} \sqcap \boxed{({\geqslant} n\, r.C)(a) < \min\{r(a,*), C\}})\big)(a) \mid a \in \mathsf{ind}(\mathcal{O})\Big\},$$

where

$$\begin{aligned}
\mathsf{red}_{z_i,z_s,m,S,\lhd}({\geqslant} n\, r.C) := {} & \{{\geqslant} m\, \mathfrak{r}.(\mathsf{AN} \sqcap \boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow \lhd \min\{r, C\}})\} \cup \\
& \{\mathsf{AN} \sqcap \boxed{({\geqslant} n\, r.C) \lhd \min\{r^-, \langle C\rangle_\uparrow\}} \mid z_i = 1\} \cup && (\mathsf{I}) \\
& \{\boxed{({\geqslant} n\, r.C) \lhd \min\{\exists r.\mathsf{Self}, C\}} \mid z_s = 1\} \cup && (\mathsf{R}) \\
& \{\exists\mathfrak{r}.(\{a\} \sqcap \neg\{b\}) \mid a, b \in S,\ a \neq b\} \cup && (\mathsf{O}) \\
& \{\boxed{({\geqslant} n\, r.C) \lhd \min\{r(*,a), C(a)\}} \mid a \in S\} && (\mathsf{O})
\end{aligned}$$

If we do not have inverse roles, local reflexivity, or nominals in the source language, then we fix the numbers z_i, z_s, or m, respectively, to 0, 0, or $n - z_i - z_s$, which effectively eliminates the conjuncts using these constructors from the above axioms.

The reduction of at-least restrictions works similarly to the one of existential restrictions: the first axiom ensures the existence of the n required witnesses, while the second one ensures that no n different elements can exceed the value of the at-least restriction. Unfortunately, the number of named successors cannot be counted using a classical at-least restriction in our encoding, since these named successors do not know about the degree of the role connection from an anonymous element; otherwise they would have to store a possibly infinite amount of information since they may have infinitely many anonymous role predecessors. For this reason, the above axioms first guess how many $(m - n)$ and which (S) named elements are connected to the current domain element to the appropriate degrees (given by $r(*, a)$). For named elements, however, this additional guess is not necessary.

Finally, we define

$$\mathsf{red}(\mathcal{O}) := \mathsf{red}(\mathcal{U}) \cup \mathsf{red}(\mathcal{A}) \cup \mathsf{red}(\mathsf{AN}) \cup \mathsf{red}(\uparrow) \cup \mathsf{red}(\mathcal{R}) \cup \mathsf{red}(\mathcal{T}) \cup \bigcup_{C\in\mathsf{sub}(\mathcal{O})} \mathsf{red}(C).$$

5.2. Correctness and Complexity

The reduction is correct in the sense that the resulting ontology $\mathsf{red}(\mathcal{O})$ has a classical model iff $\mathcal{O}$ has a G-model. As mentioned before, this holds only for the sublogics G-$\mathcal{SRIQ}$, G-$\mathcal{SROQ}$, and G-$\mathcal{SROI}$ that have the forest model property [16]. However, the correctness is not affected by the presence or absence of local reflexivity statements.

Lemma 5.5. *In G-$\mathcal{SRIQ}$, G-$\mathcal{SROQ}$, or G-$\mathcal{SROI}$, $\mathcal{O}$ has a G-model iff $\mathsf{red}(\mathcal{O})$ has a classical model.*

Proof sketch. If the $\mathcal{ALCOQ}$ ontology $\mathsf{red}(\mathcal{O})$ is consistent, then it also has a quasi-forest model [16], whose structure we exploit in our reduction. At each domain element, the satisfaction of the order concepts $\boxed{\alpha \leqslant \beta}$ uniquely defines a total preorder over the relevant concepts and assertions. Due to $\mathsf{red}(\mathcal{A})$, the preorder over $\mathcal{U}_\mathcal{A}$ is shared by all domain elements. To obtain a G-interpretation, we can thus define an instantiation of all relevant values, starting with $\mathcal{U}_\mathcal{A}$, and continuing by induction on the tree structure. Whenever a domain element d refers back to a named individual, the already fixed values are taken into account to find appropriate instantiations for the concepts and roles at d. The density of the rational numbers in $[0, 1]$ ensures that we can find enough values for all domain elements. The so defined G-interpretation is a model of $\mathcal{O}$ due to the axioms in $\mathsf{red}(\mathcal{O})$.

Conversely, from a G-model of $\mathcal{O}$ we construct a forest-shaped classical model of $\mathsf{red}(\mathcal{O})$ by the well-known unraveling technique. Again, this is only possible because of the quasi-forest model property of $\mathcal{SRIQ}$, $\mathcal{SROQ}$, and $\mathcal{SROI}$, which, as we show, is inherited by their Gödel extensions. The interpretation of the order concepts is derived from the actual values in the G-model $\mathcal{I}$, i.e. a domain element d satisfies $\boxed{\alpha \leqslant \beta}$ iff $\alpha^{\mathcal{I}}(d) \leqslant \beta^{\mathcal{I}}(d)$. The interpretation of the elements of $\mathcal{U}$ is straightforward; for example, $(r(a, b))^{\mathcal{I}}(d)$ is the value $r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$, and $r^{\mathcal{I}}(d) := r^{\mathcal{I}}(e, d)$, where e is the parent of d in the tree. The witnessed Gödel semantics of $\mathcal{O}$ then implies the satisfaction of all axioms in $\mathsf{red}(\mathcal{O})$. □

We now analyze the complexity of the reduction. As in [18], the construction of the automata $\mathbf{A}_r$ causes an exponential blowup in the size of $\mathcal{R}$, which cannot be avoided [26]. Independently of this, our reduction also involves an exponential blowup in the (binary encoding of) the largest number n involved in a number restriction in $\mathcal{O}$, and in the number of individual names occurring in $\mathcal{O}$, since the number of disjuncts in each GCI from $\mathsf{red}(\geqslant n\, r.C)$ is linear in $n \cdot 2^{|\mathsf{ind}(\mathcal{O})|}$. However, we can avoid both if either nominals or number restrictions are disallowed. Hence, we obtain the following complexity results.

Theorem 5.6. *Deciding consistency is*

- 2-ExpTime-*complete in* G-$\mathcal{SRIQ}$, G-$\mathcal{SROI}$, *and* G-$\mathcal{SROQ}$, *and*
- ExpTime-*complete in all FDLs between* G-$\mathcal{ALC}$ *and* G-$\mathcal{SHOI}$ *or* G-$\mathcal{SHIQ}$.

Proof. The consistency of the $\mathcal{ALCOQ}$ ontology $\mathsf{red}(\mathcal{O})$ is decidable in exponential time in the size of $\mathsf{red}(\mathcal{O})$ [16]. The first upper bound thus follows from the fact that the size of $\mathsf{red}(\mathcal{O})$ is exponential in the size of $\mathcal{O}$. 2-ExpTime-hardness, even without involutive negation and assertions restricted to the form $\alpha \geqslant p$, follows from classical results [26] since in this case reasoning in sublogics of G-$\mathcal{SROIQ}$ is equivalent to reasoning in the underlying classical DLs [7].

Without complex role inclusions, i.e. restricting to simple role inclusions and transitivity axioms, the size of the automata $\mathbf{A}_r$ is polynomial in the size of $\mathcal{R}$ [18]. The other exponential blowup can be avoided by disallowing nominals or number restrictions. Hence, for G-$\mathcal{SHOI}$ and G-$\mathcal{SHIQ}$, the size of $\mathsf{red}(\mathcal{O})$ is polynomial in the size of $\mathcal{O}$, and the lower bound follows again from the reduction in [7] and ExpTime-hardness of consistency in classical $\mathcal{ALC}$ [29]. □

These results hold regardless of whether the numbers in number restrictions are encoded in unary or in binary. We leave open the complexity of consistency in G-$\mathcal{SHOQ}$, which is ExpTime-complete in the classical case [16, 29].

6. A Tableau Algorithm

In this section, we extend the classical tableau construction from [23, 30] with the ideas developed in [10, 15] to produce a tableau-based reasoning algorithm capable of handling full G-$\mathcal{SROIQ}$. Although this algorithm does not allow us to derive tight complexity bounds, it constructs a G-model in a goal-oriented way and exhibits the same pay-as-you-go behavior as the classical tableau algorithm [23]. When the ontology is well-behaved, the tableau algorithm can avoid the exponential blowup arising from the combination of nominals and number restrictions.

Again, we consider $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ to be an arbitrary, but fixed, G-$\mathcal{SROIQ}$ ontology. We assume without loss of generality that the ABox $\mathcal{A}$ contains no individual (in)equality assertions. In fact, $a \approx b$ is equivalent to the assertion $\{b\}(a) \geqslant 1$, and $a \not\approx b$ to $\{b\}(a) \leqslant 0$.[7] As in Section 5, we also do not consider role assertions explicitly.

[7] We did not use these replacements in Section 5, because $a \approx b$ and $a \not\approx b$ can be expressed directly in classical $\mathcal{SROIQ}$. In contrast, for the tableau algorithm we would need to introduce additional rules, making the approach less readable.

We again use the set $\mathsf{sub}(\mathcal{O})$ defined in Section 5. However, we modify the definition of $\mathsf{sub}(C)$ for at-least restrictions as follows (this is needed for the rule (NN) in Table 6 below):

$$\mathsf{sub}(\geqslant n\, r.D) := \mathsf{sub}(D) \cup \{\geqslant m\, r.D, \neg\geqslant m\, r.D \mid 1 \leqslant m \leqslant n\}.$$

Due to this construction, the size of $\mathsf{sub}(\mathcal{O})$ is now also exponential in the largest number appearing in number restrictions in $\mathcal{O}$ (assuming that such numbers are given in binary encoding). We also need to use the larger order structure

$$\mathcal{U}(\Delta) := \mathsf{val}(\mathcal{O}) \cup \big\{C(x) \mid C \in \mathsf{sub}(\mathcal{O}),\ x \in \Delta\big\} \cup \big\{r(x,y), \neg r(x,y) \mid r \in \mathsf{rol}(\mathcal{O}),\ x,y \in \Delta\big\},$$

where Δ is a set of *nodes*, $\mathsf{inv}(C(x)) := \neg C(x)$, and $\mathsf{inv}(r(x,y)) := \neg r(x,y)$. The tableau algorithm uses sets of order assertions over this order structure in order to characterize the behavior of a model of $\mathcal{O}$.

To simplify dealing with constants, inverse roles, and local reflexivity, in the following we will treat the expressions $\overline{p}(x)$ and p as if they were the same, and likewise for $r(x,y)$ and $r^-(y,x)$, and $(\exists r.\mathsf{Self})(x)$ and $r(x,x)$. This is clearly justified by the semantics. In essence, this corresponds to implicitly adding the order assertions $\overline{p}(x) = p$, $r(x,y) = r^-(y,x)$, and $(\exists r.\mathsf{Self})(x) = r(x,x)$, respectively.

6.1. Tableaux

As in [23, 30], we first define the notion of a *tableau*, which is essentially an abstract version of a model of $\mathcal{O}$ that may still be infinite, but allows us to simplify the semantics. For example, all complex role inclusions are handled by three simple rules for the behavior of the concepts $\forall\mathbf{A}.C$. The conditions are very similar to the reduction of Section 5, but are simpler to formulate since we do not distinguish between r-successors in a tree structure, r^--predecessors, and named r-successors. As in Section 5, we will need more sophisticated techniques to deal with the interactions between nominals and number restrictions (see Section 6.2). Note that in the following the term *entailment* is used exclusively in for sets of order assertions (cf. Lemma 2.2).

Definition 6.1 (Tableau)**.** A *tableau* for $\mathcal{O}$ is a pair $(\Delta, \mathcal{A}^*)$, where Δ is a non-empty set of *nodes* and $\mathcal{A}^*$ is a satisfiable set of order assertions over $\mathcal{U}(\Delta)$, such that the following conditions hold, for all $x, y \in \Delta$, $C, D \in \mathsf{sub}(\mathcal{O})$, $r, s \in \mathsf{rol}(\mathcal{O})$, and $a \in \mathsf{ind}(\mathcal{O})$:

(T1) If $(C \sqcap D)(x)$ occurs in $\mathcal{A}^*$, then $\mathcal{A}^*$ entails $(C \sqcap D)(x) = \min\{C(x), D(x)\}$.

(T2) If $(C \to D)(x)$ occurs in $\mathcal{A}^*$, then $\mathcal{A}^*$ entails $(C \to D)(x) = C(x) \Rightarrow D(x)$.

(T3) If $(\neg C)(x)$ occurs in $\mathcal{A}^*$, then $C(x)$ also occurs in $\mathcal{A}^*$.

(T4) If $(\exists r.C)(x)$ occurs in $\mathcal{A}^*$, then there is a $y \in \Delta$ such that $\mathcal{A}^*$ entails $(\exists r.C)(x) \leqslant \min\{r(x,y), C(y)\}$.

(T5) If $(\exists r.C)(x)$ occurs in $\mathcal{A}^*$, then $\mathcal{A}^*$ entails $(\exists r.C)(x) \geqslant (\exists\mathbf{A}_r.C)(x)$.

(T6) If $(\exists\mathbf{A}^q.C)(x)$ and $r(x,y)$ occur in $\mathcal{A}^*$ and $q \xrightarrow{r,p} q' \in \mathbf{A}$, then $\mathcal{A}^*$ entails

$$(\exists\mathbf{A}^q.C)(x) \geqslant \min\{p, r(x,y), (\exists\mathbf{A}^{q'}.C)(y)\}.$$

(T7) If $(\exists\mathbf{A}^q.C)(x)$ occurs in $\mathcal{A}^*$ and $q \xrightarrow{\varepsilon,p} q' \in \mathbf{A}$, then $\mathcal{A}^*$ entails $(\exists\mathbf{A}^q.C)(x) \geqslant \min\{p, (\exists\mathbf{A}^{q'}.C)(x)\}$.

(T8) If $(\exists\mathbf{A}^q.C)(x)$ occurs in $\mathcal{A}^*$ and q is final, then $\mathcal{A}^*$ entails $(\exists\mathbf{A}^q.C)(x) \geqslant C(x)$.

(T9) If $(\forall r.C)(x)$ occurs in $\mathcal{A}^*$, then there is a $y \in \Delta$ such that $\mathcal{A}^*$ entails $(\forall r.C)(x) \geqslant r(x,y) \Rightarrow C(y)$.

(T10) If $(\forall r.C)(x)$ occurs in $\mathcal{A}^*$, then $\mathcal{A}^*$ entails $(\forall r.C)(x) \leqslant (\forall\mathbf{A}_r.C)(x)$.

(T11) If $(\forall\mathbf{A}^q.C)(x)$ and $r(x,y)$ occur in $\mathcal{A}^*$ and $q \xrightarrow{r,p} q' \in \mathbf{A}$, then $\mathcal{A}^*$ entails

$$(\forall\mathbf{A}^q.C)(x) \leqslant \min\{p, r(x,y)\} \Rightarrow (\forall\mathbf{A}^{q'}.C)(y).$$

(T12) If $(\forall \mathbf{A}^q.C)(x)$ occurs in $\mathcal{A}^*$ and $q \xrightarrow{\varepsilon,p} q' \in \mathbf{A}$, then $\mathcal{A}^*$ entails $(\forall \mathbf{A}^q.C)(x) \leqslant p \Rightarrow (\forall \mathbf{A}^{q'}.C)(x)$.

(T13) If $(\forall \mathbf{A}^q.C)(x)$ occurs in $\mathcal{A}^*$ and q is final, then $\mathcal{A}^*$ entails $(\forall \mathbf{A}^q.C)(x) \leqslant C(x)$.

(T14) If $(\geqslant n\, r.C)(x)$ occurs in $\mathcal{A}^*$, then there are at least n elements $y \in \Delta$ for which $\mathcal{A}^*$ entails

$$(\geqslant n\, r.C)(x) \leqslant \min\{r(x,y), C(y)\}.$$

(T15) If $(\geqslant n\, r.C)(x)$ occurs in $\mathcal{A}^*$, then there are at most $n-1$ elements $y \in \Delta$ for which $\mathcal{A}^*$ entails

$$(\geqslant n\, r.C)(x) < \min\{r(x,y), C(y)\}.$$

(T16) If $(\geqslant n\, r.C)(x)$ and $r(x,y)$ occur in $\mathcal{A}^*$, then $\mathcal{A}^*$ entails either

$$(\geqslant n\, r.C)(x) \geqslant \min\{r(x,y), C(y)\} \text{ or } (\geqslant n\, r.C)(x) < \min\{r(x,y), C(y)\}.$$

(T17) If $\{a\}(x)$ occurs in $\mathcal{A}^*$, then $\mathcal{A}^*$ entails either $\{a\}(x) \geqslant 1$ or $\{a\}(x) \leqslant 0$.

(T18) There is exactly one $x_a \in \Delta$ such that $\mathcal{A}^*$ entails $\{a\}(x_a) \geqslant 1$.

(T19) If $C(a) \bowtie D(b) \in \mathcal{A}$ (resp. $C(a) \bowtie p \in \mathcal{A}$), then $\mathcal{A}^*$ entails $C(x_a) \bowtie D(x_b)$ (resp. $C(x_a) \bowtie p$).

(T20) If $C \sqsubseteq D \geqslant p \in \mathcal{T}$, then $\mathcal{A}^*$ entails $C(x) \Rightarrow D(x) \geqslant p$.

(T21) If $r \sqsubseteq s \geqslant p \in \mathcal{R}$, then $\mathcal{A}^*$ entails $r(x,y) \Rightarrow s(x,y) \geqslant p$.

(T22) If $\mathsf{dis}(r,s) \geqslant p \in \mathcal{R}$, then $\mathcal{A}^*$ entails $\min\{r(x,y), s(x,y)\} \leqslant 1-p$.

The main differences to the classical tableau conditions for classical $\mathcal{SROIQ}$ from [23, 30], in addition to the use of order assertions, are the following:

- The semantics of the involutive negation is handled by the conditions that define the order structure $\mathcal{U}(\Delta)$. Hence, we do not need a dedicated condition for it beyond adding the relevant subconcept to the tableau (see (T3)). This addition is necessary in order to be able to decompose this subconcept further.
- We do not internalize any axioms of the ontology. For this reason, we need to include the dedicated conditions (T19)–(T22).
- Although we do not explicitly consider at-most restrictions here, the corresponding conditions from [23, 30] are mirrored in (T15) and (T16). The reason for this is that fuzzy at-least restrictions exhibit a behavior similar to that of at-most restrictions, as explained in Remark 2.3.

As shown in the next lemma, it suffices to construct a countable tableau to verify that the ontology $\mathcal{O}$ is consistent. The requirement on the cardinality of the tableau is the main difference to the corresponding result in [23]. This restriction is necessary to ensure that there exist enough rational values in the interval $[0,1]$ to instantiate all relevant concepts and roles.

Lemma 6.2. *If $\mathcal{O}$ is consistent, then it has a tableau, and if it has a countable tableau, then $\mathcal{O}$ is consistent.*

Proof sketch. Given any G-model $\mathcal{I}$ of $\mathcal{O}$, we obtain a tableau $(\Delta^\mathcal{I}, \mathcal{A}^*)$ by collecting all order assertions $u \bowtie v$ for which $u^\mathcal{I} \bowtie v^\mathcal{I}$ holds. For this, we consider $p^\mathcal{I} := p$, $C(d)^\mathcal{I} := C^\mathcal{I}(d)$, $r(d,e)^\mathcal{I} := r^\mathcal{I}(d,e)$, and $(\neg r(d,e))^\mathcal{I} := 1 - r^\mathcal{I}(d,e)$. By construction, $\mathcal{A}^*$ is satisfiable, and it is easy to verify that the conditions (T1)–(T22) hold. For example, if $(\forall \mathbf{A}^q.C)(d)$ occurs in $\mathcal{A}^*$ and q is final, then we have $(\|\mathbf{A}^q\|, \varepsilon) = 1$, and thus

$$(\forall \mathbf{A}^q.C)^\mathcal{I}(d) = \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^\mathcal{I}} w^\mathcal{I}(d,e) \Rightarrow C^\mathcal{I}(e) \leqslant \varepsilon^\mathcal{I}(d,d) \Rightarrow C^\mathcal{I}(d) = C^\mathcal{I}(d).$$

By our construction, this means that $(\forall \mathbf{A}^q.C)(d) \leqslant C(d)$ is contained in $\mathcal{A}^*$, and thus (T13) is satisfied.

Conversely, from any countable tableau we obtain a G-interpretation by taking all nodes as domain elements and instantiating the values for all concepts and roles according to the order assertions. Such an instantiation must exist since the set of order assertions is satisfiable and we only need to find countably many rational numbers from $[0,1]$. The tableau conditions ensure that this G-interpretation satisfies $\mathcal{O}$. $\square$

6.2. Tableau Rules

The construction of a possibly infinite tableau for $\mathcal{O}$ is hardly a decision procedure for consistency, as it might not terminate in finite time. To achieve termination, we need to appropriately lift the notion of blocking [23, 30] to sets of order assertions, in a way that guarantees that at a *finite* structure is generated. We also need to take a more fine-grained view at the structure of a tableau. First, we designate a subset $\Delta_o \subseteq \Delta$ that contains all *nominal nodes*. Furthermore, in the tableau algorithm we need to introduce new individual names that do not occur in $\mathsf{ind}(\mathcal{O})$, and we assume that the corresponding nominals are contained in $\mathsf{sub}(\mathcal{O})$. Instead of allowing to connect arbitrary pairs of individuals with roles, we will maintain a binary *neighbor relation* $\mathcal{N}$ on Δ, which represents a tree-shaped substructure of a model of $\mathcal{O}$. For a node x, we define its *neighborhood* $\mathcal{N}(x)$ as the projection to one component of the reflexive, symmetric closure of $\mathcal{N}$; that is

$$\mathcal{N}(x) := \{x\} \cup \big\{y \mid (x,y) \in \mathcal{N} \text{ or } (y,x) \in \mathcal{N}\big\}.$$

It is important to note the difference between $(x,y) \in \mathcal{N}$ and $y \in \mathcal{N}(x)$. If $(x,y) \in \mathcal{N}$, then y is called a *successor* of x, and x is a *predecessor* of y. *Ancestor* is the transitive closure of predecessor, and *descendant* the transitive closure of successor. Finally, instead of a global set $\mathcal{A}^*$, we will maintain for each node $x \in \Delta$ a local set of order assertions $\mathcal{L}(x)$ over the localized order structure

$$\begin{aligned}\mathcal{U}(x) &:= \mathcal{U}_o \cup \big\{C(y) \mid C \in \mathsf{sub}(\mathcal{O}),\ y \in \mathcal{N}(x)\big\} \cup \big\{r(x,y), \neg r(x,y) \mid r \in \mathsf{rol}(\mathcal{O}),\ y \in \mathcal{N}(x)\big\}, \text{ where}\\ \mathcal{U}_o &:= \mathsf{val}(\mathcal{O}) \cup \big\{C(a) \mid C \in \mathsf{sub}(\mathcal{O}),\ a \in \Delta_o\big\} \cup \big\{r(a,b), \neg r(a,b) \mid r \in \mathsf{rol}(\mathcal{O}),\ a,b \in \Delta_o\big\},\end{aligned}$$

and inv is defined as for $\mathcal{U}(\Delta)$. It is easy to see that $\mathcal{U}(x)$ is always a subset of $\mathcal{U}(\Delta)$. However, apart from information about nominal nodes, order assertions over $\mathcal{U}(x)$ can only talk about the concepts at neighbors of x and role connections between x and its neighbors. The goal is to keep inferences local to the nodes, and share information between two neighbors only when it is relevant for both of them.

In addition, the algorithm maintains a global set $\mathcal{L}_o$ that contains all order assertions over the designated elements in $\mathcal{U}_o$. These assertions represent the knowledge about the named individuals, which is shared by all nodes and implicitly present in all sets $\mathcal{L}(x)$, for $x \in \Delta$. More precisely, whenever we say that some expression ϕ occurs in $\mathcal{L}(x)$, it should be read as "ϕ occurs in $\mathcal{L}_o \cup \mathcal{L}(x)$", and similarly for the entailment of order assertions by $\mathcal{L}(x)$. Moreover, the process of *adding* an order assertion $\alpha \bowtie \beta$ over $\mathcal{U}(x)$ to $\mathcal{L}(x)$ distinguishes between two kinds of assertions: if α and β both belong to $\mathcal{U}_o$, then this assertion is added to the shared set $\mathcal{L}_o$; otherwise, it is directly added to $\mathcal{L}(x)$. These conventions ensure that the set $\mathcal{L}_o$ is not replicated in the label of every node.

We can now define the data structure that is used by the tableau algorithm.

Definition 6.3. A *completion graph* for $\mathcal{O}$ is a tuple $\mathcal{G} = (\Delta, \Delta_o, \mathcal{N}, \mathcal{L}, \mathcal{L}_o, \neq)$, where Δ is a finite set of *nodes*, $\Delta_o \subseteq \Delta$ contains the *nominal nodes*, $\mathcal{N}$ is a binary *neighbor* relation on Δ, $\mathcal{L}$ is a *labeling function* that assigns each node $x \in \Delta$ a set $\mathcal{L}(x)$ of oder assertions over $\mathcal{U}(x)$, $\mathcal{L}_o$ is a shared set of order assertions over $\mathcal{U}_o$, and $\neq$ is a binary relation on Δ.

The relation $\neq$ indicates the nodes from Δ that must be kept different. If $x \neq y$ does not hold, then the two nodes x and y can be merged into a single node if it is needed, e.g. in order to satisfy some number restrictions. We denote by $\doteq$ the complement of $\neq$. Notice that $x \doteq y$ does not mean that x and y will *necessarily* be merged, but only that it is possible to do so.

Nominal nodes and blockable nodes. The set Δ_o is not fixed *a priori*, but rather defined as the set of all $x \in \Delta$ such that $\mathcal{L}(x)$ entails $\{a\}(x) \geqslant 1$ for some $a \in \mathsf{N_I}$. Recall that we may need to introduce more such individual names in the construction of the completion graph, and hence the set Δ_o changes dynamically. All nodes in $\Delta \setminus \Delta_o$ are called *blockable nodes*. The idea is that nominal nodes may be arbitrarily interconnected, but blockable nodes always form a tree structure among themselves (represented by $\mathcal{N}$). Each nominal node may be the root of such a tree, and additionally all blockable nodes may have $\mathcal{N}$-successors that are nominal nodes.

Table 2: The tableau rules for the propositional constructors

$(\sqcap)$	IF $(C \sqcap D)(x)$ occurs in $\mathcal{L}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(C \sqcap D)(x) = \min\{C(x), D(x)\}$.
$(\to)$	IF $(C \to D)(x)$ occurs in $\mathcal{L}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(C \to D)(x) = C(x) \Rightarrow D(x)$.
$(\neg)$	IF $(\neg C)(x)$ occurs in $\mathcal{L}(x)$, but $C(x)$ does not occur in $\mathcal{L}(x)$, THEN add $C(x) \leqslant C(x)$ to $\mathcal{L}(x)$.
(ch_o)	IF $\{a\}(x)$ occurs in $\mathcal{L}(x)$, THEN do the following: • if $\mathcal{L}(x)$ entails $\{a\}(x) > 0$, then ensure that $\mathcal{L}(x)$ entails $\{a\}(x) \geqslant 1$, • if $\mathcal{L}(x)$ entails $\{a\}(x) < 1$, then ensure that $\mathcal{L}(x)$ entails $\{a\}(x) \leqslant 0$, • otherwise, ensure that $\mathcal{L}(x)$ entails either $\{a\}(x) \geqslant 1$ or $\{a\}(x) \leqslant 0$.
(o)	IF for some $a \in \mathsf{N_I}$ there are two nodes x, y such that $\mathcal{L}(x)$ entails $\{a\}(x) \geqslant 1$, $\mathcal{L}(y)$ entails $\{a\}(y) \geqslant 1$, and $x \doteq y$, THEN merge x into y.

Table 3: The tableau rules for axioms, and transfer of order assertions between neighbors

$(\sqsubseteq_\mathcal{T})$	IF $C \sqsubseteq D \geqslant p \in \mathcal{T}$, THEN ensure that $\mathcal{L}(x)$ entails $C(x) \Rightarrow D(x) \geqslant p$.
$(\sqsubseteq_\mathcal{R})$	IF $r \sqsubseteq s \geqslant p \in \mathcal{R}$ and $y \in \mathcal{N}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $r(x,y) \Rightarrow s(x,y) \geqslant p$.
(dis)	IF $\mathsf{dis}(r,s) \geqslant p \in \mathcal{R}$ and $y \in \mathcal{N}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $\min\{r(x,y), s(x,y)\} \leqslant 1 - p$.
$(\rightsquigarrow)$	IF 1. $\mathcal{L}(x)$ entails $\alpha \lhd \beta$, where $\lhd \in \{<, \leqslant\}$, and 2. there is a $y \in \Delta$ such that $\{\alpha, \beta\} \subseteq \mathcal{U}(y)$, THEN ensure that $\mathcal{L}(y)$ entails $\alpha \lhd \beta$.

Rule applications and clashes. The *initial completion graph* for $\mathcal{O}$ is $\mathcal{G}_0 := (\Delta_0, \Delta_{o,0}, \mathcal{N}_0, \mathcal{L}_0, \mathcal{L}_{o,0}, \not\doteq_0)$, where $\Delta_{o,0} := \Delta_0 := \mathsf{ind}(\mathcal{O})$, $\mathcal{N}_0 := \not\doteq_0 := \emptyset$, $\mathcal{L}_0(a) := \emptyset$ for all $a \in \mathsf{ind}(\mathcal{O})$, and $\mathcal{L}_{o,0}$ consists of all assertions from $\mathcal{A}$, together with the assertions $\{a\}(a) \geqslant 1$, for all $a \in \mathsf{ind}(\mathcal{O})$.

Example 6.4. For the example ontology from Section 3, the initial completion graph contains the single node a, and $\mathcal{L}_{o,0}$ contains the order assertions

$$(\exists r.A)(a) \geqslant p_A,\ (\exists r.B)(a) \geqslant p_B,\ ({\leqslant}1\, r.C)(a) \geqslant p_C,\ \{a\}(a) \geqslant 1,$$

expressing exactly the restrictions of $\mathcal{A}$. These assertions are implicitly available to each node that will be created during the course of the tableau algorithm. From this set it already follows that $({\geqslant}2\, r.C)(a) \leqslant 1 - p_C$, due to the definition of $\mathsf{order}(\mathcal{U}_o)$, the fact that ${\leqslant}1\, r.C = \neg{\geqslant}2\, r.C$, and Lemma 2.2.

Starting from $\mathcal{G}_0$, the tableau algorithm nondeterministically applies the rules listed in Tables 2–6, which modify the completion graph according to the semantics of concepts and axioms. For the following exposition, let $\mathcal{G} = (\Delta, \Delta_o, \mathcal{N}, \mathcal{L}, \mathcal{L}_o, \not\doteq)$ be an arbitrary completion graph produced in this way from $\mathcal{G}_0$.

The names of the tableau rules were chosen as close as possible to the notation in [23, 30]. However, there are some notable differences in their behavior, as well as new rules. For instance, since labels of nodes may refer to neighboring nodes, we need the dedicated rule $(\rightsquigarrow)$ to ensure that labels of neighbors

Table 4: The tableau rules for existential and value restrictions, which are dual to each other.

$(\exists)$	IF 1. $(\exists r.C)(x)$ occurs in $\mathcal{L}(x)$, x is not blocked, and 2. there is no safe neighbor $y \in \mathcal{N}(x)$ such that $\mathcal{L}(x)$ entails $(\exists r.C)(x) \leqslant \min\{r(x,y), C(y)\}$, THEN 1. introduce a new node y, add (x,y) to $\mathcal{N}$, and 2. ensure that $\mathcal{L}(x)$ entails $(\exists r.C)(x) \leqslant \min\{r(x,y), C(y)\}$.
$(\exists_1)$	IF $(\exists r.C)(x)$ occurs in $\mathcal{L}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(\exists r.C)(x) \geqslant (\exists \mathbf{A}_r.C)(x)$.
$(\exists_2)$	IF $(\exists \mathbf{A}^q.C)(x)$ and $r(x,y)$ occur in $\mathcal{L}(x)$ and $q \xrightarrow{r,p} q' \in \mathbf{A}$, THEN ensure that $\mathcal{L}(x)$ entails $(\exists \mathbf{A}^q.C)(x) \geqslant \min\{p, r(x,y), (\exists \mathbf{A}^{q'}.C)(y)\}$.
$(\exists_2^\varepsilon)$	IF $(\exists \mathbf{A}^q.C)(x)$ occurs in $\mathcal{L}(x)$ and $q \xrightarrow{\varepsilon,p} q' \in \mathbf{A}$, THEN ensure that $\mathcal{L}(x)$ entails $(\exists \mathbf{A}^q.C)(x) \geqslant \min\{p, (\exists \mathbf{A}^{q'}.C)(x)\}$.
$(\exists_3)$	IF $(\exists \mathbf{A}^q.C)(x)$ occurs in $\mathcal{L}(x)$ and q is final, THEN ensure that $\mathcal{L}(x)$ entails $(\exists \mathbf{A}^q.C)(x) \geqslant C(x)$.
$(\forall)$	IF 1. $(\forall r.C)(x)$ occurs in $\mathcal{L}(x)$, x is not blocked, and 2. there is no safe neighbor $y \in \mathcal{N}(x)$ such that $\mathcal{L}(x)$ entails $(\forall r.C)(x) \geqslant r(x,y) \Rightarrow C(y)$, THEN 1. introduce a new node y, add (x,y) to $\mathcal{N}$, and 2. ensure that $\mathcal{L}(x)$ entails $(\forall r.C)(x) \geqslant r(x,y) \Rightarrow C(y)$.
$(\forall_1)$	IF $(\forall r.C)(x)$ occurs in $\mathcal{L}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(\forall r.C)(x) \leqslant (\forall \mathbf{A}_r.C)(x)$.
$(\forall_2)$	IF $(\forall \mathbf{A}^q.C)(x)$ and $r(x,y)$ occur in $\mathcal{L}(x)$ and $q \xrightarrow{r,p} q' \in \mathbf{A}$, THEN ensure that $\mathcal{L}(x)$ entails $(\forall \mathbf{A}^q.C)(x) \leqslant \min\{p, r(x,y)\} \Rightarrow (\forall \mathbf{A}^{q'}.C)(y)$.
$(\forall_2^\varepsilon)$	IF $(\forall \mathbf{A}^q.C)(x)$ occurs in $\mathcal{L}(x)$ and $q \xrightarrow{\varepsilon,p} q' \in \mathbf{A}$, THEN ensure that $\mathcal{L}(x)$ entails $(\forall \mathbf{A}^q.C)(x) \leqslant p \Rightarrow (\forall \mathbf{A}^{q'}.C)(x)$.
$(\forall_3)$	IF $(\forall \mathbf{A}^q.C)(x)$ occurs in $\mathcal{L}(x)$ and q is final, THEN ensure that $\mathcal{L}(x)$ entails $(\forall \mathbf{A}^q.C)(x) \leqslant C(x)$.

Table 5: The tableau rules for number restrictions

$(\geqslant)$	IF 1. $(\geqslant n\, r.C)(x)$ occurs in $\mathcal{L}(x)$, x is not blocked, and 2. there do not exist n safe neighbors $y_1, \ldots, y_n \in \mathcal{N}(x)$ with $y_i \not\doteq y_j$, $1 \leqslant i < j \leqslant n$, such that $\mathcal{L}(x)$ entails $(\geqslant n\, r.C)(x) \leqslant \min\{r(x, y_i), C(y_i)\}$, $1 \leqslant i \leqslant n$, THEN 1. introduce n new nodes $y_1, \ldots, y_n$ with $(x, y_1), \ldots, (x, y_n) \in \mathcal{N}$ and $y_i \not\doteq y_j$, $1 \leqslant i < j \leqslant n$, and 2. for each i, $1 \leqslant i \leqslant n$, ensure that $\mathcal{L}(x)$ entails $(\geqslant n\, r.C)(x) \leqslant \min\{r(x, y_i), C(y_i)\}$.
$(\neg\geqslant)$	IF there exist at least n neighbors $y \in \mathcal{N}(x)$ such that $\mathcal{L}(x)$ entails $(\geqslant n\, r.C)(x) < \min\{r(x, y), C(y)\}$, THEN choose two such neighbors y, z with $y \doteq z$, and do the following: • if y is a nominal node, then merge z into y; • else if z is a nominal node or an ancestor of y, then merge y into z; • else merge z into y.
(ch)	IF 1. $(\geqslant n\, r.C)(x)$ and $r(x, y)$ occur in $\mathcal{L}(x)$ and 2. $\mathcal{L}(x)$ entails neither $(\geqslant n\, r.C)(x) < \min\{r(x, y), C(y)\}$ nor $(\geqslant n\, r.C)(x) \geqslant \min\{r(x, y), C(y)\}$, THEN ensure that $\mathcal{L}(x)$ entails either $(\geqslant n\, r.C)(x) < \min\{r(x, y), C(y)\}$ or $(\geqslant n\, r.C)(x) \geqslant \min\{r(x, y), C(y)\}$.

Table 6: The special tableau rules for nominals

(NN)	IF 1. $(\geqslant n\, r.C)(x)$ occurs in $\mathcal{L}(x)$, x is a nominal node, 2. there is a blockable node y with $(y, x) \in \mathcal{N}$ such that $\mathcal{L}(x)$ entails $(\geqslant n\, r.C)(x) < \min\{r(x, y), C(y)\}$, and 3. there do not exist $\ell \leqslant n-1$ nominal nodes $z_1, \ldots, z_\ell$ such that $\mathcal{L}(x)$ entails $(\geqslant(\ell+1)\, r.C)(x) = (\geqslant n\, r.C)(x)$ and $(\geqslant(\ell+1)\, r.C)(x) < \min\{r(x, z_i), C(z_i)\}$, $1 \leqslant i \leqslant \ell$, and $z_i \not\doteq z_j$, $1 \leqslant i < j \leqslant \ell$, THEN 1. guess a number m between 1 and $n-1$, and add $(\geqslant(m+1)\, r.C)(x) = (\geqslant n\, r.C)(x)$ to $\mathcal{L}(x)$, 2. introduce m new nodes $y_1, \ldots, y_m$ with $(x, z_1), \ldots, (x, z_m) \in \mathcal{N}$ and $z_i \not\doteq z_j$, $1 \leqslant i < j \leqslant m$, 3. introduce m new individual names $a_1, \ldots, a_m$, and for each i, $1 \leqslant i \leqslant m$, ensure that $\mathcal{L}(x)$ entails $(\geqslant(m+1)\, r.C)(x) < \min\{r(x, z_i), C(z_i)\}$ and $\{a_i\}(z_i) \geqslant 1$.
$(\neg\geqslant_o)$	IF 1. $(\geqslant n\, r.C)(x)$ occurs in $\mathcal{L}(x)$, x is a nominal node, 2. there is a blockable node $y \in \mathcal{N}(x)$ such that $\mathcal{L}(x)$ entails $(\geqslant n\, r.C)(x) < \min\{r(x, y), C(y)\}$, and 3. there are nominal nodes $z_1, \ldots, z_{n-1} \in \mathcal{N}(x)$ with $z_i \not\doteq z_j$, $1 \leqslant i < j \leqslant n-1$, such that $\mathcal{L}(x)$ entails $(\geqslant n\, r.C)(x) < \min\{r(x, z_i), C(z_i)\}$, $1 \leqslant i \leqslant n-1$, THEN 1. choose a z_i, $1 \leqslant i \leqslant n-1$, such that $y \doteq z_i$ and 2. merge y into z_i.

are synchronized. This rule is sound due to the monotonicity of inferences between order assertions: if new information is added, it cannot invalidate previous entailments.

When we extend the neighborhood of x by adding a new node y with $(x, y) \in \mathcal{N}$, the label $\mathcal{L}(x)$ still contains order assertions over the (now extended) order structure $\mathcal{U}(x)$. The same holds when we introduce new nominal nodes using the rule (NN), and in this case also $\mathcal{U}_o$ and $\mathcal{L}_o$ are updated according to their definitions.

Example 6.5. Consider the completion graph from Example 6.4. Since $(\exists r.A)(a)$ occurs in $\mathcal{L}(a)$ and there are no r-successors, the rule $(\exists)$ is applicable. We create a new node x and add the two atomic order assertions $(\exists r.A)(a) \leqslant r(a, x)$ and $(\exists r.A)(a) \leqslant A(x)$ to $\mathcal{L}(a)$, which are equivalent to the order assertion required by the rule. Note that these assertions are not part of $\mathcal{L}_o$ since they refer to the blockable node x.

However, we also need to apply rules ensuring that $(\exists r.A)(a)$ is an upper bound for $\min\{r(a, x), A(x)\}$, and hence x is a proper witness for the existential restriction. Recall that $\mathbf{A}_r$ has exactly two states and one transition $i_r \xrightarrow{r,1} f_r$. Hence, we first use $(\exists_1)$ to introduce $(\exists r.A)(a) \geqslant (\exists \mathbf{A}_r^{i_r}.A)(a)$, and then apply $(\exists_2)$ to $(\exists \mathbf{A}_r^{i_r}.A)(a)$ and $r(a, x)$. Here we have several choices as to the relative order of $(\exists \mathbf{A}_r^{i_r}.A)(a)$, $r(a, x)$, and $(\exists \mathbf{A}_r^{f_r}.A)(x)$. The precise details of this process will be described later. If we decide to add $(\exists \mathbf{A}_r^{i_r}.A)(a) \geqslant r(a, x)$, then we are finished since this entails $(\exists r.A)(a) \geqslant r(a, x) \geqslant \min\{r(a, x), A(x)\}$. The other alternative is to assert $(\exists \mathbf{A}_r^{i_r}.A)(a) \geqslant (\exists \mathbf{A}_r^{f_r}.A)(a)$, in which case we also have to apply $(\exists_3)$ to get $(\exists r.A)(a) \geqslant (\exists \mathbf{A}_r^{i_r}.A)(a) \geqslant (\exists \mathbf{A}_r^{f_r}.A)(a) \geqslant A(x) \geqslant \min\{r(a, x), A(x)\}$.

This mirrors the intuition described in Section 3, where we chose the first option. In that case, note that $\mathcal{L}(a)$ entails $(\exists r.A)(a) \geqslant r(a, x)$, which also concerns the node x. Since $a \in \mathcal{N}(x)$, we have $(\exists r.A)(a), r(a, x) \in \mathcal{U}(x)$, and hence the rule $(\rightsquigarrow)$ requires that $\mathcal{L}(x)$ also contains this assertion since it is relevant to x.

In contrast to the classical tableau algorithm, we need the rule (ch_o), which is similar to the "choose" rule (ch) for number restrictions: Since the value of $\{a\}(x)$ is not *a priori* restricted to 0 or 1, we need to make each node choose one of these two possible values. As in the classical case, the rule (o) then ensures that two nodes that choose the value 1 for the same nominal are merged. Also unlike the classical algorithm, the rules $(\neg\geqslant)$, (NN), and $(\neg\geqslant_o)$ (called $(\leqslant)$, (NN), and $(\leqslant_o)$ in [23, 30]) are applied to at-least restrictions instead of at-most restrictions; this is due to the dual nature of the semantics of number restrictions explained earlier.

Our completion graph contains a *clash* if one of the following conditions holds:[8]

- For some node $x \in \Delta$, the set $\mathcal{L}(x)$ is unsatisfiable (cf. Lemma 2.2).
- For some $({\geqslant}n\, r.C) \in \mathsf{sub}(\mathcal{O})$, there are nodes $x, y_1, \dots, y_n \in \Delta$, such that $y_i \neq y_j$, $1 \leqslant i < j \leqslant n$, and $\mathcal{L}(x)$ entails $({\geqslant}n\, r.C)(x) < \min\{r(x, y_i), C(y_i)\}$, $1 \leqslant i \leqslant n$.
- For some $a \in \mathsf{N_I}$, there are nodes $x, y \in \Delta$ such that $x \neq y$, $\mathcal{L}(x)$ entails $\{a\}(x) \geqslant 1$, and $\mathcal{L}(y)$ entails $\{a\}(y) \geqslant 1$.

A completion graph is *complete* if it contains a clash or none of the tableau rules are applicable in such a way that the completion graph is changed. If the tableau rules can be applied to $\mathcal{G}_0$ such that a complete and clash-free completion graph is obtained, then the algorithm has successfully proven the consistency of $\mathcal{O}$. If we obtain a clash, then either we have made the wrong choices in the rule applications, or $\mathcal{O}$ is inconsistent.

We now explain in detail all notions used in the tableau rules, most of which are suitably lifted variants of the definitions in [23, 30].

Ensuring entailments. Many rules need to *ensure* that a label $\mathcal{L}(x)$ entails an order assertion ϕ. We describe this process here in detail (see also the definition of complex order concepts in Section 5):

1. If ϕ is already entailed by $\mathcal{L}(x)$, then do nothing.

[8] Recall that each label $\mathcal{L}(x)$ implicitly contains the shared assertions from $\mathcal{L}_o$.

2. If ϕ is a simple order assertion, i.e. it does not use $\Rightarrow$ or min, then add it to $\mathcal{L}(x)$ (or $\mathcal{L}_o$).

3. If ϕ is of the form $\alpha \bowtie \beta$, where β is a complex expression, and $\mathcal{L}(x)$ already entails $\beta = \delta$ for some order expression δ that contains less occurrences of $\Rightarrow$ and min than β, then ensure that $\mathcal{L}(x)$ entails $\alpha \bowtie \delta$.

4. If ϕ is of the form $\alpha = \beta$, then ensure that $\mathcal{L}(x)$ entails $\alpha \leqslant \beta$ and $\alpha \geqslant \beta$.

5. If ϕ is of the form $\alpha \lhd \min\{\beta, \gamma\}$ with $\lhd \in \{<, \leqslant\}$, then ensure that $\mathcal{L}(x)$ entails both $\alpha \lhd \beta$ and $\alpha \lhd \gamma$.

6. If ϕ is of the form $\alpha \rhd \min\{\beta, \gamma\}$ with $\rhd \in \{>, \geqslant\}$, then ensure that $\mathcal{L}(x)$ entails either $\alpha \rhd \beta$ or $\alpha \rhd \gamma$.

7. If ϕ is of the form $\alpha \lhd \beta \Rightarrow \gamma$, then ensure that $\mathcal{L}(x)$ entails either $\alpha \lhd \gamma$, or both $\beta \leqslant \gamma$ and $\alpha \lhd 1$.

8. If ϕ is of the form $\alpha \rhd \beta \Rightarrow \gamma$, then ensure that $\mathcal{L}(x)$ entails either $\beta \leqslant \gamma$ and $\alpha \rhd 1$, or $\beta > \gamma$ and $\alpha \rhd \gamma$ (for $\rhd = >$, only the second option is feasible since all labels need to stay satisfiable).

Case 3 in particular deals with order assertions such as $\alpha \leqslant \min\{\beta, \gamma\}$ for which it is already known that $\beta \leqslant \gamma$: in this case, $\mathcal{L}(x)$ entails $\beta = \min\{\beta, \gamma\}$, and hence it suffices to assert that $\alpha \leqslant \beta$. This process ensures that only the minimal amount of information that is necessary to entail an order assertion is added to $\mathcal{L}(x)$. However, due to the semantics of the Gödel operators, it often includes nondeterministic choices.

Remark 6.6. In comparison with tableau algorithms for finitely valued FDLs [31, 32], one could wonder why our tableau rules do not just propagate lower and upper bounds of complex concepts to the subconcepts, but usually assert both directions. For illustration purposes, suppose that we use the following two rules instead of ($\sqcap$):

($\sqcap_<$)	IF $\mathcal{L}(x)$ entails $(C \sqcap D)(x) < \alpha$ for some $\alpha \in \mathcal{U}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(C \sqcap D)(x) \geqslant \min\{C(x), D(x)\}$.
($\sqcap_>$)	IF $\mathcal{L}(x)$ entails $(C \sqcap D)(x) > \alpha$ for some $\alpha \in \mathcal{U}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(C \sqcap D)(x) \leqslant \min\{C(x), D(x)\}$

This means that an order relation between $(C \sqcap D)(x)$ and $\min\{C(x), D(x)\}$ is only asserted if there is an actual need to do so, because there is some restriction on $(C \sqcap D)(x)$ that affects the values of the subconcepts C and D at x. This corresponds to the behavior of the rules with similar names in [31, 32].

However, this would mean that the rule ($\sqcap_<$) has to be applied also if there is a *lower* bound on $\min\{C(x), D(x)\}$, i.e. an $\alpha \in \mathcal{U}(x)$ such that $\alpha < \min\{C(x), D(x)\}$. Otherwise, this lower bound would not be transferred to $(C \sqcap D)(x)$ and any $\beta \in \mathcal{U}(x)$ with $\beta \geqslant (C \sqcap D)(x)$; but the presence of a lower bound for $\beta = (\forall r.C)(x)$ may be necessary to apply the rule ($\forall_1$), in the same way as for ($\sqcap_>$) above. Hence, we would actually need the following rules:

($\sqcap_<$)'	IF $\mathcal{L}(x)$ entails $(C \sqcap D)(x) < \alpha$ or $\min\{C(x), D(x)\} > \alpha$ for some $\alpha \in \mathcal{U}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(C \sqcap D)(x) \geqslant \min\{C(x), D(x)\}$.
($\sqcap_>$)'	IF $\mathcal{L}(x)$ entails $(C \sqcap D)(x) > \alpha$ or $\min\{C(x), D(x)\} < \alpha$ for some $\alpha \in \mathcal{U}(x)$, THEN ensure that $\mathcal{L}(x)$ entails $(C \sqcap D)(x) \leqslant \min\{C(x), D(x)\}$

In the above example, if $\min\{C(x), D(x)\}$ has a lower bound, this would be transferred via ($\sqcap_<$)' to $(C \sqcap D)(x)$, and hence ($\sqcap_>$)' also has to be applied. This means that any rule application of either ($\sqcap_<$)' or ($\sqcap_>$)' would automatically apply both. Rather than having two rule applications, we write them compactly as ($\sqcap$). Similar arguments apply to other constructors, such as $\to$, $\forall r.C$, and $\geqslant n\, r.C$. However, the rules ($\neg\geqslant$), (NN), and ($\neg\geqslant_o$) have conditions that implicitly state the existence of an upper bound. Similarly, the rule (ch_o) first checks if there are upper or lower bounds on the value of $\{a\}(x)$.

Blocking and safe neighbors. We adapt the notion of blocking from [30] to sets of order assertions. A node x is *directly blocked* if it has ancestors x', y, and y' such that

- $(x', x), (y', y) \in \mathcal{N}$;
- x, y, and all nodes on the path from y to x are blockable;
- for all order assertions ϕ over $\mathcal{U}(\Delta)$ involving only the nodes x and x', we have that $\mathcal{L}(x)$ entails ϕ iff $\mathcal{L}(y)$ entails $\sigma(\phi)$, where σ replaces x by y and x' by y'.

In this case, we say that *y blocks x.* A node is *blocked* if it is directly blocked or it is blockable and its predecessor is blocked. In the latter case, we say that it is *indirectly blocked.*

The rules $(\forall)$, $(\geqslant)$, and (NN) are called *generating*, and the rules $(\neg\geqslant)$, (o) and $(\neg\geqslant_o)$ are called *shrinking.* Note that generating rules are not applicable to any blocked nodes, but all the other rules may be applied to all nodes. The reason for this is that, due to inverse roles, by applying these rules to blocked nodes, the order assertions at unblocked nodes may change, possibly leading to a clash or the breaking of a blocking relation.

A neighbor y of a node x is *safe* if (i) x is blockable, or (ii) x is a nominal node and y is not blocked. It is important to make this distinction since only safe neighbors really count for the satisfaction of the witnessing conditions for value and number restrictions. The reason for this is that the blocked predecessor of nominal nodes do not produce individuals in the tableau that will be constructed from the completion graph.

Merging and pruning. It is sometimes necessary to merge nodes in order to satisfy the semantics of nominals or number restrictions. When merging the node y into x, we replace all occurrences of y in $\mathcal{L}(y)$ by x and add these assertions to $\mathcal{L}(x)$. Additionally, we modify the neighborhood of x such that it inherits all neighbors of y, and then remove y (and all blockable subtrees from this node) from the completion graph. We will also say that x is a *direct heir* of y. Formally, to *merge y into x*, we perform the following steps:

1. For all nodes $z \in \Delta$ with $(z, y) \in \mathcal{N}$, we replace (z, y) by (z, x) in $\mathcal{N}$.
2. For all nominal nodes $z \in \Delta$ with $(y, z) \in \mathcal{N}$, we replace (y, z) by (x, z) in $\mathcal{N}$.
3. We collect all blockable nodes $z \in \Delta$ with $(y, z) \in \mathcal{N}$ into the set Z; these nodes will be removed from the completion graph together with y.
4. For all order assertions $\phi \in \mathcal{L}(y)$ that do not involve nodes from Z, we add $\sigma(\phi)$ to $\mathcal{L}(x)$, where σ is a substitution that replaces y by x.[9]
5. For all nodes $z \in \Delta$ with $y \neq z$, we add $x \neq z$.
6. We prune y from the completion graph,

where the operation of *pruning y* is defined recursively as follows:

1. For all nodes $z \in \Delta$ with $(y, z) \in \mathcal{N}$, we remove (y, z) from $\mathcal{N}$ and, if z is blockable, prune z from the completion graph.
2. We remove y from Δ and remove all order assertions involving y from all labels.

The tree-like structure of the blockable parts of a completion graph ensures that pruning removes only subtrees, but no ancestors of y.

[9]If x is a nominal node, then $\sigma(\phi)$ may be an order assertion over $\mathcal{U}_o$. In this case, it is added to $\mathcal{L}_o$, and hence becomes shared knowledge.

Example 6.7. We continue with Example 6.5. As described in Section 3, the rule (ch) will force x to choose whether $(\geqslant 2\, r.C)(a) < \min\{r(a,x), C(x)\}$ holds, and similarly for the node y introduced as a witness for $(\exists r.B)(a)$. If we choose "<" for both nodes, then $(\neg\geqslant)$ has to be applied to merge x and y. Since neither x nor y have $\mathcal{N}$-successors, we simply rename x to y and join the sets $\mathcal{L}(x)$ and $\mathcal{L}(y)$. Note that $(\geqslant)$ enforces the existence of at least one other node z for which $(\geqslant 2\, r.C)(a) \leqslant \min\{r(a,z), C(z)\}$ holds, which we ignored in Section 3.

The rules (NN) and $(\neg\geqslant_o)$ are not relevant for our running example; they come into play when a nominal node has a blockable $\mathcal{N}$-predecessor (see [30] for details).

Strategy of rule applications. The *level* of a nominal node is defined as follows: Every nominal node x where $\mathcal{L}(x)$ entails $\{a\}(x) \geqslant 1$ for some $a \in \mathsf{ind}(\mathcal{O})$ (e.g. one of the initial nominal nodes) is of level 0; and a nominal node is of level $i > 0$ if it is not of some lower level $j < i$ and has a neighbor that is of level $i-1$. Note that merging can only decrease the levels of nodes, but not increase them.

We use a similar strategy of rule applications as in [30], i.e. the priority order between rules is as follows:

1. $(\rightsquigarrow)$
2. (ch_o) and (o)
3. (NN) and $(\neg\geqslant_o)$ (first applied to nominal nodes with lower levels)
4. all other rules.

Now that we have introduced all relevant definitions, we can prove termination and correctness of the algorithm.

Theorem 6.8. *Every sequence of applications of the tableau rules to $\mathcal{G}_0$ terminates. Moroever, if the tableau rules can be applied to $\mathcal{G}_0$ in such a way that a complete and clash-free completion graph is obtained, then $\mathcal{O}$ has a countable tableau. Finally, if $\mathcal{O}$ has a tableau, then the tableau rules can be applied to $\mathcal{G}_0$ in such a way that a complete and clash-free completion graph is obtained.*

Proof sketch. Termination can be shown using similar arguments to the ones in [30]. The key observation is that the tableau algorithm can generate at most $2(o+2m+4k+2)^2$ order assertions, where $m := |\mathsf{sub}(\mathcal{O})|$, $k := |\mathsf{rol}(\mathcal{O})|$, and $o := |\mathsf{val}(\mathcal{O})|$. This means that blocking limits the length of chains of (blockable) role successors to $\lambda := 2^{2(o+2m+4k+2)^2}+1$. Furthermore, the rules (NN) and $(\neg\geqslant_o)$ ensure that there are at most $O(\ell(mn)^\lambda)$ nominal nodes, where $\ell := |\mathsf{ind}(\mathcal{O})|$.

From any complete and clash-free completion graph, we obtain a countable tableau by *unraveling* the neighbor relation between the nodes along the blocking relation. That is, we iteratively replace each blocked node x with a copy of an ancestor that blocks it, together with all blockable successors of that ancestor. While it is easy to verify the tableau conditions, some work has to be put into proving that the resulting set of order assertions is still satisfiable. By Lemma 2.2, unsatisfiability must be caused by a $<$-cycle in the induced graph, and we can show that such a cycle cannot exist due to the tableau rules.

Finally, we can use any tableau for $\mathcal{O}$ to guide the application of the tableau rules in such a way that no clash is obtained. Due to termination, at some point this must result in a complete and clash-free completion graph. □

The bound on the number of nodes is triply exponential in the size of $\mathcal{O}$, and hence this proves a 3-NExpTime upper bound on the complexity of consistency in G-$\mathcal{SROIQ}$, which is the same bound that is obtained from the classical tableau algorithm for $\mathcal{SROIQ}$ [23]. This is in contrast to 2-NExpTime-completeness of classical $\mathcal{SROIQ}$, which is shown in [26] using a reduction to the two-variable fragment of first-order logic with counting quantifiers. The 2-NExpTime-hardness can be transferred to our setting via the reduction in [7].

7. Related Work

Research on FDLs started with the papers [31, 33, 34], which extended classical $\mathcal{ALC}$ with the traditional fuzzy semantics introduced by Zadeh [9], based on the minimum function to interpret conjunction, but using an implication function different from the residuum. It was shown early on, however, that this choice for interpreting the DL constructors leads to essentially *finitely valued* reasoning: One can restrict the considerations to the values occurring in the ontologies plus their involutive negations [11, 31]. In [22], Mathematical Fuzzy Logic was introduced to FDLs, and with it the infinitely valued t-norms and residua forming the Gödel, Łukasiewicz, and Product semantics, as well as finitely valued variants of the former two. Unfortunately, expressive FDLs under infinite Łukasiewicz or Product semantics turned out to be undecidable [7, 35, 36]. Reasoning procedures developed for the remaining FDLs can be classified into two mainstream branches: crispification approaches that reduce finitely valued ontologies to classical ontologies, and tableau algorithms for infinitely valued FDLs where the TBox is restricted to be *unfoldable* (intuitively, forbidding cyclic relationships between concepts such as $A \sqsubseteq \exists r.A$).

The former approach was pioneered in [11] for $\mathcal{ALC}$ under Zadeh semantics, and extended to lattice-based Zadeh semantics [37], $\mathcal{SROIQ}$ under finitely valued Łukasiewicz [28] and Gödel semantics [38], $\mathcal{ALCH}$ with combinations of finite Gödel and Łukasiewicz semantics [39, 40], and $\mathcal{SROIQ}$ with arbitrary finite t-norms [12]. Some of these reductions haven been implemented in the *DeLorean* system [41]. Although most of them incur an exponential blowup in the size of the ontology, it was recently shown in [13] that this blowup can be avoided through a pre-processing step. In that paper, it was also pointed out that the reduction for number restrictions proposed in [12, 28] for finitely valued Łukasiewicz semantics was incorrect; so far, the only known correct reduction encodes number restrictions using non-standard *role disjunctions* [13]. Our own reduction for infinitely valued Gödel FDLs is based on similar ideas as these crispifications, but incurs different exponential blowups (see Section 5).

At the same time, tableau algorithms have been developed for FDLs with unfoldable TBoxes by combining classical tableau algorithms with mixed integer optimization problems. As in the classical case, they usually do not allow to derive tight complexity bounds. Tableau algorithms were demonstrated for $\mathcal{ALC}$ under Łukasiewicz [42] and Product semantics [43], and $\mathcal{SI}$ [44] and $\mathcal{ALC}$ [45, 46] with arbitrary t-norms. A tableau algorithm for Zadeh, Gödel, and Łukasiewicz semantics has been implemented in the *fuzzyDL* reasoner, which uses an external library to solve the optimization problems [47]. The tableau algorithm presented in [48] for $\mathcal{ALC}$ under Łukasiewicz semantics can also deal with order assertions, and even linear inequalities over assertions, but does not consider TBoxes. Similar order assertions have been used to decide propositional Gödel logic, albeit without involutive negation [24].

The traditional Zadeh semantics takes up a special position in FDL research, as it underlies the only existing tableau algorithms that can deal with full GCIs and the expressivity of $\mathcal{SHOIQ}$, without needing to solve mixed integer optimization problems [32, 49–51]. This is due to the fact that reasoning can be restricted to finitely many values, as described earlier. In contrast, our algorithms deal with truly infinitely valued FDLs via the abstraction of order assertions. In addition, it is straightforward to extend them to the Zadeh semantics, obtaining an algorithm for a combined logic as in the crispification of [12].

8. Conclusions

In this paper, we have presented two algorithms for reasoning in (fragments of) fuzzy $\mathcal{SROIQ}$ under Gödel semantics. To the best of our knowledge, these are the first crispification and tableau methods for FDLs that can deal with infinitely valued semantics and general concept inclusions. The two procedures are based on an insight that was first observed in the context of an automata-based method [10]: that it is possible to abstract from the precise degrees associated to concepts and roles, and consider only their relative order when deciding consistency of an ontology. However, the new methods presented in this paper are better suited for implementing an efficient reasoner than the automata construction from [10]. Through the reduction to classical description logics, it is possible to use available highly-optimized reasoners (e.g. [52, 53]), but at the cost of incurring in an exponential blowup whenever the input (fuzzy) ontology uses both nominals and number restrictions. In addition, it includes an explicit representation of the exponentially large automata $\mathbf{A}_r$ constructed to recognize relevant role chains. This means that further work needs to be done before the reduction approach can be used for efficient reasoning in expressive FDLs under Gödel semantics.

In contrast, the tableau approach considers only those parts of the automata that are needed to extend the completion graph, and hence they can be constructed on-the-fly. This algorithm exhibits a larger amount of nondeterminism than its classical counterpart [23, 30], but this is inherent in the Gödel semantics and cannot be avoided. For a practical implementation, it is crucial to use a fast algorithm for deciding entailments of order assertions, as well as incorporate optimizations developed for classical tableau algorithms [54], where possible. Further *ad hoc* optimizations and data structures will also need to be developed to obtain an efficient implementation of this approach.

While the practical applicability of these approaches still needs to be confirmed, we believe that they represent a large step in the direction of efficient reasoning in expressive fuzzy DLs. Indeed, with the help of order structures, we obtain methods that closely resemble those currently in use for classical DLs. This similarity is an asset that allows developers to focus only on the new features, while exploiting the progress made in classical DLs over the last decade.

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References

[1] F. Baader, D. Calvanese, D. L. McGuinness, D. Nardi, P. F. Patel-Schneider (Eds.), The Description Logic Handbook: Theory, Implementation, and Applications, 2nd Edition, Cambridge University Press, 2007.

[2] W3C OWL Working Group, OWL 2 web ontology language: Document overview, W3C recommendation, W3C, available at `http://www.w3.org/TR/owl2-overview/` (Oct. 2012).

[3] P. Hájek, Metamathematics of Fuzzy Logic (Trends in Logic), Springer-Verlag, 2001.

[4] T. Lukasiewicz, U. Straccia, Managing uncertainty and vagueness in description logics for the semantic web, Journal of Web Semantics 6 (4) (2008) 291–308. `doi:10.1016/j.websem.2008.04.001`.

[5] P. Cintula, P. Hájek, C. Noguera (Eds.), Handbook of Mathematical Fuzzy Logic, Vol. 37–38 of Studies in Logic, College Publications, 2011, 2 volumes.

[6] F. Bobillo, M. Cerami, F. Esteva, À. García-Cerdaña, R. Peñaloza, U. Straccia, Fuzzy description logic, in: P. Cintula, C. G. Fermüller, C. Noguera (Eds.), Handbook of Mathematical Fuzzy Logic, Vol. 3 of Studies in Logic, College Publications, 2016.

[7] S. Borgwardt, F. Distel, R. Peñaloza, The limits of decidability in fuzzy description logics with general concept inclusions, Artificial Intelligence 218 (2015) 23–55. `doi:10.1016/j.artint.2014.09.001`.

[8] K. Gödel, Zum intuitionistischen Aussagenkalkül, Anzeiger der Akademie der Wissenschaften in Wien 69 (1932) 65–66.

[9] L. A. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338–353. `doi:10.1016/S0019-9958(65)90241-X`.

[10] S. Borgwardt, F. Distel, R. Peñaloza, Decidable Gödel description logics without the finitely-valued model property, in: C. Baral, G. De Giacomo, T. Eiter (Eds.), Proc. of the 14th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'14), AAAI Press, 2014, pp. 228–237.
URL `http://www.aaai.org/ocs/index.php/KR/KR14/paper/view/7803`

[11] U. Straccia, Transforming fuzzy description logics into classical description logics, in: J. J. Alferes, J. A. Leite (Eds.), Proc. of the 9th Eur. Conf. on Logics in Artificial Intelligence (JELIA'04), Vol. 3229 of Lecture Notes in Computer Science, Springer-Verlag, 2004, pp. 385–399. `doi:10.1007/978-3-540-30227-8_33`.

[12] F. Bobillo, M. Delgado, J. Gómez-Romero, U. Straccia, Joining Gödel and Zadeh fuzzy logics in fuzzy description logics, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 20 (4) (2012) 475–508. `doi:10.1142/S0218488512500249`.

[13] S. Borgwardt, T. Mailis, R. Peñaloza, A.-Y. Turhan, Answering fuzzy conjunctive queries over finitely valued fuzzy ontologies, Journal on Data SemanticsIn press. DOI: 10.1007/s13740-015-0055-y. `doi:10.1007/s13740-015-0055-y`.

[14] S. Borgwardt, R. Peñaloza, Reasoning in fuzzy description logics using automata, Fuzzy Sets and SystemsIn press. `doi:10.1016/j.fss.2015.07.013`.

[15] S. Borgwardt, R. Peñaloza, Reasoning in expressive description logics under infinitely valued Gödel semantics, in: C. Lutz, S. Ranise (Eds.), Proc. of the 10th Int. Symp. on Frontiers of Combining Systems (FroCoS'15), Vol. 9322 of Lecture Notes in Artificial Intelligence, Springer-Verlag, 2015, pp. 49–65. `doi:10.1007/978-3-319-24246-0_4`.

[16] D. Calvanese, T. Eiter, M. Ortiz, Regular path queries in expressive description logics with nominals, in: C. Boutilier (Ed.), Proc. of the 21st Int. Joint Conf. on Artificial Intelligence (IJCAI'09), AAAI Press, 2009, pp. 714–720.
URL http://ijcai.org/papers09/Papers/IJCAI09-124.pdf
[17] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Trends in Logic, Studia Logica Library, Springer-Verlag, 2000.
[18] I. Horrocks, U. Sattler, Decidability of $\mathcal{SHIQ}$ with complex role inclusion axioms, Artificial Intelligence 160 (1–2) (2004) 79–104. doi:10.1016/j.artint.2004.06.002.
[19] F. Bobillo, M. Delgado, J. Gómez-Romero, Optimizing the crisp representation of the fuzzy description logic $\mathcal{SROIQ}$, in: P. C. G. da Costa, C. d'Amato, N. Fanizzi, K. B. Laskey, K. J. Laskey, T. Lukasiewicz, M. Nickles, M. Pool (Eds.), Uncertainty Reasoning for the Semantic Web I, Vol. 5327 of Lecture Notes in Artificial Intelligence, Springer-Verlag, 2008, pp. 189–206. doi:10.1007/978-3-540-89765-1_12.
[20] I. Horrocks, U. Sattler, S. Tobies, Practical reasoning for very expressive description logics, Logic Journal of the Interest Group in Pure and Applied Logic 8 (3) (2000) 239–263. doi:10.1093/jigpal/8.3.239.
[21] F. Bobillo, M. Delgado, J. Gómez-Romero, A crisp representation for fuzzy $\mathcal{SHOIN}$ with fuzzy nominals and general concept inclusions, in: P. C. G. da Costa, C. d'Amato, N. Fanizzi, K. B. Laskey, K. J. Laskey, T. Lukasiewicz, M. Nickles, M. Pool (Eds.), Uncertainty Reasoning for the Semantic Web I, Vol. 5327 of Lecture Notes in Artificial Intelligence, Springer-Verlag, 2008, pp. 174–188. doi:10.1007/978-3-540-89765-1_11.
[22] P. Hájek, Making fuzzy description logic more general, Fuzzy Sets and Systems 154 (1) (2005) 1–15. doi:10.1016/j.fss.2005.03.005.
[23] I. Horrocks, O. Kutz, U. Sattler, The even more irresistible $\mathcal{SROIQ}$, in: P. Doherty, J. Mylopoulos, C. Welty (Eds.), Proc. of the 10th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'06), AAAI Press, 2006, pp. 57–67.
URL http://www.aaai.org/Library/KR/2006/kr06-009.php
[24] D. Guller, On the satisfiability and validity problems in the propositional Gödel logic, Computational Intelligence 399 (2012) 211–227. doi:10.1007/978-3-642-27534-0_14.
[25] M. Droste, W. Kuich, H. Vogler, Handbook of Weighted Automata, 1st Edition, Springer-Verlag, 2009.
[26] Y. Kazakov, $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are harder than $\mathcal{SHOIQ}$, in: G. Brewka, J. Lang (Eds.), Proc. of the 11th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'08), AAAI Press, 2008, pp. 274–284.
URL http://www.aaai.org/Library/KR/2008/kr08-027.php
[27] S. Tobies, The complexity of reasoning with cardinality restrictions and nominals in expressive description logics, Journal of Artificial Intelligence Research 12 (2000) 199–217. doi:10.1613/jair.705.
[28] F. Bobillo, U. Straccia, Reasoning with the finitely many-valued Łukasiewicz fuzzy description logic $\mathcal{SROIQ}$, Information Sciences 181 (2011) 758–778. doi:10.1016/j.ins.2010.10.020.
[29] K. Schild, A correspondence theory for terminological logics: Preliminary report, in: J. Mylopoulos, R. Reiter (Eds.), Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI'91), Morgan Kaufmann, 1991, pp. 466–471.
URL http://ijcai.org/Past%20Proceedings/IJCAI-91-VOL1/PDF/072.pdf
[30] I. Horrocks, U. Sattler, A tableau decision procedure for $\mathcal{SHOIQ}$, Journal of Automated Reasoning 39 (3) (2007) 249–276. doi:10.1007/s10817-007-9079-9.
[31] U. Straccia, Reasoning within fuzzy description logics, Journal of Artificial Intelligence Research 14 (2001) 137–166. doi:10.1613/jair.813.
[32] G. Stoilos, U. Straccia, G. B. Stamou, J. Z. Pan, General concept inclusions in fuzzy description logics, in: G. Brewka, S. Coradeschi, A. Perini, P. Traverso (Eds.), Proc. of the 17th Eur. Conf. on Artificial Intelligence (ECAI'06), IOS Press, 2006, pp. 457–461.
URL http://ebooks.iospress.nl/publication/2731
[33] C. B. Tresp, R. Molitor, A description logic for vague knowledge, in: H. Prade (Ed.), Proc. of the 13th Eur. Conf. on Artificial Intelligence (ECAI'98), John Wiley and Sons, 1998, pp. 361–365.
[34] U. Straccia, A fuzzy description logic, in: Proc. of the 15th Nat. Conf. on Artificial Intelligence (AAAI'98), 1998, pp. 594–599.
URL http://www.aaai.org/Library/AAAI/1998/aaai98-084.php
[35] F. Baader, R. Peñaloza, On the undecidability of fuzzy description logics with GCIs and product t-norm, in: C. Tinelli, V. Sofronie-Stokkermans (Eds.), Proc. of the 8th Int. Symp. on Frontiers of Combining Systems (FroCoS'11), Vol. 6989 of Lecture Notes in Computer Science, Springer-Verlag, 2011, pp. 55–70. doi:10.1007/978-3-642-24364-6_5.
[36] M. Cerami, U. Straccia, On the (un)decidability of fuzzy description logics under Łukasiewicz t-norm, Information Sciences 227 (2013) 1–21. doi:10.1016/j.ins.2012.11.019.
[37] U. Straccia, Description logics over lattices, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 14 (1) (2006) 1–16. doi:10.1142/S0218488506003807.
[38] F. Bobillo, M. Delgado, J. Gómez-Romero, U. Straccia, Fuzzy description logics under Gödel semantics, International Journal of Approximate Reasoning 50 (3) (2009) 494–514. doi:10.1016/j.ijar.2008.10.003.
[39] F. Bobillo, U. Straccia, Finite fuzzy description logics: A crisp representation for finite fuzzy $\mathcal{ALCH}$, in: F. Bobillo, R. Carvalho, P. C. G. da Costa, C. D'Amato, N. Fanizzi, K. B. Laskey, K. J. Laskey, T. Lukasiewicz, T. Martin, M. Nickles, M. Pool (Eds.), Proc. of the 6th Int. Workshop on Uncertainty Reasoning for the Semantic Web (URSW'10), Vol. 654 of CEUR Workshop Proceedings, 2010, pp. 61–72.
URL http://ceur-ws.org/Vol-654/paper6.pdf
[40] F. Bobillo, U. Straccia, Finite fuzzy description logics and crisp representations, in: F. Bobillo, P. C. G. da Costa, C. d'Amato, N. Fanizzi, K. Laskey, K. Laskey, T. Lukasiewicz, M. Nickles, M. Pool (Eds.), Uncertainty Reasoning for the Semantic Web II, Vol. 7123 of Lecture Notes in Computer Science, Springer-Verlag, 2013, pp. 102–121. doi:

10.1007/978-3-642-35975-0_6.
[41] F. Bobillo, M. Delgado, J. Gómez-Romero, DeLorean: A reasoner for fuzzy OWL 2, Expert Systems with Applications 39 (1) (2012) 258–272. doi:10.1016/j.eswa.2011.07.016.
[42] U. Straccia, F. Bobillo, Mixed integer programming, general concept inclusions and fuzzy description logics, Mathware & Soft Computing 14 (3) (2007) 247–259.
URL http://ic.ugr.es/Mathware/index.php/Mathware/article/view/21
[43] F. Bobillo, U. Straccia, A fuzzy description logic with product t-norm, in: Proc. of the 2007 IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'07), IEEE Computer Society Press, 2007, pp. 1–6. doi:10.1109/FUZZY.2007.4295443.
[44] G. Stoilos, G. B. Stamou, A framework for reasoning with expressive continuous fuzzy description logics, in: B. C. Grau, I. Horrocks, B. Motik, U. Sattler (Eds.), Proc. of the 22nd Int. Workshop on Description Logics (DL'09), Vol. 477 of CEUR Workshop Proceedings, 2009.
URL http://ceur-ws.org/Vol-477/paper_5.pdf
[45] F. Bobillo, U. Straccia, Fuzzy description logics with general t-norms and datatypes, Fuzzy Sets and Systems 160 (23) (2009) 3382–3402. doi:10.1016/j.fss.2009.03.006.
[46] V. Haarslev, H.-I. Pai, N. Shiri, A formal framework for description logics with uncertainty, International Journal of Approximate Reasoning 50 (9) (2009) 1399–1415. doi:10.1016/j.ijar.2009.04.009.
[47] F. Bobillo, U. Straccia, The fuzzy ontology reasoner fuzzyDL, Knowledge-Based Systems 95 (2016) 12–34. doi:10.1016/j.knosys.2015.11.017.
[48] A. Kułacka, D. Pattinson, L. Schröder, Syntactic labelled tableaux for Łukasiewicz fuzzy $\mathcal{ALC}$, in: F. Rossi (Ed.), Proc. of the 23rd Int. Joint Conf. on Artificial Intelligence (IJCAI'13), AAAI Press, 2013, pp. 762–768.
URL http://ijcai.org/papers13/Abstracts/147.html
[49] G. Stoilos, G. B. Stamou, J. Z. Pan, V. Tzouvaras, I. Horrocks, Reasoning with very expressive fuzzy description logics, Journal of Artificial Intelligence Research 30 (2007) 273–320. doi:10.1613/jair.2279.
[50] G. Stoilos, G. B. Stamou, S. D. Kollias, Reasoning with qualified cardinality restrictions in fuzzy description logics, in: Proc. of the 2008 IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'08), IEEE Computer Society Press, 2008, pp. 637–644. doi:10.1109/FUZZY.2008.4630436.
[51] G. Stoilos, G. B. Stamou, Reasoning with fuzzy extensions of OWL and OWL 2, Knowledge and Information Systems 40 (2014) 205–242. doi:10.1007/s10115-013-0641-y.
[52] B. Glimm, I. Horrocks, B. Motik, G. Stoilos, Z. Wang, HermiT: An OWL 2 reasoner, Journal of Automated Reasoning 53 (2014) 245–269. doi:10.1007/s10817-014-9305-1.
[53] A. Steigmiller, T. Liebig, B. Glimm, Konclude: System description, Journal of Web Semantics 27–28 (2014) 78–85. doi:10.1016/j.websem.2014.06.003.
[54] A. Steigmiller, T. Liebig, B. Glimm, Extended caching, backjumping and merging for expressive description logics, in: B. Gramlich, D. Miller, U. Sattler (Eds.), Proc. of the 6th Int. Joint Conf. on Automated Reasoning (IJCAR'12), Vol. 7364 of Lecture Notes in Computer Science, Springer-Verlag, 2012, pp. 514–529. doi:10.1007/978-3-642-31365-3_40.

Appendix A. Proofs

Lemma 2.2. *A set Φ of order assertions over an order structure S is satisfiable iff G_Φ has no $<$-cycle.*

Proof. Assume that G_Φ has a $<$-cycle involving an element $\delta \in S$, and consider an arbitrary model $\preccurlyeq_*$ of Φ. By the definitions of E_Φ and $\mathsf{order}(S)$, for every edge $(\alpha, \lhd, \beta) \in E_\Phi$, we have $\alpha \lhd_* \beta$. Since both $\alpha \leqslant \beta < \gamma$ and $\alpha < \beta < \gamma$ imply $\alpha < \gamma$, this means that $\delta <_* \delta$, which is a contradiction.

Conversely, assume that G_Φ has no $<$-cycle. We eliminate all other cycles by considering the equivalence relation $\sim$ on S that is defined by $\alpha \sim \beta$ iff α and β are involved in a cycle in G_Φ. Note that such a cycle can only involve $\leqslant$-edges and we also consider cycles of length 0. Furthermore, since E_Φ is symmetric w.r.t. inv, we have $\alpha \sim \beta$ iff $\mathsf{inv}(\alpha) \sim \mathsf{inv}(\beta)$. We now construct the quotient graph $G_\sim := (S/_\sim, E_\sim)$, where

$$E_\sim := \{([\alpha]_\sim, [\beta]_\sim) \mid (\alpha, \lhd, \beta) \in E_\Phi,\ \alpha \not\sim \beta\}.$$

It is easy to verify that G_Φ is acyclic, and hence there exists a strict total order $<_\sim$ on $S/_\sim$ such that $[\alpha]_\sim <_\sim [\beta]_\sim$ whenever $(\alpha, \lhd, \beta) \in E_\Phi$ and $\alpha \not\sim \beta$. Since E_Φ is symmetric w.r.t. inv, we can assume w.l.o.g. that $<_\sim$ also has this property, i.e. that $[\alpha]_\sim <_\sim [\beta]_\sim$ iff $[\mathsf{inv}(\beta)]_\sim <_\sim [\mathsf{inv}(\alpha)]_\sim$. Thus, the relation

$$\preccurlyeq_* := \{(\alpha, \beta) \mid \alpha \sim \beta \text{ or } [\alpha]_\sim <_\sim [\beta]_\sim\}$$

is a total preorder on S.

For any edge $(\alpha, <, \beta) \in E_\Phi$, we have $\alpha \not\sim \beta$ and $[\alpha]_\sim <_\sim [\beta]_\sim$, and thus $\alpha \prec_* \beta$. Similarly, if $(\alpha, \leqslant, \beta) \in E_\Phi$, then either $\alpha \sim \beta$ or $[\alpha]_\sim <_\sim [\beta]_\sim$, and hence $\alpha \preccurlyeq_* \beta$. By the definition of E_Φ, this shows that $\preccurlyeq_*$ is an element of $\mathsf{order}(S)$ and satisfies all order assertions in Φ. $\square$

Lemma 4.4. *A* G*-interpretation* $\mathcal{I}$ *satisfies* $\mathcal{R}_h$ *iff for every* $r \in \mathsf{rol}(\mathcal{O})$*, every* $w \in \mathsf{rol}(\mathcal{O})^+$*, and all* $d, e \in \Delta^{\mathcal{I}}$*, it holds that*

$$w^{\mathcal{I}}(d,e) \Rightarrow r^{\mathcal{I}}(d,e) \geqslant (\|\mathbf{A}_r\|, w).$$

Proof. If $\mathcal{I}$ violates any $w \sqsubseteq r \geqslant p \in \mathcal{R}$, then there are $d, e \in \Delta^{\mathcal{I}}$ such that $w^{\mathcal{I}}(d,e) \Rightarrow r^{\mathcal{I}}(d,e) < p$. Since $(\|\mathbf{A}_r\|, w) \geqslant p$ by construction of $\mathbf{A}_r$, we get $w^{\mathcal{I}}(d,e) \Rightarrow r^{\mathcal{I}}(d,e) < (\|\mathbf{A}_r\|, w)$.

For the converse direction, assume that $\mathcal{I}$ satisfies $\mathcal{R}$, and let $r \in \mathsf{rol}(\mathcal{O})$, $w \in \mathsf{rol}(\mathcal{O})^+$, and $d, e \in \Delta^{\mathcal{I}}$. We prove the claim through well-founded induction on $\prec$. It suffices to show the claim for all role names r since $\mathbf{A}_{r^-}$ is a mirrored copy of $\mathbf{A}_r$.

If $(\|\mathbf{A}_r\|, w) = 0$ or $w^{\mathcal{I}}(d,e) = 0$, then the claim is trivially satisfied. If both values are > 0, then by the construction of $\mathbf{A}_r$ there must be

- a word $w' = r_1 \dots r_n \in \mathsf{rol}(\mathcal{O})^+$ such that $r_i \prec r$ or $r_i = r$ holds for all $1 \leqslant i \leqslant n$, and
- words $w_1, \dots, w_n \in \mathsf{rol}(\mathcal{O})^*$ such that $w = w_1 \dots w_n$ and
$$(\|\mathbf{A}_r\|, w) = \min\left\{(\|\mathbf{A}_r^1\|, w'), (\|\mathbf{A}_{r_1}\|, w_1), \dots, (\|\mathbf{A}_{r_n}\|, w_n)\right\} > 0, \tag{A.1}$$
where, if $r_i = r$, then $w_i = r$, and thus $(\|\mathbf{A}_{r_i}\|, w_i) = 1$. Since we have $(\|\mathbf{A}_{r_i}\|, w_i) > 0$, $1 \leqslant i \leqslant n$, we know by the construction of $\mathbf{A}_{r_i}$ that all w_i are non-empty.

Since $w = w_1 \dots w_n$, we have

$$w^{\mathcal{I}}(d,e) = \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}}} \min_{i=1}^{n} w_i^{\mathcal{I}}(d_{i-1}, d_i),$$

where we set $d_0 := d$ and $d_n := e$. For any such choice of $d_1, \dots, d_{n-1} \in \Delta^{\mathcal{I}}$, it holds that, if $r_i \prec r$, then $w_i^{\mathcal{I}}(d_{i-1}, d_i) \Rightarrow r_i^{\mathcal{I}}(d_{i-1}, d_i) \geqslant (\|\mathbf{A}_{r_i}\|, w_i)$, by the induction hypothesis. But this also holds for $r_i = r$ since then $w_i = r$. Hence, we obtain

$$\begin{aligned}
(w')^{\mathcal{I}}(d,e) &= \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}}} \min_{i=1}^{n} r_i^{\mathcal{I}}(d_{i-1}, d_i) \\
&\geqslant \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}}} \min_{i=1}^{n} \min\{w_i^{\mathcal{I}}(d_{i-1}, d_i),\ (\|\mathbf{A}_{r_i}\|, w_i)\} \\
&= \min\{w^{\mathcal{I}}(d,e), (\|\mathbf{A}_{r_1}\|, w_1), \dots, (\|\mathbf{A}_{r_n}\|, w_n)\}.
\end{aligned} \tag{A.2}$$

We proceed by a case distinction on the transitivity and symmetry properties of r in $\mathcal{R}$.

1. Assume that no role inclusions of the form $rr \sqsubseteq r \geqslant p$ or $r^- \sqsubseteq r \geqslant p$ occur in $\mathcal{R}$. Since $(\|\mathbf{A}_r^1\|, w') > 0$, by construction of $\mathbf{A}_r^1 = \mathbf{A}_r^0$ we know that w' is of the form $w' = u_1 \dots u_m t v_1 \dots v_k$ where
   - $u_i r \sqsubseteq r \geqslant p_i \in \mathcal{R}$ for all $1 \leqslant i \leqslant m$,
   - either $t \sqsubseteq r \geqslant p \in \mathcal{R}$ or $t = r$ (and then we set $p := 1$),
   - $r v_j \sqsubseteq r \geqslant p'_j \in \mathcal{R}$ for all $1 \leqslant j \leqslant k$, and
   - $(\|\mathbf{A}_r^1\|, w') = \min\{p_1, \dots, p_m, p, p'_1, \dots, p'_k\}$.

   Hence, we get
$$\begin{aligned}
r^{\mathcal{I}}(d,e) &\geqslant \min\{p'_k, (rv_k)^{\mathcal{I}}(d,e)\} \\
&= \min\left\{p'_k, \sup_{e'_k \in \Delta^{\mathcal{I}}} \min\{r^{\mathcal{I}}(d, e'_k), v_k^{\mathcal{I}}(e'_k, e)\}\right\} \\
&\dots \\
&\geqslant \min\Big\{p_1, \dots, p_m, p, p'_1, \dots, p'_k, \\
&\qquad \sup_{e_1,\dots,e_m,e'_1,\dots,e'_k \in \Delta^{\mathcal{I}}} \min\{u_1^{\mathcal{I}}(d, e_1), \dots, t^{\mathcal{I}}(e_m, e'_1), \dots, v_k^{\mathcal{I}}(e'_k, e)\}\Big\} \\
&= \min\{(\|\mathbf{A}_r^1\|, w'), (w')^{\mathcal{I}}(d,e)\}
\end{aligned}$$

   The claim now follows from this inequation together with (A.1) and (A.2).

2. Consider the case that $rr \sqsubseteq r \geqslant p_t \in \mathcal{R}$, but there is no role inclusion $r^- \sqsubseteq r \geqslant p \in \mathcal{R}$. Then w' must be of the form

$$w' = \big(u_1^{(1)}, \dots, u_{m_1}^{(1)} t^{(1)} v_1^{(1)}, \dots, v_{k_1}^{(1)}\big) \dots \big(u_1^{(\ell)}, \dots, u_{m_\ell}^{(\ell)} t^{(\ell)} v_1^{(\ell)}, \dots, v_{k_\ell}^{(\ell)}\big)$$

with $\ell \geqslant 1$ and

- $u_i^{(o)} r \sqsubseteq r \geqslant p_i^{(o)} \in \mathcal{R}$ for all $1 \leqslant o \leqslant \ell$ and $1 \leqslant i \leqslant m_o$,
- for each $1 \leqslant o \leqslant \ell$, either $t^{(o)} \sqsubseteq r \geqslant p^{(o)} \in \mathcal{R}$ or $t^{(o)} = r$ (and then we set $p^{(o)} := 1$),
- $rv_j^{(o)} \sqsubseteq r \geqslant (p'_j)^{(o)} \in \mathcal{R}$ for all $1 \leqslant o \leqslant \ell$ and $1 \leqslant j \leqslant k_o$, and
- $(\|\mathbf{A}_r^1\|, w') = \min\{p_t, p_0\}$ if $\ell > 1$, and $(\|\mathbf{A}_r^1\|, w') = p_0$ if $\ell = 1$, where

$$p_0 := \min\{p_i^{(o)}, p^{(o)}, (p'_j)^{(o)} \mid 1 \leqslant o \leqslant \ell,\ 1 \leqslant i \leqslant m_o,\ 1 \leqslant j \leqslant k_o\}.$$

The claim can be obtained by the same arguments as in Case 1. Note that the axiom $rr \sqsubseteq r \geqslant p_t$ is only needed if $\ell > 1$.

3. If $r^- \sqsubseteq r \geqslant p_s \in \mathcal{R}$, but there is no role inclusion $rr \sqsubseteq r \geqslant p \in \mathcal{R}$, then w' is of the form $w' = u_1 \dots u_m t v_1 \dots v_k$, where

- $u_i r \sqsubseteq r \geqslant p_i \in \mathcal{R}$ or $ru_i^- \sqsubseteq r \geqslant p_i \in \mathcal{R}$ for all $1 \leqslant i \leqslant m$,
- $t \sqsubseteq r \geqslant p \in \mathcal{R}$, $t^- \sqsubseteq r \geqslant p \in \mathcal{R}$, $t = r$, or $t = r^-$ (in the latter two cases we set $p := 1$),
- $rv_j \sqsubseteq r \geqslant p'_j \in \mathcal{R}$ or $v_j^- r \sqsubseteq r \geqslant p'_j \in \mathcal{R}$ for all $1 \leqslant j \leqslant k$, and
- $(\|\mathbf{A}_r^1\|, w') = \min\{p_s, p_0\}$ if one of the "inverse" cases applies, and $(\|\mathbf{A}_r^1\|, w') = p_0$ otherwise, where $p_0 := \min\{p_1, \dots, p_m, p, p'_1, \dots, p'_k\}$.

The claim can be obtained as in Case 1.

4. If both $rr \sqsubseteq r \geqslant p_t$ and $r^- \sqsubseteq r \geqslant p_s$ are present in $\mathcal{R}$, then w' is a non-empty sequence of words of the form described in Case 3, and the claim can be shown as before. $\square$

Lemma 5.5. *In* G-$\mathcal{SRIQ}$, G-$\mathcal{SROQ}$, *or* G-$\mathcal{SROI}$, $\mathcal{O}$ *has a* G*-model iff* $\mathsf{red}(\mathcal{O})$ *has a classical model.*

For readability, we prove the two directions of this claim in two separate lemmas.

Lemma A.1. *If* $\mathsf{red}(\mathcal{O})$ *has a classical model, then* $\mathcal{O}$ *has a* G*-model.*

Proof. Since $\mathsf{red}(\mathcal{O})$ contains only the role name $\mathfrak{r}$ and no inverses, and hence is in $\mathcal{ALCOQ}$, we can assume that it has a *quasi-forest model* $\mathcal{I}$ with the following properties [16]:

- $\Delta^{\mathcal{I}} \subseteq \mathsf{ind}(\mathcal{O}) \times \mathbb{N}^*$;
- for each $a \in \mathsf{ind}(\mathcal{O})$, the set $\{u \in \mathbb{N}^* \mid (a, u) \in \Delta^{\mathcal{I}}\}$ is prefix-closed;
- for each $a \in \mathsf{ind}(\mathcal{O})$, we have $a^{\mathcal{I}} = (a, \varepsilon)$;
- for all $a \in \mathsf{ind}(\mathcal{O})$, $u \in \mathbb{N}^*$, and $i \in \mathbb{N}$ with $(a, ui) \in \Delta^{\mathcal{I}}$, we have $((a, u), (a, ui)) \in \mathfrak{r}^{\mathcal{I}}$; and
- whenever $((a, u), (b, u')) \in \mathfrak{r}^{\mathcal{I}}$, then
  a) $a = b$ and $u' = ui$ for some $i \in \mathbb{N}$ or
  b) $u' = \varepsilon$.

We assume here that all named individuals in $\mathsf{ind}(\mathcal{O})$ are interpreted by distinct elements in $\mathcal{I}$. In general, we would have to consider sets of names from $\mathsf{ind}(\mathcal{O})$ as the roots of $\mathcal{I}$. Since this is relevant only for number restrictions and (in)equality assertions, we ignore this in the following and only mention it at the appropriate places.

For any $u = n_1 \dots n_k \in \mathbb{N}^*$ with $k \geqslant 1$, we denote by $u_\uparrow := n_1 \dots n_{k-1}$ its predecessor. We denote by $\preccurlyeq_\mathcal{A}$ the binary relation on $\mathcal{U}_\mathcal{A}$ defined by $\alpha \preccurlyeq_\mathcal{A} \beta$ iff $c^\mathcal{I} \in \boxed{\alpha \leqslant \beta}^\mathcal{I}$ for an arbitrary $c \in \mathsf{ind}(\mathcal{O})$. This is a total preorder due to the axioms in $\mathsf{red}(\mathcal{U})$. We similarly define $\alpha \preccurlyeq_u^a \beta$ iff $(a, u) \in \boxed{\alpha \leqslant \beta}^\mathcal{I}$, for all $\alpha, \beta \in \mathcal{U}$. Since $\mathcal{I}$ satisfies $\mathsf{red}(\mathcal{A})$ and all domain elements are connected via $\mathfrak{r}$, we have $\preccurlyeq_\mathcal{A} \subseteq \preccurlyeq_u^a$ for all $(a, u) \in \Delta^\mathcal{I}$. We further denote by $\equiv_\mathcal{A}$ ($\equiv_u^a$) the equivalence relation induced by $\preccurlyeq_\mathcal{A}$ ($\preccurlyeq_u^a$).

As a first step in the construction of a G-model of $\mathcal{O}$, we now construct a function $v \colon \mathcal{U}_\mathcal{A} \cup (\mathcal{U} \times \Delta^\mathcal{I}) \to [0, 1]$ such that

(P1) for all $p \in \mathsf{val}(\mathcal{O})$, we have $v(p) = p$,

(P2) for all $\alpha, \beta \in \mathcal{U}_\mathcal{A}$, we have $v(\alpha) \leqslant v(\beta)$ iff $\alpha \preccurlyeq_\mathcal{A} \beta$,

(P3) for all $\alpha \in \mathcal{U}_\mathcal{A}$, we have $v(\mathsf{inv}(\alpha)) = 1 - v(\alpha)$,

(P4) for every $C \in \mathsf{sub}(\mathcal{O})$ and all $a \in \mathsf{ind}(\mathcal{O})$, we have $v(C(a)) = v(C, a, \varepsilon)$,

(P5) for all $(a, u) \in \Delta^\mathcal{I}$,

   a) for all $p \in \mathsf{val}(\mathcal{O})$, we have $v(p, a, u) = p$,
   b) for all $\alpha, \beta \in \mathcal{U}$, we have $v(\alpha, a, u) \leqslant v(\beta, a, u)$ iff $\alpha \preccurlyeq_u^a \beta$,
   c) for all $\alpha \in \mathcal{U}$, we have $v(\mathsf{inv}(\alpha), a, u) = 1 - v(\alpha, a, u)$, and
   d) if $u \neq \varepsilon$, then for all $C \in \mathsf{sub}(\mathcal{O})$ we have $v(C, a, u_\uparrow) = v(\langle C \rangle_\uparrow, a, u)$.

We start defining v on $\mathcal{U}_\mathcal{A}$. Let $\mathcal{U}_\mathcal{A}/_{\equiv_\mathcal{A}}$ be the set of all equivalence classes of $\equiv_\mathcal{A}$. Then $\preccurlyeq_\mathcal{A}$ yields a total order $\leqslant_\mathcal{A}$ on $\mathcal{U}_\mathcal{A}/_{\equiv_\mathcal{A}}$. If $0 = p_0 < p_1 < \dots < p_{k-1} < p_k = 1$ are the elements of $\mathsf{val}(\mathcal{O})$, then since $\mathcal{I}$ satisfies $\mathsf{red}(\mathcal{U})$, we have

$$[0]_\mathcal{A} <_\mathcal{A} [p_1]_\mathcal{A} <_\mathcal{A} \dots <_\mathcal{A} [p_{k-1}]_\mathcal{A} <_\mathcal{A} [1]_\mathcal{A}$$

w.r.t. this order. For $[\alpha]_\mathcal{A} \in \mathcal{U}_\mathcal{A}/{\equiv_\mathcal{A}}$, we set $\mathsf{inv}([\alpha]_\mathcal{A}) := [\mathsf{inv}(\alpha)]_\mathcal{A}$, which is well-defined due to the axioms in $\mathsf{red}(\mathcal{U})$. On all $\alpha \in [p]_\mathcal{A}$ for $p \in \mathsf{val}(\mathcal{O})$, we now define $v(\alpha) := p$, which ensures that (P1) holds. For the equivalence classes that do not contain a value from $\mathsf{val}(\mathcal{O})$, note that by $\mathsf{red}(\mathcal{U})$, every such class must be strictly between $[p_i]_\mathcal{A}$ and $[p_{i+1}]_\mathcal{A}$ for some $p_i, p_{i+1} \in \mathsf{val}(\mathcal{O})$. We denote the n_i equivalence classes between $[p_i]_\mathcal{A}$ and $[p_{i+1}]_\mathcal{A}$ as follows:

$$[p_i]_\mathcal{A} <_\mathcal{A} E_1^i <_\mathcal{A} \dots <_\mathcal{A} E_{n_i}^i <_\mathcal{A} [p_{i+1}]_\mathcal{A}.$$

For every $\alpha \in E_j^i$, $1 \leqslant j \leqslant n_i$, we set $v(\alpha) := p_i + \frac{j}{n_i+1}(p_{i+1} - p_i)$, which ensures that (P2) is also satisfied. Furthermore, $1 - p_{i+1}$ and $1 - p_i$ are also adjacent in $\mathsf{val}(\mathcal{O})$ and we have

$$[1 - p_{i+1}]_\mathcal{A} <_\mathcal{A} \mathsf{inv}(E_{n_i}^i) <_\mathcal{A} \dots <_\mathcal{A} \mathsf{inv}(E_1^i) <_\mathcal{A} [1 - p_i]_\mathcal{A}$$

by the axioms in $\mathsf{red}(\mathcal{U})$. Hence, it follows from the definition of $v(\alpha)$ that (P3) holds.

We now construct the values of $v(\alpha, a, \varepsilon)$ using a similar technique. However, we now start by setting $v(\alpha, a, \varepsilon) := v(\alpha)$ for all $\alpha \in [\beta]_\varepsilon^a$ with $\beta \in \mathcal{U}_\mathcal{A}$. To see that this is well-defined, consider the case that $[\beta]_\varepsilon^a = [\beta']_\varepsilon^a$ for two different elements $\beta, \beta' \in \mathcal{U}_\mathcal{A}$. Since $\preccurlyeq_\varepsilon^a$ contains the preorder $\preccurlyeq_\mathcal{A}$, we know that $[\beta]_\mathcal{A} = [\beta']_\mathcal{A}$, and hence $v(\beta) = v(\beta')$ by (P2). We can now arrange all other values between those already fixed as shown above, thereby satisfying the first three conditions of (P5). Since $a^\mathcal{I} = (a, \varepsilon)$ and $\mathcal{I}$ satisfies $\mathsf{red}(\mathcal{A})$, this construction also ensures that (P4) is satisfied.

We now proceed to define $v(\alpha, a, u)$ by induction on the structure of the tree $\{u \mid (a, u) \in \Delta^\mathcal{I}\}$. Assume that $v(\alpha, a, u_\uparrow)$ has already been defined for all $\alpha \in \mathcal{U}$, and satisfies (P5)a)–c). By assumption, we have $((a, u_\uparrow), (a, u)) \in \mathfrak{r}^\mathcal{I}$, and by $\mathsf{red}(\mathsf{AN})$ we know that $(a, u) \in \mathsf{AN}^\mathcal{I}$. We again use the same construction as before, but this time fixing all values $v(\alpha,a,u) := v(\alpha,a,u_\uparrow)$ for all $\alpha \in \mathcal{U}_\mathcal{A}$ and $v(\alpha,a,u) := v(C,u_\uparrow)$ for all $C \in \mathsf{sub}(\mathcal{O})$ and all $\alpha \in [\langle C\rangle_\uparrow]_u$. This is well-defined by the same arguments as above and the fact that $\mathcal{I}$ satisfies $\mathsf{red}(\uparrow)$. We then fix the remaining values as before. This construction ensures that all the points in (P5) are satisfied.

Based on v, we now define the G-interpretation $\mathcal{I}_\mathsf{f}$ over the domain $\Delta^{\mathcal{I}_\mathsf{f}} := \Delta^\mathcal{I}$, where $a^{\mathcal{I}_\mathsf{f}} := a^\mathcal{I} = (a,\varepsilon)$ for all $a \in \mathsf{ind}(\mathcal{O})$[10] and $A^{\mathcal{I}_\mathsf{f}}(d) := v(A,d)$ for all $A \in \mathsf{N_C} \cap \mathsf{sub}(\mathcal{O})$ and $d \in \Delta^{\mathcal{I}_\mathsf{f}}$. The values for all other concept names are irrelevant and can be fixed arbitrarily. For all $r \in \mathsf{N_R} \cap \mathsf{rol}(\mathcal{O})$, we first define a "simple" interpretation $\mathcal{I}_\mathsf{f}^0$ as follows.

$$r^{\mathcal{I}_\mathsf{f}^0}((a,u),(b,u')) := \begin{cases} v(r,a,u') & \text{if } a=b \text{ and } u=u'_\uparrow,\\ v(r^-,a,u) & \text{if } a=b \text{ and } u'=u_\uparrow,\\ v(\exists r.\mathsf{Self},a,u) & \text{if } (a,u)=(b,u') \text{ and } r \text{ is simple},\\ v(r(a,b)) & \text{if } u=u'=\varepsilon \text{ and } a\neq b,\\ v(r(a,*),b,u') & \text{if } u=\varepsilon,\ u'\neq\varepsilon, \text{ and } ((b,u'),(a,\varepsilon))\in\mathfrak{r}^\mathcal{I},\\ v(r(*,b),a,u) & \text{if } u'=\varepsilon,\ u\neq\varepsilon, \text{ and } ((a,u),(b,\varepsilon))\in\mathfrak{r}^\mathcal{I},\\ 0 & \text{otherwise} \end{cases}$$

In the absence of inverse roles, we set the second and fifth lines to 0; and if (local) reflexivity is not allowed, then the third line is 0; finally, if there are no nominals in our source logic, then the fifth and sixth lines are 0. Due to $\mathsf{red}(\mathcal{A})$ and $\mathsf{red}(\exists r.\mathsf{Self})$, the same expressions as for role names can be used to evaluate inverse roles. In the case that r is simple, this already suffices. Otherwise, we use the automaton $\mathbf{A}_r$ to complete $\mathcal{I}_\mathsf{f}^0$ by additional links as follows: we set

$$r^{\mathcal{I}_\mathsf{f}}(d,e) := \sup_{w\in\mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|,w), w^{\mathcal{I}_\mathsf{f}^0}(d,e)\} \tag{A.3}$$

for all $d,e \in \Delta^{\mathcal{I}_\mathsf{f}}$. Note that this expression is also valid for simple roles r: by Proposition 4.5, we have $(\|\mathbf{A}_r\|,r) = 1$, $(\|\mathbf{A}_r\|,s) = p$ whenever $s \sqsubseteq_p r$, and $(\|\mathbf{A}_r\|,w) = 0$ for all other words w, and moreover $\mathsf{red}(\mathcal{R})$ yields

$$\min\{(\|\mathbf{A}_r\|,r), r^{\mathcal{I}_\mathsf{f}^0}(d,e)\} = r^{\mathcal{I}_\mathsf{f}^0}(d,e) \geqslant \min\{p, s^{\mathcal{I}_\mathsf{f}^0}(d,e)\} = \min\{(\|\mathbf{A}_r\|,s), s^{\mathcal{I}_\mathsf{f}^0}(d,e)\}.$$

The expression (A.3) can also be used to evaluate inverse roles due to the semantics of role chains and Proposition 4.3.

We now show by induction on the structure of C that

$$C^{\mathcal{I}_\mathsf{f}}(d) = v(C,d) \text{ for all } C \in \mathsf{sub}(\mathcal{O}) \text{ and } d \in \Delta^\mathcal{I}, \tag{A.4}$$

where we exclude the auxiliary concepts of the form $\forall\mathbf{A}.C$ and $\exists\mathbf{A}.C$.

For nominals $\{a\}$, we have $\{a\}^{\mathcal{I}_\mathsf{f}}(d) = 1$ if $d = (a,\varepsilon)$, and $\{a\}^{\mathcal{I}_\mathsf{f}}(d) = 0$ otherwise. By $\mathsf{red}(\{a\})$ and (P5)a)–b), in the former case we have $v(\{a\},d) = 1$, while in the latter case it holds that $v(\{a\},d) = 0$. For local reflexivity concepts $\exists r.\mathsf{Self}$, we have $(\exists r.\mathsf{Self})^{\mathcal{I}_\mathsf{f}}(d) = r^{\mathcal{I}_\mathsf{f}}(d,d) = r^{\mathcal{I}_\mathsf{f}^0}(d,d) = v(\exists r.\mathsf{Self},d)$ since r is simple. For $\neg C$, we have

$$(\neg C)^{\mathcal{I}_\mathsf{f}}(d) = 1 - C^{\mathcal{I}_\mathsf{f}}(d) = 1 - v(C,d) = v(\neg C,d)$$

by the induction hypothesis and (P5)c). The claim for $\top$, $\overline{q}$, $\sqcap$, and $\to$ can also be shown using the induction hypothesis, the semantics of these constructors, and the properties (P5)a)–b) of v.

For value restrictions $\forall r.C$, consider first the case that $d = (a,u)$ with $u \neq \varepsilon$, and hence $d \in \mathsf{AN}^\mathcal{I}$. By the second axiom in $\mathsf{red}(\forall r.C)$, one of the following must hold:

[10] If we are dealing with equivalence classes of individuals as the roots of $\mathcal{I}$, then $a^{\mathcal{I}_\mathsf{f}}$ is interpreted using the equivalence class containing a.

- There is an anonymous $\mathfrak{r}$-successor of d that satisfies $\boxed{\langle \forall r.C\rangle_{\uparrow} \geqslant r \Rightarrow C}$, which must be of the form (a, ui) for $i \in \mathbb{N}$ due to our assumption on the structure of $\mathcal{I}$. We get

$$v(\forall r.C, d) \geqslant v(r, a, ui) \Rightarrow v(C, a, ui) \geqslant r^{\mathcal{I}_{\mathsf{f}}}(d, (a, ui)) \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(a, ui).$$

- We have $d \in \boxed{\langle \forall r.C\rangle \geqslant r^- \Rightarrow \langle C\rangle_{\uparrow}}$, and thus (P5)b) and d) and the induction hypothesis yield

$$\begin{aligned} v(\forall r.C, d) \geqslant v(r^-, a, u) \Rightarrow v(\langle C\rangle_{\uparrow}, a, u) \\ &= r^{\mathcal{I}_{\mathsf{f}}^0}((a, u), (a, u_{\uparrow})) \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(a, u_{\uparrow}) \\ &\geqslant r^{\mathcal{I}_{\mathsf{f}}}(d, (a, u_{\uparrow})) \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(a, u_{\uparrow}). \end{aligned}$$

  Hence, $(a, u_{\uparrow})$ can take the role of the witness for $\forall r.C$ at d if we can show that the latter implication is $\geqslant v(\forall r.C, d)$ for all successors.

- The role r is simple and we have

$$v(\forall r.C, d) \geqslant v(\exists r.\mathsf{Self}, a, u) \Rightarrow v(C, a, u) \geqslant r^{\mathcal{I}_{\mathsf{f}}}(d, d) \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(d)$$

  by similar arguments as above. In this case, we can choose d itself as the witness.

- There is a $b \in \mathsf{ind}(\mathcal{O})$ such that $(d, (b, \varepsilon)) \in \mathfrak{r}^{\mathcal{I}}$, and again we have

$$v(\forall r.C, d) \geqslant v(r(*, b), a, u) \Rightarrow v(C(b), a, u) \geqslant r^{\mathcal{I}_{\mathsf{f}}}(d, (b, \varepsilon)) \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(b, \varepsilon)$$

  due to (P5)b) and d), $\mathsf{red}(\mathcal{A})$, and the induction hypothesis.

The assertions in $\mathsf{red}(\forall r.C)$ similarly ensure the existence of witnesses for $\forall r.C$ at all named domain elements. For the remainder of the claim, consider any $e \in \Delta^{\mathcal{I}_{\mathsf{f}}}$. Due to the first axiom in $\mathsf{red}(\forall r.C)$, we have

$$\begin{aligned} r^{\mathcal{I}_{\mathsf{f}}}(d, e) \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(e) &= \Big(\sup_{w \in \mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}_{\mathsf{f}}^0}(d, e)\}\Big) \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(e) \\ &= \inf_{w \in \mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}_{\mathsf{f}}^0}(d, e)\} \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(e) \\ &\overset{(*)}{\geqslant} v(\forall \mathbf{A}_r.C, d) \\ &\geqslant v(\forall r.C, d). \end{aligned}$$

as required, if we can show $(*)$, i.e. it remains to show that

$$\min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}_{\mathsf{f}}^0}(d_0, d_n)\} \Rightarrow C^{\mathcal{I}_{\mathsf{f}}}(d_n) \geqslant v(\forall \mathbf{A}_r.C, d_0)$$

holds for all $d_0, d_n \in \Delta^{\mathcal{I}_{\mathsf{f}}}$ and $w = r_1 \dots r_n \in \mathsf{rol}(\mathcal{O})^+$. Since $(\|\mathbf{A}_r\|, w)$ and $w^{\mathcal{I}_{\mathsf{f}}^0}(d_0, d_n)$ are defined as suprema, it suffices to consider any run $\mathsf{r} = (w_i, q_i)_{0 \leqslant i \leqslant m}$ with $(w_0, q_0) = (w, i_r)$, $(w_m, q_m) = (\varepsilon, f_r)$, and transitions $q_{i-1} \xrightarrow{x_i, p_i} q_i$ in $\mathbf{A}_r$, and any sequence $d_1, \dots, d_m \in \Delta^{\mathcal{I}_{\mathsf{f}}}$. To synchronize these two sequences, we define the mapping $\gamma\colon \{0, \dots, m\} \to \{0, \dots, n\}$, where $\gamma(0) := 0$, and

$$\gamma(i) := \begin{cases} \gamma(i-1) & \text{if } x_i = \varepsilon, \\ \gamma(i-1) + 1 & \text{if } x_i \neq \varepsilon. \end{cases}$$

Since $x_1 \dots x_m = w = r_1 \dots r_n$, we know that γ is surjective and $\gamma(m) = n$. We now show by induction on i that we have

$$v(\forall \mathbf{A}_r.C, d_0) \leqslant \min\Big\{ \min_{j=1}^{i} p_j, \min_{j=1}^{\gamma(i)} r_j^{\mathcal{I}_{\mathsf{f}}^0}(d_{j-1}, d_j)\Big\} \Rightarrow v(\mathbf{A}_r^{q_i}, d_{\gamma(i)}) \tag{A.5}$$

For all i, $0 \leqslant i \leqslant m$. By the axiom $\top \sqsubseteq \boxed{(\forall \mathbf{A}_r^{q_m}.C) \leqslant C}$ in $\mathsf{red}(\forall \mathbf{A}_r^{q_m}.C)$ and the induction hypothesis, the claim for m implies $(*)$.

For $i = 0$, (A.5) trivially holds. Assume now that it holds for $i-1$. To show the claim for i, by Proposition 2.1 it suffices to show that

$$v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) \leqslant p_i \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \tag{A.6}$$

whenever $x_i = \varepsilon$ (and hence $\gamma(i) = \gamma(i-1)$), and

$$v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) \leqslant \min\{p_i, r_{\gamma(i)}^{\mathcal{I}_\mathfrak{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \tag{A.7}$$

for all $x_i \neq \varepsilon$ (for which we have $\gamma(i) = \gamma(i-1)+1$).

For the former case, the axioms in $\mathsf{red}_{\varepsilon,p_i,q_i}(\forall \mathbf{A}_r^{q_m}.C)$ and (P5) directly yield the claim (A.6). If $x_i \notin \varepsilon$, we make a case distinction on $d_{\gamma(i)}$. If $r_{\gamma(i)}^{\mathcal{I}_\mathfrak{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)}) = 0$, then the claim is trivially satisfied; otherwise, we must have one of the following cases:

- If $d_{\gamma(i)} = (a, u)$ and $d_{\gamma(i-1)} = (a, u_\uparrow)$, then we have $(d_{\gamma(i-1)}, d_{\gamma(i)}) \in \mathfrak{r}^{\mathcal{I}}$ and $d_{\gamma(i)} \in \mathsf{AN}^{\mathcal{I}}$. Hence, the first axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$ and (P5) yield

$$\begin{aligned} v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &= v(\langle \forall \mathbf{A}_r^{q_{i-1}}.C\rangle_\uparrow, d_{\gamma(i)}) \\ &\leqslant \min\{p, v(r_{\gamma(i)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \\ &= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathfrak{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}), \end{aligned}$$

  as claimed in (A.7).

- If $d_{\gamma(i-1)} = (a, u)$ and $d_{\gamma(i)} = (a, u_\uparrow)$, then $d_{\gamma(i-1)} \in \mathsf{AN}^{\mathcal{I}}$ and inverse roles are allowed. Hence, the second axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$ and (P5) yield

$$\begin{aligned} v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &\leqslant \min\{p, v(r_{\gamma(i)}^-, d_{\gamma(i-1)})\} \Rightarrow v(\langle \forall \mathbf{A}_r^{q_i}.C\rangle_\uparrow, d_{\gamma(i-1)}) \\ &= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathfrak{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}). \end{aligned}$$

- If $d_{\gamma(i-1)} = d_{\gamma(i)}$, then $r_{\gamma(i)}$ is simple and local reflexivity is allowed, which means that we similarly get

$$\begin{aligned} v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &\leqslant \min\{p, v(\exists r_{\gamma(i)}.\mathsf{Self}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \\ &= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathfrak{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \end{aligned}$$

  by the third axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$.

- If $d_{\gamma(i-1)} = (a, \varepsilon)$ and $d_{\gamma(i)} = (b, \varepsilon)$, then $((a,\varepsilon),(b,\varepsilon)) \in \mathfrak{r}^{\mathcal{I}}$, and thus

$$\begin{aligned} v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &\leqslant \min\{p, v(r_{\gamma(i)}(*, b), d_{\gamma(i-1)})\} \Rightarrow v((\forall \mathbf{A}_r^{q_i}.C)(b), d_{\gamma(i-1)}) \\ &= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathfrak{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \end{aligned}$$

  by the corresponding axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$.

- If $d_{\gamma(i-1)} = (a, \varepsilon)$, $d_{\gamma(i)} \in \mathsf{AN}^{\mathcal{I}}$, and $(d_{\gamma(i)}, d_{\gamma(i-1)}) \in \mathfrak{r}^{\mathcal{I}}$, then nominals and inverse roles are allowed and

$$\begin{aligned} v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &= v((\forall \mathbf{A}_r^{q_{i-1}})(a), d_{\gamma(i)}) \\ &\leqslant \min\{p, v(r_{\gamma(i)}^-(*, a), d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \\ &= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathfrak{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}). \end{aligned}$$

- Finally, the case that $d_{\gamma(i-1)} \in \mathsf{AN}^{\mathcal{I}}$, $d_{\gamma(i)} = (b, \varepsilon)$, and $(d_{\gamma(i-1)}, d_{\gamma(i)}) \in \mathfrak{r}^{\mathcal{I}}$ can be handled as in the previous two cases.

This concludes the proof of (A.7), and hence that of (A.5) and of $(*)$, which shows that (A.4) holds for all value restrictions. The proof for existential restrictions can be done using dual arguments.

For at-least restrictions $\geqslant n\, r.C$, note first that r must be simple, and hence we have $r^{\mathcal{I}_\mathsf{f}}(d, e) = r^{\mathcal{I}_\mathsf{f}^0}(d, e)$ for all $d, e \in \Delta^{\mathcal{I}_\mathsf{f}}$. We first consider the case that $d \in \mathsf{AN}^{\mathcal{I}}$, i.e. it is of the form (a, u) with $u \neq \varepsilon$. By the first axiom in $\mathsf{red}(\geqslant n\, r.C)$, there are $z_i, z_s \in \{0, 1\}$, $m \in \{0, \dots, n - z_i - z_s\}$ and $S \subseteq \mathsf{ind}(\mathcal{O})$ with $|S| = n - m - z_i - z_s$ such that $\mathsf{red}_{z_i,z_s,m,S,\leqslant}(\geqslant n\, r.C)$ is satisfied by d.

- If $z_i = 1$, then inverse roles are allowed and $d \in \boxed{(\geqslant n\, r.C) \leqslant \min\{r^-, \langle C\rangle_\uparrow\}}^{\mathcal{I}}$. Further, we obtain

$$v(\geqslant n\, r.C, d) \leqslant \min\{v(r^-, d), v(\langle C\rangle_\uparrow, d)\} = \min\{r^{\mathcal{I}_\mathsf{f}}(d, (a, u_\uparrow)), C^{\mathcal{I}_\mathsf{f}}(a, u_\uparrow)\}$$

  by the induction hypothesis.

- If $z_s = 1$, then local reflexivity is allowed and we have

$$v(\geqslant n\, r.C, d) \leqslant \min\{v(\exists r.\mathsf{Self}, d), v(C, d)\} = \min\{r^{\mathcal{I}_\mathsf{f}}(d, d), C^{\mathcal{I}_\mathsf{f}^0}(d)\}.$$

- Consider now any $b \in \mathsf{ind}(\mathcal{O})$. We know that nominals are allowed and $(d, (b, \varepsilon)) \in \mathfrak{r}^{\mathcal{I}}$, and thus

$$v(\geqslant n\, r.C, d) \leqslant \min\{v(r(*, b), d), v(C(b), d)\} = \min\{r^{\mathcal{I}_\mathsf{f}}(d, (b, \varepsilon)), C^{\mathcal{I}_\mathsf{f}}(b, \varepsilon)\}.$$

  Furthermore, all elements of S must be interpreted by different elements in $\mathcal{I}$, and hence also in $\mathcal{I}_\mathsf{f}$, even if we consider sets of individual names in the roots of $\mathcal{I}$.

- Additionally, there are m different elements $e_1, \dots, e_m \in \Delta^{\mathcal{I}}$ such that $(d, e_j) \in \mathfrak{r}^{\mathcal{I}}$ and $e_j \in \mathsf{AN}^{\mathcal{I}}$ for each e_j, and hence they must be of the form (a, ui_j). We obtain, for every j, $1 \leqslant j \leqslant m$,

$$v(\geqslant n\, r.C, d) = v(\langle \geqslant n\, r.C\rangle_\uparrow, e_j) \leqslant \min\{v(r, e_j), v(C, e_j)\} = \min\{r^{\mathcal{I}_\mathsf{f}}(d, e_j), C^{\mathcal{I}_\mathsf{f}}(e_j)\}.$$

Note that all r-successors of d considered above, i.e. $(a, u_\uparrow)$, d, (b, ε), and e_j, $1 \leqslant j \leqslant m$, must be different; in particular, we do not consider nominals and inverse roles at the same time (since obviously $\mathcal{O}$ contains at-least restrictions), and thus even if $u_\uparrow = \varepsilon$, we do not have $a \in S$. Hence, these elements can serve as the witnesses for $\geqslant n\, r.C$ at d (assuming that its value is exactly $v(\geqslant n\, r.C, d)$, which is shown below). For named domain elements, the witnesses are created by the first kind of assertions in $\mathsf{red}(\geqslant n\, r.C)$, where only two cases need to be considered (named and unnamed successors); note that all unnamed $\mathfrak{r}$-successors of (a, ε) must be of the form (a, i) due to our assumptions on the structure of $\mathcal{I}$.

Assume now again that $d = (a, u) \in \mathsf{AN}^{\mathcal{I}}$ and that the elements found above are not witnesses for $(\geqslant n\, r.C)^{\mathcal{I}_\mathsf{f}}(d) = v(\geqslant n\, r.C, d)$. Then there must be n different elements $e_1, \dots, e_n \in \Delta^{\mathcal{I}_\mathsf{f}}$ such that

$$v(\geqslant n\, r.C, d) < \min\{r^{\mathcal{I}_\mathsf{f}}(d, e_j), C^{\mathcal{I}_\mathsf{f}}(e_j)\}$$

for all j, $1 \leqslant j \leqslant n$. We show that we can find suitable z_i, z_s, m, and S such that $\mathsf{red}_{z_i,z_s,m,S,<}(\geqslant n\, r.C)$ is satisfied by d, which contradicts our assumption that $\mathcal{I}$ is a model of $\mathsf{red}(\mathcal{O})$.

- If inverse roles are allowed and there is an index j, $1 \leqslant j \leqslant n$, such that $e_j = (a, u_\uparrow)$, then we set $z_i := 1$. By the induction hypothesis and our assumption above, we have

$$v(\geqslant n\, r.C, d) < \min\{r^{\mathcal{I}_\mathsf{f}}(d, e_j), C^{\mathcal{I}_\mathsf{f}}(e_j)\} = \min\{v(r^-, d), v(C, e_j)\} = \min\{v(r^-, d), v(\langle C\rangle_\uparrow, d)\}.$$

- If local reflexivity is allowed and there is an index j, $1 \leqslant j \leqslant n$, such that $e_j = d$, then we set $z_s := 1$, and get

$$v(\geqslant n\, r.C, d) < \min\{v(\exists r.\mathsf{Self}, d), v(C, d)\}.$$

- If nominals are allowed, then we collect from the remaining elements those e_j that are equal to (b,ε) for some $b \in \mathsf{ind}(\mathcal{O})$. Let S be the set all of all those individual names. Since they are interpreted by different elements in $\mathcal{I}_\mathsf{f}$, they are also distinct in $\mathcal{I}$, even if we consider sets of individual names in the roots of $\mathcal{I}$. Furthermore, for any $b \in S$, since $r^{\mathcal{I}_\mathsf{f}}(d,(b,\varepsilon)) > v({\geqslant} n\, r.C) \geqslant 0$, we have $(d,(b,\varepsilon)) \in \mathfrak{r}^{\mathcal{I}}$ and

$$v({\geqslant} n\, r.C, d) < \min\{v(r(*,b),d), v(C(b),d)\}.$$

- There are exactly $m := n - |S| - z_i - z_s$ remaining elements e_j. If nominals are not allowed, then no e_j can be of the form (b,ε) for $b \in \mathsf{ind}(\mathcal{O})$ since $r^{\mathcal{I}_\mathsf{f}}(d,e_j) > 0$ and d is anonymous. If inverse roles are not allowed, then $e_j \neq (a, u_\uparrow)$ due to the same reason. Similarly, if local reflexivity is not allowed, it cannot be the case that $e_j = d$. Thus, we know for each of the remaining e_j that $e_j = (a, ui_j)$ for some $i_j \in \mathbb{N}$ and

$$v(\langle{\geqslant} n\, r.C\rangle_\uparrow, e_j) = v({\geqslant} n\, r.C, d) < \min\{v(r,e_j), v(C,e_j)\}.$$

Hence, the final part of $\mathsf{red}_{z_i,z_s,m,S,<}({\geqslant} n\, r.C)$, the restriction ${\geqslant} m\, \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow < \min\{r,C\}}\big)$, is satisfied.

For named elements $d = (a,\varepsilon)$, we can use a similar argument to contradict the second kind of assertions in $\mathsf{red}({\geqslant} n\, r.C)$. Note that there can be no anonymous element e_j satisfying $r^{\mathcal{I}_\mathsf{f}}(d,e_j) > 0$ that is not of the form $e_j = (a,i_j)$ for some $i_j \in \mathbb{N}$, since otherwise we know from the definition of $r^{\mathcal{I}_\mathsf{f}}$ that both inverse roles and nominals must be allowed, which cannot be the case since obviously number restrictions are allowed.

This concludes the proof of (A.4). It remains to show that $\mathcal{I}_\mathsf{f}$ is a model of $\mathcal{O}$. For every $\alpha \bowtie \beta \in \mathcal{A}$, we have $v(\alpha) \bowtie v(\beta)$ by $\mathsf{red}(\mathcal{A})$ and (P2). In the case that $\alpha = q \in \mathsf{val}(\mathcal{O})$, we know that $v(\alpha) = q$ by (P5)a); and if $\alpha = C(a)$, then $v(\alpha) = v(C,a,\varepsilon) = C^{\mathcal{I}_\mathsf{f}}(a,\varepsilon) = C^{\mathcal{I}_\mathsf{f}}(a^{\mathcal{I}_\mathsf{f}})$ by (P4) and (A.4).[11] Since the same holds for β, we conclude that $\alpha^{\mathcal{I}_\mathsf{f}} \bowtie \beta^{\mathcal{I}_\mathsf{f}}$.

All (in)equality assertions $a \approx b$ ($a \not\approx b$) in $\mathcal{A}$ are satisfied due to $\mathsf{red}(\mathcal{A})$ and the construction of $\mathcal{I}_\mathsf{f}$.

Consider a GCI $C \sqsubseteq D \geqslant p \in \mathcal{T}$ and $d \in \Delta^{\mathcal{I}}$. By $\mathsf{red}(\mathcal{T})$ and (P5)b), we have $v(p,d) \leqslant v(C,d) \Rightarrow v(D,d)$. Thus, (A.4) and (P5)a) yield $C^{\mathcal{I}_\mathsf{f}}(d) \Rightarrow D^{\mathcal{I}_\mathsf{f}}(d) \geqslant p$.

For any $\mathsf{dis}(r,s) \geqslant p \in \mathcal{R}$, r and s are simple, and thus we can restrict our analysis to $r^{\mathcal{I}_\mathsf{f}^0}$ and $s^{\mathcal{I}_\mathsf{f}^0}$. We have $\min\{v(r(a,b),c,u), v(s(a,b),c,u)\} \leqslant 1-p$ for all $(c,u) \in \Delta^{\mathcal{I}_\mathsf{f}}$ and $a,b \in \mathsf{ind}(\mathcal{O}) \cup \{*\}$. Hence, $\min\{r^{\mathcal{I}_\mathsf{f}}(a^{\mathcal{I}_\mathsf{f}}, b^{\mathcal{I}_\mathsf{f}}), s^{\mathcal{I}_\mathsf{f}}(a^{\mathcal{I}_\mathsf{f}}, b^{\mathcal{I}_\mathsf{f}})\} \leqslant 1-p$, where $*^{\mathcal{I}_\mathsf{f}} := (c,u)$. This takes care of all role connections involving named domain elements. Furthermore, we obtain $\min\{v(r,c,u), v(s,c,u)\} \leqslant 1-p$ in case $u \neq \varepsilon$, and thus $\min\{r^{\mathcal{I}_\mathsf{f}}((c,u_\uparrow),(c,u)), s^{\mathcal{I}_\mathsf{f}}((c,u_\uparrow),(c,u))\} \leqslant 1-p$. Similarly, if inverse roles are allowed, then $\min\{v(r^-,c,u), v(s^-,c,u)\} \leqslant 1-p$, i.e. $\min\{r^{\mathcal{I}_\mathsf{f}}((c,u),(c,u_\uparrow)), s^{\mathcal{I}_\mathsf{f}}((c,u),(c,u_\uparrow))\} \leqslant 1-p$. Finally, we know that $\min\{v(\exists r.\mathsf{Self},c,u), v(\exists s.\mathsf{Self},c,u)\} \leqslant 1-p$, and thus $\min\{r^{\mathcal{I}_\mathsf{f}}((c,u),(c,u)), s^{\mathcal{I}_\mathsf{f}}((c,u),(c,u))\} \leqslant 1-p$.

For the complex role inclusions in $\mathcal{R}$, by Lemma 4.4 it suffices to show $w^{\mathcal{I}_\mathsf{f}}(d,e) \Rightarrow r^{\mathcal{I}_\mathsf{f}}(d,e) \geqslant (\|\mathbf{A}_r\|, w)$ for all $r \in \mathsf{rol}(\mathcal{O})$, $w \in \mathsf{rol}(\mathcal{O})^+$, and $d,e \in \Delta^{\mathcal{I}_\mathsf{f}}$. We can assume that $w^{\mathcal{I}_\mathsf{f}}(d,e) > 0$ and $(\|\mathbf{A}_r\|, w) > 0$ since otherwise this inequation is trivially satisfied. Let now $w = r_1 \dots r_n$, $n \geqslant 1$. Then we have

$$\begin{aligned} w^{\mathcal{I}_\mathsf{f}}(d,e) &= \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \min_{i=1}^{n} r_i^{\mathcal{I}_\mathsf{f}}(d_{i-1},d_i) \\ &= \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \min_{i=1}^{n} \sup_{w_i \in \mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1},d_i)\} \\ &= \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \sup_{w_1,\dots,w_n \in \mathsf{rol}(\mathcal{O})^+} \min_{i=1}^{n} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1},d_i)\}, \end{aligned}$$

[11] Recall that we have eliminated all crisp role assertions from the ABox.

where we set $d_0 := d$ and $d_n := e$. Furthermore, for any choice of elements $d_1, \dots, d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}$ and words $w_1, \dots, w_n \in \mathsf{rol}(\mathcal{O})^+$, we have

$$\begin{aligned}
r^{\mathcal{I}_\mathsf{f}}(d,e) &\geqslant \min\{(\|\mathbf{A}_r\|, w_1 \dots w_n), (w_1 \dots w_n)^{\mathcal{I}_\mathsf{f}^0}(d,e)\} \\
&\geqslant \min\big\{(\|\mathbf{A}_r\|, w_1 \dots w_n), \min_{i=1}^{n} w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\big\} \\
&\geqslant \min\big\{(\|\mathbf{A}_r\|, w), \min_{i=1}^{n} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\}\big\}
\end{aligned}$$

by the construction of $\mathbf{A}_r$. Hence,

$$\begin{aligned}
(\|\mathbf{A}_r\|, w) \Rightarrow r^{\mathcal{I}_\mathsf{f}}(d,e) &\geqslant \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \sup_{w_1,\dots,w_n \in \mathsf{rol}(\mathcal{O})^+} \min_{i=1}^{n} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\} \\
&= w^{\mathcal{I}_\mathsf{f}}(d,e),
\end{aligned}$$

as required. □

We have thus shown the first direction of Lemma 5.5. We now consider the converse.

Lemma A.2. *If $\mathcal{O}$ has a* G*-model, then* $\mathsf{red}(\mathcal{O})$ *has a classical model.*

Proof. Given a G-model $\mathcal{I}$ of $\mathcal{O}$, we construct the classical interpretation $\mathcal{I}_\mathsf{c}$, whose domain consists of all sequences of the form $ad_1 \dots d_k$, where

- $a \in \mathsf{N_I}$ and $k \geqslant 0$;
- all d_i are elements of $\Delta^{\mathcal{I}}$;
- if number restrictions are allowed, then we have to put some restrictions on this sequence of domain elements:
  - d_1 is not equal to $b^{\mathcal{I}}$ for any $b \in \mathsf{N_I}$;
  - if nominals are allowed, then no d_i is equal to $b^{\mathcal{I}}$ for any $b \in \mathsf{N_I}$; and
  - if inverse roles are allowed, then $d_2 \neq a^{\mathcal{I}}$, and there is no index i such that $d_i = d_{i+2}$.

For ease of presentation, we assume that all individual names are interpreted by distinct elements of $\Delta^{\mathcal{I}}$. In general, however, we would have to consider equivalence classes of individual names as roots for $\mathcal{I}_\mathsf{c}$, where $a, b \in \mathsf{N_I}$ are equivalent iff $a^{\mathcal{I}} = b^{\mathcal{I}}$. Since this is only relevant for number restrictions and (in)equality assertions, we will ignore this for most of the proof and only mention it at the appropriate places.

We now set $a^{\mathcal{I}_\mathsf{c}} := a$ for all $a \in \mathsf{N_I}$, and

$$\begin{aligned}
\mathfrak{r}^{\mathcal{I}_\mathsf{c}} := {} & \{(\varrho, \varrho d) \mid \varrho d \in \Delta^{\mathcal{I}_\mathsf{c}}\} \cup \\
& \{(a,b) \mid a, b \in \mathsf{N_I}\} \cup \\
& \begin{cases} \{(\varrho, a) \mid \varrho \in \Delta^{\mathcal{I}_\mathsf{c}}, a \in \mathsf{N_I}\} & \text{if nominals are present,} \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}$$

We denote by $\mathsf{tail}(ad_1 \dots d_k)$ the element d_k if $k > 0$, and $a^{\mathcal{I}}$ if $k = 0$. Similarly, we set $\mathsf{prev}(ad_1 \dots d_k)$ to

d_{k-1} if $k > 1$, and to $a^{\mathcal{I}}$ if $k = 1$. For any $\alpha \in \mathcal{U}$ and $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$, we define

$$\alpha^{\mathcal{I}}(\varrho) := \begin{cases} C^{\mathcal{I}}(\mathsf{tail}(\varrho)) & \text{if } \alpha = C \in \mathsf{sub}(\mathcal{O}); \\ C^{\mathcal{I}}(\mathsf{prev}(\varrho)) & \text{if } \alpha = \langle C \rangle_{\uparrow} \text{ for } C \in \mathsf{sub}(\mathcal{O}); \\ q & \text{if } \alpha = q \in \mathsf{val}(\mathcal{O}); \\ 0 & \text{if } \alpha = r \text{ and } \varrho \in \mathsf{N_I}; \\ 0 & \text{if } \alpha = r,\ \varrho \notin \mathsf{N_I},\ \mathsf{tail}(\varrho) = \mathsf{prev}(\varrho),\ r \text{ is simple, and } (\mathsf{R}); \\ r^{\mathcal{I}}(\mathsf{prev}(\varrho), \mathsf{tail}(\varrho)) & \text{if } \alpha = r,\ \varrho \notin \mathsf{N_I}, \text{ and either } \mathsf{tail}(\varrho) \neq \mathsf{prev}(\varrho),\ r \text{ is non-simple, or } \neg(\mathsf{R}); \\ C^{\mathcal{I}}(a^{\mathcal{I}}) & \text{if } \alpha = C(a); \\ r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) & \text{if } \alpha = r(a,b); \\ r^{\mathcal{I}}(a^{\mathcal{I}}, \mathsf{tail}(\varrho)) & \text{if } \alpha = r(a,*); \\ r^{\mathcal{I}}(\mathsf{tail}(\varrho), a^{\mathcal{I}}) & \text{if } \alpha = r(*,a); \\ 1 - \mathsf{inv}(\alpha)^{\mathcal{I}}(\varrho) & \text{if } \alpha \text{ involves a negated role } \neg r, \end{cases}$$

where

$$(\forall \mathbf{A}.C)^{\mathcal{I}}(d) := \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e),$$

where $\varepsilon^{\mathcal{I}}(d,e) := 1$ if $d = e$, and $\varepsilon^{\mathcal{I}}(d,e) := 0$ otherwise. Note that $\alpha^{\mathcal{I}}(\varrho)$ is not defined for $\alpha \in \mathsf{sub}_{\uparrow}(\mathcal{O})$ if $\varrho \in \mathsf{N_I}$. We fix these values $\alpha^{\mathcal{I}}(\varrho)$ arbitrarily, in such a way that for all $\alpha, \beta \in \mathcal{U}$ we have $\alpha^{\mathcal{I}}(\varrho) \leqslant \beta^{\mathcal{I}}(\varrho)$ iff $\mathsf{inv}(\beta)^{\mathcal{I}}(\varrho) \leqslant \mathsf{inv}(\alpha)^{\mathcal{I}}(\varrho)$. We now define $\mathsf{AN}^{\mathcal{I}_\mathsf{c}} := \Delta^{\mathcal{I}_\mathsf{c}} \setminus \mathsf{N_I}$ and, for all concept names $\boxed{\alpha \leqslant \beta}$ with $\alpha, \beta \in \mathcal{U}$,

$$\boxed{\alpha \leqslant \beta}^{\mathcal{I}_\mathsf{c}} := \{\varrho \mid \alpha^{\mathcal{I}}(\varrho) \leqslant \beta^{\mathcal{I}}(\varrho)\}.$$

It is easy to see that we also have $\varrho \in \boxed{\alpha \bowtie \beta}^{\mathcal{I}_\mathsf{c}}$ iff $\alpha^{\mathcal{I}}(\varrho) \bowtie \beta^{\mathcal{I}}(\varrho)$ for all other order expressions $\bowtie$, and that $\mathcal{I}_\mathsf{c}$ satisfies $\mathsf{red}(\mathsf{AN})$ and $\mathsf{red}(\mathcal{U})$. We now show that $\mathcal{I}_\mathsf{c}$ satisfies the remaining parts of $\mathsf{red}(\mathcal{O})$.

For any $\alpha \bowtie \beta \in \mathcal{A}$, we have $\alpha^{\mathcal{I}} \bowtie \beta^{\mathcal{I}}$ since $\mathcal{I}$ satisfies $\mathcal{A}$. From this it follows that $\alpha^{\mathcal{I}}(c) \bowtie \beta^{\mathcal{I}}(c)$ for any $c \in \mathsf{ind}(\mathcal{O})$. All (in)equalities $a \approx b$ ($a \not\approx b$) in $\mathsf{red}(\mathcal{A})$ are satisfied if we consider the generalized construction with equivalence classes of individual names as roots for $\mathcal{I}_\mathsf{c}$. It is straightforward to verify the remaining axioms in $\mathsf{red}(\mathcal{A})$.

For any GCI $C \sqsubseteq D \geqslant p \in \mathcal{T}$ and every $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$, we know that

$$C^{\mathcal{I}}(\varrho) \Rightarrow D^{\mathcal{I}}(\varrho) = C^{\mathcal{I}}(\mathsf{tail}(\varrho)) \Rightarrow D^{\mathcal{I}}(\mathsf{tail}(\varrho)) \geqslant p,$$

and hence $\varrho \in \boxed{p \leqslant C \Rightarrow D}^{\mathcal{I}_\mathsf{c}}$.

For $\mathsf{red}(\uparrow)$, consider any $\varrho \in \boxed{\alpha \bowtie \beta}^{\mathcal{I}_\mathsf{c}}$ and $\varrho' \in \mathsf{AN}$ with $(\varrho, \varrho') \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$. Then ϱ' must be of the form ϱd for some $d \in \Delta^{\mathcal{I}}$, and we have

$$\langle \alpha \rangle_{\uparrow}^{\mathcal{I}}(\varrho d) = \alpha^{\mathcal{I}}(\mathsf{tail}(\varrho)) \bowtie \beta^{\mathcal{I}}(\mathsf{tail}(\varrho)) = \langle \beta \rangle_{\uparrow}^{\mathcal{I}}(\varrho d),$$

and hence $\varrho' \in \boxed{\langle \alpha \rangle_{\uparrow} \bowtie \langle \beta \rangle_{\uparrow}}^{\mathcal{I}_\mathsf{c}}$.

For $\mathsf{red}(\mathcal{R})$, consider first any role inclusion $r \sqsubseteq s \geqslant p \in \mathcal{R}$ where r and s are both simple. Then we have $r^{\mathcal{I}}(\mathsf{prev}(\varrho), \mathsf{tail}(\varrho)) \Rightarrow s^{\mathcal{I}}(\mathsf{prev}(\varrho), \mathsf{tail}(\varrho)) \geqslant p$ for every $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}} \setminus \mathsf{N_I}$ with $\mathsf{tail}(\varrho) \neq \mathsf{prev}(\varrho)$; furthermore, every $a \in \mathsf{N_I}$ also satisfies $\boxed{r \Rightarrow s \geqslant p}$ since $0 \leqslant 0$. Similar arguments can be used if $\mathsf{tail}(\varrho) = \mathsf{prev}(\varrho)$, for $\boxed{r(a,b) \Rightarrow s(a,b) \geqslant p}$ and $\boxed{\exists r.\mathsf{Self} \Rightarrow \exists s.\mathsf{Self} \geqslant p}$, and for the reduction of disjoint role axioms.

It remains to consider the axioms in $\mathsf{red}(C)$ for $C \in \mathsf{sub}(\mathcal{O})$. The reductions for $\top$, $\{a\}$, $\overline{q}$, $\exists r.\mathsf{Self}$, $\sqcap$, and $\rightarrow$ obviously reflect the semantics of these constructors and are easy to verify.

We now consider the axioms in $\mathsf{red}(\forall r.C)$. By Lemma 4.4, we have

$$\begin{aligned} (\forall \mathbf{A}_r.C)^{\mathcal{I}}(d) &= \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e) \\ &\geqslant \inf_{e \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(d,e) \Rightarrow C^{\mathcal{I}}(e) \\ &= (\forall r.C)^{\mathcal{I}}(d) \end{aligned}$$

for all $d \in \Delta^{\mathcal{I}}$. Lemma 4.4 talks only about $w \in \mathsf{rol}(\mathcal{O})^+$, but it holds also for $w = \varepsilon$ since then $(\|\mathbf{A}_r\|, w) = 0$ due to the construction of $\mathbf{A}_r$. Hence, the axiom $\top \sqsubseteq \boxed{(\forall r.C) \leqslant (\forall \mathbf{A}_r.C)}$ is satisfied by $\mathcal{I}_\mathsf{c}$. We consider the axiom

$$\begin{aligned}\mathsf{AN} \sqsubseteq \exists \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}\big) \sqcup \\ \boxed{(\forall r.C) \geqslant r^- \Rightarrow \langle C\rangle_\uparrow} \sqcup \\ \boxed{(\forall r.C) \geqslant (\exists r.\mathsf{Self}) \Rightarrow C} \sqcup \\ \bigsqcup_{a \in \mathsf{ind}(\mathcal{O})} \big(\exists \mathfrak{r}.\{a\} \sqcap \boxed{(\forall r.C) \geqslant r(*, a) \Rightarrow C(a)}\big),\end{aligned}$$

where the third disjunct is only present if local reflexivity is allowed and r is simple, likewise for the second disjunct and inverse roles, and the last disjunct is contingent on the presence of nominals. Let further $\varrho \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$, i.e. $\varrho = ad_1 \dots d_k$ with $k \geqslant 1$. Since $\mathcal{I}$ is witnessed, there is an $e \in \Delta^{\mathcal{I}}$ with $(\forall r.C)^{\mathcal{I}}(d_k) = r^{\mathcal{I}}(d_k, e) \Rightarrow C^{\mathcal{I}}(e)$. If $\varrho e \in \Delta^{\mathcal{I}_\mathsf{c}}$, then $(\varrho, \varrho e) \in \mathfrak{r}^{\mathcal{I}}$ and $\varrho e \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$

- If $d_k \neq e$, r is non-simple, or local reflexivity is not allowed, then

  $$\langle \forall r.C\rangle_\uparrow^{\mathcal{I}}(\varrho e) = r^{\mathcal{I}}(d_k, e) \Rightarrow C^{\mathcal{I}}(e) = r^{\mathcal{I}}(\varrho e) \Rightarrow C^{\mathcal{I}}(\varrho e),$$

  and hence $\varrho e \in \boxed{\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}^{\mathcal{I}_\mathsf{c}}$.

- Otherwise, we have $d_k = e$, r is simple, and local reflexivity is allowed, and hence $r^{\mathcal{I}}(\varrho e) = 0$. However, we have

  $$(\forall r.C)^{\mathcal{I}}(\varrho) = r^{\mathcal{I}}(d_k, d_k) \Rightarrow C^{\mathcal{I}}(d_k) = (\exists r.\mathsf{Self})^{\mathcal{I}}(\varrho) \Rightarrow C^{\mathcal{I}}(\varrho)$$

  instead, and hence ϱ satisfies the third disjunct.

Otherwise, i.e. in the case that $\varrho e \notin \Delta^{\mathcal{I}_\mathsf{c}}$, there are two more cases to consider:

- Nominals are allowed and $e = b^{\mathcal{I}}$ for some $b \in \mathsf{N_I}$. Then we have $(\varrho, b) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$ and

  $$(\forall r.C)^{\mathcal{I}}(\varrho) = r^{\mathcal{I}}(d_k, b^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(b^{\mathcal{I}}) = (r(*, b))^{\mathcal{I}}(\varrho) \Rightarrow (C(b))^{\mathcal{I}}(\varrho).$$

- Inverse roles are allowed and either (i) $k > 1$ and $d_{k-1} = b^{\mathcal{I}}$ or (ii) $k = 1$ and $a = b$. In both cases, we have $\mathsf{prev}(\varrho) = b^{\mathcal{I}}$, and hence

  $$(\forall r.C)^{\mathcal{I}}(\varrho) = r^{\mathcal{I}}(k_m, \mathsf{prev}(\varrho)) \Rightarrow C^{\mathcal{I}}(\mathsf{prev}(\varrho)) = (r^-)^{\mathcal{I}}(\varrho) \Rightarrow \langle C\rangle_\uparrow^{\mathcal{I}}(\varrho).$$

Consider now the axiom

$$\exists \mathfrak{r}.\Big(\big(\mathsf{AN} \sqcap \boxed{\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}\big) \sqcup \big(\neg \mathsf{AN} \sqcap \boxed{(\forall r.C)(a) \geqslant r(a, *) \Rightarrow C}\big)\Big)(a)$$

for some $a \in \mathsf{ind}(\mathcal{O})$. Since $\mathcal{I}$ is witnessed, there is an $e \in \Delta^{\mathcal{I}}$ such that $(\forall r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = r^{\mathcal{I}}(a^{\mathcal{I}}, e) \Rightarrow C^{\mathcal{I}}(e)$.

- If $e = b^{\mathcal{I}}$ for some $b \in \mathsf{N_I}$, then we have $(a, b) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$ and $b \notin \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$. Furthermore,

  $$(\forall r.C)(a)^{\mathcal{I}}(b) = (\forall r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(b^{\mathcal{I}}),$$

  which is equal to $(r(a, *))^{\mathcal{I}}(b) \Rightarrow C^{\mathcal{I}}(b)$, and hence we have $b \in \boxed{(\forall r.C)(a) \geqslant r(a, *) \Rightarrow C}^{\mathcal{I}_\mathsf{c}}$.

- If $e \neq b^{\mathcal{I}}$ for all $b \in \mathsf{N_I}$, then we have $ae \in \Delta^{\mathcal{I}_\mathsf{c}}$, and thus $(a, ae) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$. Moreover, $ae \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$ and $\langle \forall r.C\rangle_\uparrow^{\mathcal{I}}(ae) = (\forall r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = r^{\mathcal{I}}(a^{\mathcal{I}}, e) \Rightarrow C^{\mathcal{I}}(e)$, which is equal to $r^{\mathcal{I}}(ae) \Rightarrow C^{\mathcal{I}}(ae)$.

For $\mathsf{red}(\forall\mathbf{A}^q.C)$, we first consider the axiom $\top \sqsubseteq \boxed{(\forall\mathbf{A}^q.C) \leqslant C}$ for a final state q of $\mathbf{A}$. We have

$$(\forall\mathbf{A}^q.C)^\mathcal{I}(d) \leqslant \min\{(\|\mathbf{A}^q\|, \varepsilon), \varepsilon^\mathcal{I}(d,d)\} \Rightarrow C^\mathcal{I}(d) = C^\mathcal{I}(d)$$

for all $d \in \Delta^\mathcal{I}$, and hence $\mathcal{I}_\mathsf{c}$ satisfies this axiom. For any transition $q \xrightarrow{\varepsilon,p} q'$ in $\mathbf{A}^q$, we have to satisfy the axiom $\top \sqsubseteq \boxed{(\forall\mathbf{A}^q.C) \leqslant p \Rightarrow (\forall\mathbf{A}^{q'}.C)}$. By Proposition 4.2, we get

$$\begin{aligned}(\forall\mathbf{A}^q.C)^\mathcal{I}(d) &= \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{(\|\mathbf{A}^q\|, w), w^\mathcal{I}(d,e)\} \Rightarrow C^\mathcal{I}(e) \\ &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{p, (\|\mathbf{A}^{q'}\|, w), w^\mathcal{I}(d,e)\} \Rightarrow C^\mathcal{I}(e) \\ &= p \Rightarrow \Big(\inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{(\|\mathbf{A}^{q'}\|, w), w^\mathcal{I}(d,e)\} \Rightarrow C^\mathcal{I}(e)\Big) \\ &= p \Rightarrow (\forall\mathbf{A}^{q'}.C)^\mathcal{I}(d).\end{aligned}$$

Consider now the axiom $\mathsf{AN} \sqsubseteq \boxed{(\forall\mathbf{A}^q.C) \leqslant \min\{p, s^-\} \Rightarrow \langle\forall\mathbf{A}^{q'}.C\rangle_\uparrow}$ for a transition $q \xrightarrow{s,p} q'$ in $\mathbf{A}$, and any $\varrho \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$, which must be of the form $ad_1 \dots d_k$ with $k \geqslant 1$. We have

$$\begin{aligned}(\forall\mathbf{A}^q.C)^\mathcal{I}(\varrho) &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{(\|\mathbf{A}^q\|, sw), (sw)^\mathcal{I}(d_k,e)\} \Rightarrow C^\mathcal{I}(e) \\ &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\Big\{p, (\|\mathbf{A}^{q'}\|, w), \sup_{e'\in\Delta^\mathcal{I}} \min\{s^\mathcal{I}(d_k,e'), w^\mathcal{I}(e',e)\}\Big\} \Rightarrow C^\mathcal{I}(e) \\ &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^\mathcal{I}(d_k, \mathsf{prev}(\varrho)), w^\mathcal{I}(\mathsf{prev}(\varrho), e)\} \Rightarrow C^\mathcal{I}(e) \\ &\leqslant \min\{p, (s^-)^\mathcal{I}(\varrho)\} \Rightarrow \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{(\|\mathbf{A}^{q'}\|, w), w^\mathcal{I}(\mathsf{prev}(\varrho), e)\} \Rightarrow C^\mathcal{I}(e) \\ &= \min\{p, (s^-)^\mathcal{I}(\varrho)\} \Rightarrow \langle\forall\mathbf{A}^{q'}.C\rangle_\uparrow^\mathcal{I}(\varrho).\end{aligned}$$

by Propositions 2.1 and 4.2.

For $\top \sqsubseteq \boxed{(\forall\mathbf{A}^q.C) \leqslant \min\{p, \exists s.\mathsf{Self}\} \Rightarrow (\forall\mathbf{A}^{q'}.C)}$ and any $d \in \Delta^\mathcal{I}$, we know that s must be simple and get

$$\begin{aligned}(\forall\mathbf{A}^q.C)^\mathcal{I}(d) &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^\mathcal{I}(d,d), w^\mathcal{I}(d,e)\} \Rightarrow C^\mathcal{I}(e) \\ &= \min\{p, (\exists s.\mathsf{Self})^\mathcal{I}(d)\} \Rightarrow (\forall\mathbf{A}^{q'}.C)^\mathcal{I}(d)\end{aligned}$$

by similar arguments.

For $\top \sqsubseteq \forall\mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle\forall\mathbf{A}^q.C\rangle_\uparrow \leqslant \min\{p, s\} \Rightarrow (\forall\mathbf{A}^{q'}.C)}\big)$, consider any $\varrho, \varrho' \in \Delta^{\mathcal{I}_\mathsf{c}}$ with $(\varrho, \varrho') \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$ and $\varrho' \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$. Thus, we must have $\varrho' = \varrho d$ for some $d \in \Delta^\mathcal{I}$, and we know that $\mathsf{prev}(\varrho d) = \mathsf{tail}(\varrho)$. We obtain

$$\begin{aligned}\langle\forall\mathbf{A}^q.C\rangle_\uparrow^\mathcal{I}(\varrho d) &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^\mathcal{I}(\mathsf{tail}(\varrho), d), w^\mathcal{I}(d,e)\} \Rightarrow C^\mathcal{I}(e) \\ &\leqslant \min\{p, s^\mathcal{I}(\varrho d)\} \Rightarrow (\forall\mathbf{A}^{q'}.C)^\mathcal{I}(\varrho d).\end{aligned}$$

Consider now the axiom $\exists\mathfrak{r}.\{a\} \sqsubseteq \boxed{(\forall\mathbf{A}^q.C) \leqslant \min\{p, s(*,a)\} \Rightarrow (\forall\mathbf{A}^{q'}.C)(a)}$ for any $a \in \mathsf{ind}(\mathcal{O})$, and $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$ with $(\varrho, a) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$. We get

$$\begin{aligned}(\forall\mathbf{A}^q.C)^\mathcal{I}(\varrho) &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^\mathcal{I}(\mathsf{tail}(\varrho), a^\mathcal{I}), w^\mathcal{I}(a^\mathcal{I}, e)\} \Rightarrow C^\mathcal{I}(e) \\ &= \min\{p, (s(*,a))^\mathcal{I}(\varrho)\} \Rightarrow ((\forall\mathbf{A}^{q'}.C)(a))^\mathcal{I}(\varrho).\end{aligned}$$

Finally, for $\exists\mathfrak{r}.\{a\} \sqsubseteq \boxed{(\forall\mathbf{A}^q.C)(a) \leqslant \min\{p, s^-(*,a)\} \Rightarrow (\forall\mathbf{A}^{q'}.C)}$ and any $a \in \mathsf{ind}(\mathcal{O})$ and $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$ with $(\varrho, a) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$, we obtain

$$\begin{aligned}((\forall\mathbf{A}^q.C)(a))^\mathcal{I}(\varrho) &\leqslant \inf_{w\in\mathsf{rol}(\mathcal{O})^*} \inf_{e\in\Delta^\mathcal{I}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^\mathcal{I}(a^\mathcal{I}, \mathsf{tail}(\varrho)), w^\mathcal{I}(\mathsf{tail}(\varrho), e)\} \Rightarrow C^\mathcal{I}(e) \\ &= \min\{p, (s^-(*,a))^\mathcal{I}(\varrho)\} \Rightarrow (\forall\mathbf{A}^{q'}.C)^\mathcal{I}(\varrho).\end{aligned}$$

This concludes the analysis of the reduction of value restrictions. Again, the case of existential restrictions can be handled by dual arguments.

If number restrictions are present, we also have to satisfy $\mathsf{red}(\geqslant n\,r.C)$. Consider the first axiom and any $ad_1 \dots d_k \in \Delta^{\mathcal{I}_\mathsf{c}}$ with $k \geqslant 1$. Since $\mathcal{I}$ is witnessed, there must be n different elements $e_1, \dots, e_n \in \Delta^{\mathcal{I}}$ such that

$$(\geqslant n\,r.C)^{\mathcal{I}}(d_k) = \min_{j=1}^{n} \min\{r^{\mathcal{I}}(d_k, e_j), C^{\mathcal{I}}(e_j)\}.$$

If local reflexivity is allowed and we have $e_j = d_k$ for one of these elements, we set $z_s := 1$. If inverse roles are allowed and we have (i) $k > 1$ and $e_j = d_{k-1}$ or (ii) $k = 1$ and $e_j = a^{\mathcal{I}}$, we set $z_i := 1$. Note that the previous two elements identified for z_s and z_i must be different since otherwise we would have $d_k = d_{k-1}$ or $d_1 = a^{\mathcal{I}}$. If nominals are allowed, we define S to be the set of all individual names $b \in \mathsf{ind}(\mathcal{O})$ for which $b^{\mathcal{I}}$ is among the remaining elements from $e_1, \dots, e_n$; otherwise we set $S := \emptyset$. We thus have $m := n - z_i - z_s - |S|$ remaining elements e_j, and have uniquely identified one of the disjuncts of the axiom. We now show that for each of these elements e_j the corresponding conjunct in this disjunct is satisfied, thus showing that the whole axiom is satisfied.

- If $z_s = 1$, let e_j be the element equal to d_k. We have

  $$(\geqslant n\,r.C)^{\mathcal{I}}(\varrho) \leqslant \min\{r^{\mathcal{I}}(d_k, d_k), C^{\mathcal{I}}(d_k)\} = \min\{(\exists r.\mathsf{Self})^{\mathcal{I}}(\varrho), C^{\mathcal{I}}(\varrho)\},$$

  and thus the conjunct $\boxed{(\geqslant n\,r.C) \leqslant \min\{\exists r.\mathsf{Self}, C\}}$ is satisfied by ϱ.

- If $z_i = 1$, let e_j be the element equal to d_{k-1} or $a^{\mathcal{I}}$. Then $\mathsf{prev}(\varrho) = e_j$ and

  $$(\geqslant n\,r.C)^{\mathcal{I}}(\varrho) \leqslant \min\{r^{\mathcal{I}}(d_k, e_j), C^{\mathcal{I}}(e_j)\} = \min\{(r^-)^{\mathcal{I}}(\varrho), \langle C\rangle_\uparrow^{\mathcal{I}}(\varrho)\},$$

  validating the conjunct $\boxed{(\geqslant n\,r.C) \leqslant \min\{r^-, \langle C\rangle_\uparrow\}}$.

- Consider any $a \in S$. Then nominals are present, and thus $(\varrho, a) \in \mathfrak{r}^{\mathcal{I}}$ and

  $$(\geqslant n\,r.C)^{\mathcal{I}}(\varrho) \leqslant \min\{r^{\mathcal{I}}(d_k, a^{\mathcal{I}}), C^{\mathcal{I}}(a^{\mathcal{I}})\} = \min\{(r(*, a))^{\mathcal{I}}(\varrho), (C(a))^{\mathcal{I}}(\varrho)\},$$

  which corresponds to $\boxed{(\geqslant n\,r.C) \leqslant \min\{r(*, a), C(a)\}}$.

- For any e_j not corresponding to any of the previous cases, we know by the construction of $\Delta^{\mathcal{I}_\mathsf{c}}$ that $\varrho e_j \in \Delta^{\mathcal{I}_\mathsf{c}}$, and hence $(\varrho, \varrho e_j) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$, $\mathsf{tail}(\varrho e_j) = e_j \neq d_k = \mathsf{prev}(\varrho e_j)$, and

  $$\langle \geqslant n\,r.C\rangle_\uparrow^{\mathcal{I}}(\varrho e_j) \leqslant \min\{r^{\mathcal{I}}(d_k, e_j), C^{\mathcal{I}}(e_j)\} = \min\{r^{\mathcal{I}}(\varrho e_j), C^{\mathcal{I}}(\varrho e_j)\}.$$

  Since there are m different such elements e_j, the corresponding elements ϱe_j are also different, and $\geqslant m\,\mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \geqslant n\,r.C\rangle_\uparrow \leqslant \min\{r, C\}}\big)$ is satisfied by ϱ.

For the second axiom in $\mathsf{red}(\geqslant n\,r.C)$, assume to the contrary that there is a $\varrho = ad_1 \dots d_k \in \Delta^{\mathcal{I}_\mathsf{c}}$, and numbers z_i (if there are inverse roles), z_s (if local reflexivity is allowed), $0 \leqslant m \leqslant n - z_i - z_s$, and a set $S \subseteq \mathsf{ind}(\mathcal{O})$ of cardinality $n - m - z_i - z_s$ (which is 0 unless there are nominals) such that the corresponding conjunction is satisfied by ϱ in $\mathcal{I}_\mathsf{c}$.

- If $z_i = 1$, then $\varrho \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$, i.e. we have $k \geqslant 1$, and

  $$(\geqslant n\,r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) = (\geqslant n\,r.C)^{\mathcal{I}}(\varrho) < \min\{(r^-)^{\mathcal{I}}(\varrho), \langle C\rangle_\uparrow^{\mathcal{I}}(\varrho)\} = \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), \mathsf{prev}(\varrho)), C^{\mathcal{I}}(\mathsf{prev}(\varrho))\}.$$

  Note that $(r^-)^{\mathcal{I}}(\varrho)$ cannot be 0, and hence must be equal to $r^{\mathcal{I}}(\mathsf{tail}(\varrho), \mathsf{prev}(\varrho))$.

- If $z_s = 1$, then

  $$(\geqslant n\,r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) < \min\{(\exists r.\mathsf{Self})^{\mathcal{I}}(\varrho), C^{\mathcal{I}}(\varrho)\} = \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), \mathsf{tail}(\varrho)), C^{\mathcal{I}}(\mathsf{tail}(\varrho))\}.$$

- For each $a \in S$, we have
$$(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) < \min\{(r(*,a))^{\mathcal{I}}(\varrho), (C(a))^{\mathcal{I}}(\varrho)\} = \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), a^{\mathcal{I}}), C^{\mathcal{I}}(a^{\mathcal{I}})\}.$$
Furthermore, even if we consider equivalence classes of individual names as roots for $\mathcal{I}_{\mathsf{c}}$, all $a \in S$ are interpreted by different domain elements.

- Additionally, there are m different $\mathfrak{r}$-successors $\varrho_1, \ldots, \varrho_m$ that all satisfy AN, i.e. are of the form $\varrho e_1, \ldots, \varrho e_m$ for different elements $e_1, \ldots, e_m \in \Delta^{\mathcal{I}}$, and, for all $1 \leqslant j \leqslant m$,
$$(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) = \langle \geqslant n\, r.C\rangle^{\mathcal{I}}_{\uparrow}(\varrho e_j) < \min\{r^{\mathcal{I}}(\varrho e_j), C^{\mathcal{I}}(\varrho e_j)\} = \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), e_j), C^{\mathcal{I}}(e_j)\}.$$
Again, each $r^{\mathcal{I}}(\varrho e_j)$ must be strictly greater than 0, and hence if $z_s = 1$, then each e_j must be different from $\mathsf{tail}(\varrho)$.

Due to the construction of $\Delta^{\mathcal{I}_{\mathsf{c}}}$, the above elements of $\Delta^{\mathcal{I}}$ ($\mathsf{prev}(\varrho)$, $\mathsf{tail}(\varrho)$, $a^{\mathcal{I}}$ for $a \in S$, and e_j, $1 \leqslant j \leqslant m$) must be different. But this contradicts the semantics of $(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho))$.

For the first kind of assertions in $\mathsf{red}(\geqslant n\, r.C)$, consider any $a \in \mathsf{ind}(\mathcal{O})$. Since $\mathcal{I}$ is witnessed, there must be n different elements $e_1, \ldots, e_n \in \Delta^{\mathcal{I}}$ such that
$$(\geqslant n\, r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = \min_{j=1}^{n} \min\{r^{\mathcal{I}}(a^{\mathcal{I}}, e_j), C^{\mathcal{I}}(e_j)\}.$$
For each e_j, $1 \leqslant j \leqslant n$, we make a case distinction on whether it is named or not.

- If $e_j = b^{\mathcal{I}}$ for some $b \in \mathsf{ind}(\mathcal{O})$, then we have $(a,b) \in \mathfrak{r}^{\mathcal{I}_{\mathsf{c}}}$, $b \in (\neg\mathsf{AN})^{\mathcal{I}_{\mathsf{c}}}$, and
$$((\geqslant n\, r.C)(a))^{\mathcal{I}}(b) = (\geqslant n\, r.C)^{\mathcal{I}}(a^{\mathcal{I}}) \leqslant \min\{r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}), C^{\mathcal{I}}(b^{\mathcal{I}})\} = \min\{(r(a,*))^{\mathcal{I}}(b), C^{\mathcal{I}}(b)\}.$$

- If $e_j \neq b^{\mathcal{I}}$ for all $b \in \mathsf{ind}(\mathcal{O})$, then we have $ae_j \in \Delta^{\mathcal{I}_{\mathsf{c}}}$, and thus $(a, ae_j) \in \mathfrak{r}^{\mathcal{I}_{\mathsf{c}}}$. Furthermore, we know that $\mathsf{tail}(ae_j) = e_j \neq a^{\mathcal{I}} = \mathsf{prev}(ae_j)$, and hence
$$\langle \geqslant n\, r.C\rangle^{\mathcal{I}}_{\uparrow}(ae_j) = (\geqslant n\, r.C)^{\mathcal{I}}(a^{\mathcal{I}}) \leqslant \min\{r^{\mathcal{I}}(a^{\mathcal{I}}, e_j), C^{\mathcal{I}}(e_j)\} = \min\{r^{\mathcal{I}}(ae_j), C^{\mathcal{I}}(ae_j)\}.$$

Since different elements of $\Delta^{\mathcal{I}}$ induce different elements of $\Delta^{\mathcal{I}_{\mathsf{c}}}$, this shows that the required at-least restriction is satisfied by a.

For the second kind of assertions in $\mathsf{red}(\geqslant n\, r.C)$, assume to the contrary that there are n different $\mathfrak{r}$-successors $\varrho_1, \ldots, \varrho_n$ of a that are either anonymous and satisfy $\boxed{\langle \geqslant n\, r.C\rangle_{\uparrow} < \min\{r, C\}}$, or named and satisfy $\boxed{((\geqslant n\, r.C)(a)) < \min\{r(a,*), C\}}$.

- If ϱ_j satisfies AN, then it must be of the form ae_j for some $e_j \in \Delta^{\mathcal{I}}$ and we have
$$(\geqslant n\, r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = \langle \geqslant n\, r.C\rangle^{\mathcal{I}}_{\uparrow}(ae_j) < \min\{r^{\mathcal{I}}(ae_j), C^{\mathcal{I}}(ae_j)\} = \min\{r^{\mathcal{I}}(a^{\mathcal{I}}, e_j), C^{\mathcal{I}}(e_j)\}$$
since $\mathsf{tail}(ae_j) = e_j \neq a^{\mathcal{I}} = \mathsf{prev}(ae_j)$.

- If ϱ_j does not satisfy AN, then it is of the form b and we obtain
$$(\geqslant n\, r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = ((\geqslant n\, r.C)(a))^{\mathcal{I}}(b) < \min\{(r(a,*))^{\mathcal{I}}(b), C^{\mathcal{I}}(b)\} = \min\{r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}), C^{\mathcal{I}}(b^{\mathcal{I}})\}.$$
If we consider equivalence classes of individual names as roots for $\mathcal{I}_{\mathsf{c}}$, all such $b^{\mathcal{I}}$ are different.

This again contradicts the semantics of $(\geqslant n\, r.C)^{\mathcal{I}}(a^{\mathcal{I}})$. □

Lemma 6.2. *If $\mathcal{O}$ is consistent, then it has a tableau, and if it has a countable tableau, then $\mathcal{O}$ is consistent.*

Proof. Let first $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a G-model of $\mathcal{O}$. We construct the tableau $(\Delta^{\mathcal{I}}, \mathcal{A}^*)$, where $\mathcal{A}^*$ is the set of all order assertions $u \bowtie v$ such that $u, v \in \mathcal{U}(\Delta^{\mathcal{I}})$ with $u^{\mathcal{I}} \bowtie v^{\mathcal{I}}$, where $p^{\mathcal{I}} := p$, $C(d)^{\mathcal{I}} := C^{\mathcal{I}}(d)$, $r(d,e)^{\mathcal{I}} := r^{\mathcal{I}}(d,e)$, and $(\neg r(d,e))^{\mathcal{I}} := 1 - r^{\mathcal{I}}(d,e)$. By construction, $\mathcal{A}^*$ is satisfiable, and hence we only need to prove that it satisfies the conditions (T1)–(T22) of Definition 6.1. This can be verified by a trivial, but lengthy, case analysis, very similar to the proof of Lemma A.2. For example, if $(\forall \mathbf{A}^q.C)(d)$ occurs in $\mathcal{A}^*$ and q is final, then we know that $(\|\mathbf{A}^q\|, \varepsilon) = 1$, and hence

$$(\forall \mathbf{A}^q.C)^{\mathcal{I}}(d) = \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} w^{\mathcal{I}}(d,e) \Rightarrow C^{\mathcal{I}}(e) \leqslant \varepsilon^{\mathcal{I}}(d,d) \Rightarrow C^{\mathcal{I}}(d) = C^{\mathcal{I}}(d).$$

By our construction, this means that $(\forall \mathbf{A}^q.C)(d) \leqslant C(d)$ is contained in $\mathcal{A}^*$, and thus (T13) is satisfied.

For the second part of the lemma, let $(\Delta, \mathcal{A}^*)$ be a tableau where Δ is countable, and hence $\mathcal{U}(\Delta)$ and $\mathcal{A}^*$ are also countable. Since $\mathcal{A}^*$ is satisfiable, it has a model $\preceq_*$. Since this model is an element of $\mathsf{order}(\mathcal{U}(\Delta))$, there must exist a mapping $v \colon \mathcal{U}(\Delta) \to [0,1]$ with the following properties:

(P1) for all $p \in \mathsf{val}(\mathcal{O})$, we have $v(p) = p$;

(P2) for all $\alpha, \beta \in \mathcal{U}(\Delta)$, we have $v(\alpha) \leqslant v(\beta)$ iff $\alpha \preceq_* \beta$; and

(P3) for all $\alpha \in \mathcal{U}(\Delta)$, we have $v(\mathsf{inv}(\alpha)) = 1 - v(\alpha)$.

We now define the G-interpretation $\mathcal{I}$ as follows, for all $a \in \mathsf{ind}(\mathcal{O})$, $A \in \mathsf{sub}(\mathcal{O}) \cap \mathsf{N_C}$, and $x \in \Delta$:

- $\Delta^{\mathcal{I}} := \Delta$;
- $a^{\mathcal{I}} := a$; and
- $A^{\mathcal{I}}(x) := v(A(x))$ if $A(x)$ occurs in $\mathcal{A}^*$, and $A^{\mathcal{I}}(x) := 0$ otherwise.

The interpretation of all other individual names and concept names can be fixed arbitrarily. For the role names $r \in \mathsf{rol}(\mathcal{O}) \cap \mathsf{N_R}$, we first define a "simple" G-interpretation $\mathcal{I}_0$ as follows: $r^{\mathcal{I}_0}(x,y) := v(r(x,y))$ if $r(x,y)$ occurs in $\mathcal{A}^*$, and $r^{\mathcal{I}_0}(x,y) := 0$ otherwise. By (P2), for every inverse role $r^- \in \mathsf{rol}(\mathcal{O})$ for which $r^-(x,y)$ occurs in $\mathcal{A}^*$, we have

$$(r^-)^{\mathcal{I}_0}(x,y) = r^{\mathcal{I}_0}(y,x) = v(r(y,x)) = v(r^-(x,y)),$$

since we identify the latter two assertions; this is similar to the definition of $r^{\mathcal{I}_0}$. We now use the automaton $\mathbf{A}_r$ to "complete" $\mathcal{I}_0$ with additional links as follows: we set

$$r^{\mathcal{I}}(x,y) := \sup_{w \in \mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}_0}(x,y)\} \tag{A.8}$$

for all $x, y \in \Delta$. Note that this expression is equal to $r^{\mathcal{I}_0}(x,y)$ if r is simple: by Proposition 4.5, we have $(\|\mathbf{A}_r\|, s) = p$ whenever $s \sqsubseteq_p r$, $(\|\mathbf{A}_r\|, r) = 1$, and $(\|\mathbf{A}_r\|, w) = 0$ for all other words w, and moreover (T21) yields

$$\min\{(\|\mathbf{A}_r\|, r), r^{\mathcal{I}_0}(x,y)\} = r^{\mathcal{I}_0}(x,y) \geqslant \min\{p, s^{\mathcal{I}_0}(x,y)\} = \min\{(\|\mathbf{A}_r\|, s), s^{\mathcal{I}_0}(x,y)\}.$$

The expression (A.8) can also be used to evaluate inverse roles due to the semantics of role chains, the fact that $\mathbf{A}_{r^-}$ is a mirrored copy of $\mathbf{A}_r$ (see Section 4), and Proposition 4.3.

To show that $\mathcal{I}$ is a G-model of $\mathcal{O}$, we first prove the following claim by induction on the structure of C:

$$\text{For all } x \in \Delta \text{ and } C \in \mathsf{sub}(\mathcal{O}) \text{ for which } C(x) \text{ occurs in } \mathcal{A}^*, \text{ we have } C^{\mathcal{I}}(x) = v(C(x)). \tag{A.9}$$

For most concept constructors, this easily follows from the conditions in Definition 6.1 and the fact that $\preceq_*$, and hence v, satisfies all entailments of $\mathcal{A}^*$.

For negation, assume that $(\neg C)(x)$ occurs in $\mathcal{A}^*$. We get

$$(\neg C)^{\mathcal{I}}(x) = 1 - C^{\mathcal{I}}(x) = 1 - v(C(x)) = v((\neg C)(x))$$

by (T3), (P3), and the induction hypothesis.

For any $\exists s.\mathsf{Self} \in \mathsf{sub}(\mathcal{O})$ such that $(\exists s.\mathsf{Self})(x)$ occurs in $\mathcal{A}^*$, we have

$$(\exists s.\mathsf{Self})^{\mathcal{I}}(x) = s^{\mathcal{I}}(x,x) = s^{\mathcal{I}_0}(x,x) = v(s(x,x)) = v((\exists s.\mathsf{Self})(x))$$

since s is simple and we treat $s(x,x)$ and $(\exists s.\mathsf{Self})(x)$ synonymously.

Assume now that $(\forall r.C)(x)$ occurs in $\mathcal{A}^*$. By (T14), there must be a $y_0 \in \Delta$ such that

$$v((\forall r.C)(x)) \geqslant v(r(x,y_0)) \Rightarrow v(C(y_0)) = r^{\mathcal{I}_0}(x,y_0) \Rightarrow C^{\mathcal{I}}(y_0) \geqslant r^{\mathcal{I}}(x,y_0) \Rightarrow C^{\mathcal{I}}(y_0).$$

Hence, y_0 can act as a witness for $(\forall r.C)^{\mathcal{I}}(x)$ if we can show that the latter implication is $\geqslant v((\forall r.C)(x))$ for all elements $y \in \Delta$. For this purpose, we consider the remaining tableau conditions for value restrictions. By (T10), we get

$$\begin{aligned}
r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y) &= \Big(\sup_{w \in \mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}_0}(x,y)\}\Big) \Rightarrow C^{\mathcal{I}}(y) \\
&= \inf_{w \in \mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}_0}(x,y)\} \Rightarrow C^{\mathcal{I}}(y) \\
&\overset{(*)}{\geqslant} v((\forall \mathbf{A}_r.C)(x)) \\
&\geqslant v((\forall r.C)(x)).
\end{aligned}$$

as required, if we can show $(*)$, i.e. it remains to show that

$$\min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}_0}(x,y)\} \Rightarrow C^{\mathcal{I}}(y) \geqslant v((\forall \mathbf{A}_r.C)(x))$$

holds for all $w = r_1 \dots r_n \in \mathsf{rol}(\mathcal{O})^+$. If y is not connected to x, i.e. we have $w^{\mathcal{I}_0}(x,y) = 0$ for all such w, then this is trivial. The claim for all other y can be shown exactly as in the proof of Lemma A.1.

Consider now a number restriction for which $({\geqslant}n\, r.C)(x)$ occurs in $\mathcal{A}^*$. Recall that r must be simple, and hence we have $r^{\mathcal{I}} = r^{\mathcal{I}_0}$. If, for some $y \in \Delta$, $r(x,y)$ does not occur in $\mathcal{A}^*$, then we know that $v(({\geqslant}n\, r.C)(x)) \geqslant 0 = \min\{r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y)\}$. By (T15), (T16), and the induction hypothesis, we know that there are at most $n-1$ elements $y \in \Delta$ for which $\min\{r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y)\}$ is strictly greater than $v(({\geqslant}n\, r.C)(x))$. This means that for any n different elements $y_1, \dots, y_n \in \Delta$, we have

$$v(({\geqslant}n\, r.C)(x)) \geqslant \min_{i=1}^{n} \min\{r^{\mathcal{I}}(x,y_i), C^{\mathcal{I}}(y_i)\}.$$

Hence, to prove $v(({\geqslant}n\, r.C)(x)) = ({\geqslant}n\, r.C)^{\mathcal{I}}(x)$, it suffices to find n witnessing elements where the latter inequation holds with $=$ instead of only $\geqslant$. Their existence follows directly from (T14).

With the help of (A.9), it is now easy to show that $\mathcal{I}$ satisfies all axioms of $\mathcal{O}$. $\square$

Theorem 6.8. *Every sequence of applications of the tableau rules to $\mathcal{G}_0$ terminates. Moroever, if the tableau rules can be applied to $\mathcal{G}_0$ in such a way that a complete and clash-free completion graph is obtained, then $\mathcal{O}$ has a countable tableau. Finally, if $\mathcal{O}$ has a tableau, then the tableau rules can be applied to $\mathcal{G}_0$ in such a way that a complete and clash-free completion graph is obtained.*

We prove the three parts of this result separately.

Lemma A.3. *Every sequence of applications of the tableau rules to $\mathcal{G}_0$ terminates.*

Proof. Let $m := |\mathsf{sub}(\mathcal{O})|$, $k := |\mathsf{rol}(\mathcal{O})|$, n be the maximal number occurring in number restrictions in $\mathsf{sub}(\mathcal{O})$, $\ell := |\mathsf{ind}(\mathcal{O})|$, and $o := |\mathsf{val}(\mathcal{O})|$. Recall that in the worst case m is exponential in the size of the role hierarchy, and exponential in n if numbers are given in binary encoding. However, the exponential blowup in n is irrelevant since each set $\mathcal{L}(x)$ can contain at most one additional at-least restriction ${\geqslant}m\, r.C$ for each ${\geqslant}n\, r.C$ that occurs in $\mathcal{O}$.

Observe first that the relation $\mathcal{N}$ restricted to the blockable nodes is always tree-shaped. More precisely, such trees are rooted in nominal nodes and leaves may have outgoing $\mathcal{N}$-edges to nominal nodes. To see this, assume that the application of one of the tableau rules destroys this property, i.e. creates a completion graph with a blockable node x that has two different predecessors, i.e. $(y_1, x), (y_2, x) \in \mathcal{N}$. Then the rule that was applied must be a shrinking rule, and it must further be the case that y_1 and y_2 each had a blockable successor, x and x', respectively, and x' was merged into x by the rule $(\neg\geqslant)$ (the other two shrinking rules do merge two blockable nodes). But then there must be a common neighbor z of x and x' such that $\mathcal{L}(z)$ entails $(\geqslant n\, r.C)(z) < \min\{r(z,x), C(x)\}$ and $(\geqslant n\, r.C)(z) < \min\{r(z,x'), C(x')\}$. This means that z is a nominal node since otherwise the structure among blockable nodes would already have been non-tree-shaped before. Furthermore, we must have either $(x,z) \in \mathcal{N}$ or $(x',z) \in \mathcal{N}$ since otherwise either $z = y_1 = y_2$ or either x or x' already had two different predecessors. This means that there must be $m \leqslant n-1$ nominal neighbors $z_1, \dots, z_m$ with $z_i \neq z_j$, $1 \leqslant i < j \leqslant m$, and $\mathcal{L}(z)$ entails $(\geqslant(m+1)\, r.C)(z) < \min\{r(z,z_i), C(z_i)\}$, $1 \leqslant i \leqslant m$; otherwise the rule (NN) would be applicable and would have been applied before $(\neg\geqslant)$. But then immediately afterwards the rule $(\neg\geqslant_o)$ would have to be applied, and would merge x and x' into one of the nominal nodes $z_1, \dots, z_m$, thus invalidating our assumption.

Further observe that nodes and elements from node labels can only be removed by the shrinking rules, and new nodes can only be added by the generating rules. Moreover, each generating rule can be triggered at most once for each concept in $\mathsf{sub}(\mathcal{O})$ occurring in the label of a node x. For the rules involving role connections to neighboring nodes, this observation is due to the fact that, if a neighbor y of x is merged into another node z, then z inherits all relevant order assertions from y, and either z is then a neighbor of x (if x is a nominal node or y is a successor of x) or x is removed by pruning (if x is a blockable node and x is a successor of y). This means that each node can have at most mn blockable successors.

The next crucial observation is that blocking, which can occur only within a path consisting only of blockable nodes, occurs after at most $\lambda := 2^{2(o+2m+4k+2)^2} + 1$ steps. For this, it suffices to determine the total number of possible order assertions (not involving min or $\Rightarrow$, and using only $\leqslant$ or $>$) that can be formulated about two neighboring nodes. The underlying order structure contains $o+2m+4k+2$ elements, and hence there are $2(o+2m+4k+2)^2$ such order assertions. This means that each $\mathcal{N}$-chain of blockable nodes must contain a directly blocked node after at most λ steps. This implies that all blockable subtrees of the completion graph have branching degree at most mn and depth at most λ.

The last step is to show that the number of nominal nodes is bounded by $O(\ell(mn)^\lambda)$. The proof of this proceeds as in [30]: The rule (NN) can initially only be triggered due to a newly created nominal node x, which must be of level 0 since the only individual names occurring in $\mathsf{sub}(\mathcal{O})$ are those in $\mathsf{ind}(\mathcal{O})$. Afterwards, it may be applied to predecessors of x that were originally blockable but were then merged into nominal neighbors of x. Since the length of such a chain of blockable nodes is at most λ, the rule (NN) can be applied only to nominal nodes of level below λ. Furthermore, this rule can be applied at most m times to each node of level i (or its heirs), each time generating at most n new nominals, and hence at most $\ell(mn)^{i+1}$ nominal nodes of level $i+1$. Since it can only be applied up to level λ, this gives an upper bound of $O(\ell(mn)^\lambda)$ new nominal nodes. Additionally, each nominal node may be the root of a blockable tree of size $O((mn)^\lambda)$. Hence, the total number of nodes in a completion graph is finite, and thus each completion graph must become complete after finitely many steps. □

We now prove that the algorithm correctly decides consistency of $\mathcal{O}$ (cf. Lemma 6.2).

Lemma A.4. *If the tableau rules can be applied to $\mathcal{G}_0$ in such a way that a complete and clash-free completion graph is obtained, then there exists a countable tableau for $\mathcal{O}$.*

Proof. Assume that the tableau rules have been applied to $\mathcal{G}_0$, resulting in a complete and clash-free completion graph $\mathcal{G} = (\Delta, \Delta_o, \mathcal{N}, \mathcal{L}, \mathcal{L}_o, \neq)$. We first modify the labeling function for all nominal nodes x, by removing all assertions from $\mathcal{L}(x)$ that refer to blockable nodes, and adding all entailments about nominal nodes that may have been lost in this process. More formally, we first add all order assertions over $\mathcal{U}_o$ that are entailed by $\mathcal{L}(x)$ to $\mathcal{L}_o$, and then define $\mathcal{L}'(x) := \emptyset$; note that this new label still implicitly includes all order assertions from $\mathcal{L}_o$. The labels of the blockable nodes are not affected.

This construction allows a better separation of the behavior of the nominal nodes from that of the blockable nodes. Observe that all relevant order assertions that refer to the connection between a nominal node x and a blockable neighbor y have already been transferred to $\mathcal{L}(y)$ by the rule ($\rightsquigarrow$). Furthermore, $\mathcal{G}$ is still clash-free after this modification; however, it may not be complete anymore for the nominal nodes. For example, although a nominal node may have forgotten about a blockable witness for a value restriction, the witness itself still has all relevant information. The blocking relationships between nodes are not affected since they do not involve nominal nodes.

We now construct a countable tableau $(\Delta', \mathcal{A}^*)$ of $\mathcal{O}$ by following the structure of $\mathcal{N}$ and, at each directly blocked node x, unraveling the structure by replacing x with a copy of an ancestor that blocks it. At the same time, we will construct a function $f\colon \Delta' \to \Delta$ that specifies which node was used to construct each element of Δ'.

We initially set

- $\Delta' := \{x \in \Delta \mid x \text{ is not blocked}\}$,
- $\mathcal{A}^* := \bigcup_{x\in\Delta'} \mathcal{L}'(x)$, and
- $f(x) := x$ for all $x \in \Delta'$,

and assume first that $\mathcal{A}^*$ is unsatisfiable. According to Lemma 2.2, this can be the case only because of a sequence of elements $\alpha_1 \trianglelefteq_1 \cdots \trianglelefteq_{n-1} \alpha_n$, where $\trianglelefteq_i \in \{<, \leqslant\}$, each order assertion $\alpha_i \trianglelefteq_i \alpha_{i+1}$, $1 \leqslant i \leqslant n-1$, is entailed by some $\mathcal{L}'(x_i)$ with $x_i \in \Delta'$, we have $\alpha_1 = \alpha_n$, and at least one of the $\trianglelefteq_i$ is a strict inequality ($<$). Let this be a sequence that has minimal length among all sequences with this property. We show the following properties:

(a) We have $n > 2$. Otherwise, $\alpha_1 <_1 \alpha_1$ would be entailed by $\mathcal{L}'(x_1)$, which contradicts the clash-freeness of $\mathcal{G}$.

(b) We have $x_i \neq x_{i+1}$ for all i, $1 \leqslant i \leqslant n-2$, and $x_{n-1} \neq x_1$. Otherwise, by (a) we would have a situation like $\alpha_1 \trianglelefteq_1 \alpha_2 \trianglelefteq_2 \alpha_3$ (modulo cyclic index shifts), where both order assertions are entailed by the same $\mathcal{L}'(x)$. But then also $\alpha_1 \trianglelefteq \alpha_3$ would be entailed by $\mathcal{L}'(x)$, where $\trianglelefteq$ is $<$ if $< \in \{\trianglelefteq_1, \trianglelefteq_2\}$, and otherwise it is $\leqslant$. But this shows the existence of a shorter sequence with the same properties as before, in contradiction to our minimality assumption.

(c) None of the α_i that belongs to $\mathcal{U}_o$. Otherwise, by (a) we would have a situation like $\alpha_1 \trianglelefteq_1 \alpha_2 \trianglelefteq_2 \alpha_3$, where $\alpha_1 \in \mathcal{U}_o$. But then we would have $\alpha_1, \alpha_2 \in \mathcal{U}(x_2)$; in particular, if x_2 is a nominal node, then we would have $\alpha_1, \alpha_2 \in \mathcal{U}_o$ due to our modification of the labels. But then the rule ($\rightsquigarrow$) implies that $\alpha_1 \trianglelefteq_1 \alpha_2$ would also entailed by $\mathcal{L}'(x_2)$, which contradicts (b).

Since the (modified) labels of nominal nodes are restricted to $\mathcal{L}_o$, (c) implies that the x_i are all blockable nodes. Furthermore, since $\alpha_i \in \mathcal{U}(x_{i-1}) \cap \mathcal{U}(x_i)$, $2 \leqslant i \leqslant n-1$, and $\alpha_1 = \alpha_n \in \mathcal{U}(x_1) \cap \mathcal{U}(x_{n-1})$, each pair (x_{i-1}, x_i), $2 \leqslant i \leqslant n-1$, and (x_1, x_{n-1}) must be neighbors. The tree structure of $\mathcal{N}$ on the blockable nodes and (b) imply that there is a situation like $\alpha_1 \trianglelefteq_1 \alpha_2 \trianglelefteq_2 \alpha_3 \trianglelefteq_3 \alpha_4$ such that $x_1 = x_3$. Hence, we have $\alpha_2, \alpha_3 \in \mathcal{U}(x_1)$, which shows by ($\rightsquigarrow$) that $\alpha_1 \trianglelefteq \alpha_4$ is already entailed by $\mathcal{L}'(x_1)$, where $\trianglelefteq$ is obtained as in the proof of (b). But this again contradicts the minimality of n.

This shows that the initial tableau constructed above is satisfiable. We now iteratively expand it by unraveling $\mathcal{N}$ at the blocked nodes. Let $x' \in \Delta'$ be such that $f(x')$ has a directly blocked successor x in $\mathcal{G}$ that is not yet represented in our tableau, i.e. there is no $x'' \in \Delta'$ such that $f(x'') = x$ and x'' is connected to x' by some role assertions. Let further $x^@$ be the name that is used in $\mathcal{A}^*$ to refer to this still missing successor of x', $y \in \Delta$ be a node that blocks x in $\mathcal{G}$, and y' be the predecessor of y in $\mathcal{N}$. Recall that y', y, $f(x')$, x, and all nodes in between are blockable. Consider now the subtree $\Delta_y \subseteq \Delta$ consisting of all blockable descendants of y that are not blocked. To distinguish these nodes from those already present in Δ', for each blockable node z occurring in $\mathcal{L}'(v)$ for some $v \in \Delta_y$, let $z^@$ be a unique new node name that does not yet occur in Δ'. We now do the following:

- add $\{z^@ \mid z \in \Delta_y\}$ to Δ';
- replace all occurrences $x^@$ in $\mathcal{A}^*$ by $y^@$ and add $\bigcup_{z\in\Delta_y} \mathcal{L}'(z)^@$ to $\mathcal{A}^*$, where $\mathcal{L}'(z)^@$ is obtained from $\mathcal{L}'(z)$ by replacing all occurrences of $f(x')$ by x', and of any other blockable nodes z by $z^@$; and
- set $f(z^@) := z$ for all $z \in \Delta_y$.

The resulting set $\mathcal{A}^*$ is such that it looks as if x has never been blocked, but rather that the tableau rules have been applied to it and its successors without restrictions. Assume now that $\mathcal{A}^*$ has become unsatisfiable by this construction, and hence there is a sequence $\alpha_1 \trianglelefteq_1 \cdots \trianglelefteq_{n-1} \alpha_n$ as above, where n is minimal. Since the original $\mathcal{A}^*$, and hence also $\bigcup_{z\in\Delta_y} \mathcal{L}'(z)^@$, are satisfiable, it must be the case that this sequence involves nodes from the previous Δ' as well some of the form $z^@$ for $z \in \Delta_y$. We can show the properties (a)–(c) as before. Moreover, the tree-shape of the connections between blockable nodes is maintained by our construction. To derive a contradiction in the same way as above, it suffices to note that due to the blocking condition all order assertions shared by $f(x')$ and x' are also shared by x' and $y^@$ (after renaming $x^@$ to $y^@$ and $f(x')$ to x'), and hence $\mathcal{L}'(y)^@$ and the set corresponding to $\mathcal{L}'(f(x'))$ in $\mathcal{A}^*$ behave as if the rule $(\rightsquigarrow)$ has been applied exhaustively.

If we continue this process infinitely, taking care that every directly blocked node is unraveled eventually, we obtain the final tableau $(\Delta', \mathcal{A}^*)$. The set $\mathcal{A}^*$ is satisfiable due to the compactness theorem of first-order logic. It remains to verify the tableau conditions. It is easy to verify that the local conditions (T1)–(T3), (T5), (T7), (T8), (T10), (T12), (T13), (T17), and (T20) are satisfied due to the corresponding tableau rules. We consider the remaining ones:

(T9) If $x \in \Delta'$ is such that $(\forall r.C)(x)$ occurs in $\mathcal{A}^*$, then $f(x)$ is not blocked in $\mathcal{G}$. Hence, by the rule $(\forall)$ there must be a safe neighbor $y \in \mathcal{N}(x)$ such that $\mathcal{L}(x)$ entails $(\forall r.C)(f(x)) \geqslant r(f(x), y) \Rightarrow C(y)$.

Consider first the case that x is blockable. If y is not blocked, then we have directly introduced (a copy y' of) y into Δ', together with (the copy x of) $f(x)$, and the (renamed) entailment still holds in $\mathcal{A}^*$. If y is a successor of x, then it may be the case that y is directly blocked in $\mathcal{G}$. But then we have introduced a node y' into Δ' that can serve as a replacement for this missing successor, i.e. $\mathcal{A}^*$ entails $(\forall r.C)(x) \geqslant r(x, y') \Rightarrow C(y')$ due to the blocking condition.

If x is a nominal node, then we know that y is not blocked since it is a safe neighbor of x. Nevertheless, it may be that we have removed from $\mathcal{L}(x)$ some assertions that were necessary to derive the above entailment; this can only be the case if y is blockable. However, by the rule $(\rightsquigarrow)$, this entailment has been transferred to $\mathcal{L}(y)$, and is still present in $\mathcal{L}'(y)$, which is why it is still entailed by $\mathcal{A}^*$.

(T11) If $(\forall \mathbf{A}.C)(x)$ and $r(x, y)$ occur in $\mathcal{A}^*$, then the required entailment is provided by the rule $(\forall_2)$. Again, the modification of $\mathcal{L}$ to $\mathcal{L}'$ for the nominal nodes is rendered irrelevant by the rule $(\rightsquigarrow)$.

(T14) This case can be handled by similar arguments as for (T9). Additionally, the n safe neighbors created by the rule $(\geqslant)$ are still distinct in Δ' since they can never be merged.

(T15) Assume that $(\geqslant n\, r.C)(x)$ occurs in $\mathcal{A}^*$ and there are different $y_1, \dots, y_n \in \Delta'$ such that $\mathcal{A}^*$ entails $(\geqslant n\, r.C)(x) < \min\{r(x, y_i), C(y_i)\}$. By our construction, we know that $(\geqslant n\, r.C)(f(x))$ occurs in $\mathcal{L}(f(x))$ and there exist n neighbors $y'_1, \dots, y'_n$ of $f(x)$ (which are possibly blocked) for which similar assertions are entailed by $\mathcal{L}(f(x))$. Since $\mathcal{G}$ is clash-free, there must be two of these neighbors that are not in the relation $\neq$, and hence the rule $(\neg\geqslant)$ is applicable to $\mathcal{G}$. This contradicts our assumption that $\mathcal{G}$ is complete.

(T18) For each $a \in \mathsf{ind}(\mathcal{O})$, the existence of exactly one nominal node for a is due to the definition of the initial completion graph $\mathcal{G}_0$, clash-freeness of $\mathcal{G}$, the rule (o), and our construction of the tableau.

(T19) We consider the example of an assertion $r(a, b) \geqslant C(c)$ in $\mathcal{A}$. In $\mathcal{G}_0$, there exist nodes a, b, c that are all neighbors, and each label entails $r(a, b) \geqslant C(c)$. Due to merging, in $\mathcal{G}$ there exist heirs x_a, x_b, x_c of these original nodes, which inherit the neighbor relationships as well as the required entailment. Hence, $\mathcal{A}^*$ also entails $r(x_a, x_b) \geqslant C(x_c)$. The proofs for the other kinds of assertions are similar.

Finally, (T4), (T6), (T16), (T21), and (T22) can be shown using similar arguments. $\square$

The other direction is easier to show.

Lemma A.5. *If there is a tableau for $\mathcal{O}$, then the tableau rules can be applied to $\mathcal{G}_0$ in such a way that a complete and clash-free completion graph is obtained.*

Proof. We use the tableau $(\Delta', \mathcal{A}^*)$ for $\mathcal{O}$ to guide the application of the completion rules to $\mathcal{G}_0$. We will maintain a function $f\colon \Delta \to \Delta'$ that matches the nodes of our completion graph to the nodes of the tableau, such that the following conditions are satisfied:

(i) If $\alpha \bowtie \beta$ occurs in $\mathcal{L}(x)$ and α, β do not involve number restrictions or nominals that do not occur in $\mathcal{O}$, then $\mathcal{A}^*$ entails $f(\alpha) \bowtie f(\beta)$, where $f(\alpha)$ is obtained from α by replacing all nodes according to f.

(ii) If $x \neq y$, then $f(x) \neq f(y)$.

(iii) If ${\geqslant} m\, r.C \in \mathsf{sub}(\mathcal{O})$ does not occur in $\mathcal{O}$ and $({\geqslant} m\, r.C)(x)$ occurs in $\mathcal{L}(x)$, then there are exactly $m-1$ elements $y \in \Delta'$ such that $\mathcal{A}^*$ entails

$$({\geqslant} n\, r.C)(f(x)) < \min\{r(f(x), y), C(y)\}.$$

For each $({\geqslant} m\, r.C)(x)$ occurring in $\mathcal{L}(x)$ for which ${\geqslant} m\, r.C$ does not occur in $\mathcal{O}$, we know that $\mathcal{L}(x)$ entails $({\geqslant} m\, r.C)(x) = ({\geqslant} n\, r.C)(x)$ for some ${\geqslant} n\, r.C$ that does occur in $\mathcal{O}$. Hence, (i) and the satisfiability of $\mathcal{A}^*$ imply that all node labels of our completion graph will be satisfiable. Furthermore, clashes due to number restrictions are ruled out by (i)–(iii) and (T15). Finally, nominals behave correctly due to (i), (ii), and (T18). Hence, our final completion graph will be clash-free.

For the initial completion graph, we set $f(a) := x_a$ for all $\mathsf{ind}(\mathcal{O})$, where x_a is the nominal node that exists by (T18). Due to (T18) and (T19), this mapping satisfies all our conditions. We now show by induction on the sequence of rule applications how the tableau rules can be applied in order to maintain the conditions (i)–(iii). For most of the tableau rules, it is trivial to show that they can be applied in such a way that the conditions remain satisfied. For the rules that have to make nondeterministic choices because of the semantics of $\Rightarrow$ and min (i.e. the rules $(\sqcap)$, $(\to)$, $(\sqsubseteq_\mathcal{T})$, $(\sqsubseteq_\mathcal{R})$, (dis), $(\forall_2)$, $(\forall_2^\varepsilon)$, $(\exists_2)$, $(\exists_2^\varepsilon)$, (ch), and (ch_o)), we know by the corresponding conditions of Definition 6.1 and our semantics that we can always choose one of the alternatives such that (i) is not violated. It is also clear that the rule $(\rightsquigarrow)$ does not affect this condition.

Consider now the rule $(\geqslant)$; the arguments for $(\exists)$ and $(\forall)$ are similar. Assume that we have to apply this rule because $({\geqslant} n\, r.C)(x)$ occurs in $\mathcal{L}(x)$, and hence by (i) the element $({\geqslant} n\, r.C)(f(x))$ occurs in $\mathcal{A}^*$. Due to (T14), there are at least n elements $y_1, \dots, y_n \in \Delta'$ such that $\mathcal{A}^*$ entails $({\geqslant} n\, r.C)(x) \leqslant \min\{r(x, y), C(y)\}$, and hence we can introduce n new neighbors $y_1', \dots, y_n'$ according to $(\geqslant)$ and set $f(y_i') := y_i$, $1 \leqslant i \leqslant n$, in order to keep the conditions (i)–(iii) satisfied.

For the shrinking rule $(\neg{\geqslant}_o)$, consider any $({\geqslant} n\, r.C)(x)$, y, and $z_1, \dots, z_{n-1}$ as in the preconditions of this rule. Then by (iii) or (i) and (T15), we know that Δ' contains at most $n-1$ nodes z for which $\mathcal{A}^*$ entails $({\geqslant} n\, r.C)(f(x)) < \min\{r(f(x), z), C(z)\}$. Furthermore, by (i), the nodes $f(y), f(z_1), \dots, f(z_{n-1})$ all satisfy this condition. By (ii), this implies that there is an index i, $1 \leqslant i \leqslant n-1$, such that $f(y) = f(z_i)$, and hence $y \doteq z_i$. This shows that the rule $(\neg{\geqslant}_o)$ can be applied in such a way that all conditions remain satisfied. The same can be shown for $(\neg{\geqslant})$ using similar arguments.

For (o), assume that there exist an $a \in \mathsf{N_I}$ and two nodes x, y whose labels entail $\{a\}(x) \geqslant 1$ and $\{a\}(y) \geqslant 1$, respectively. This can only be the case for $a \in \mathsf{ind}(\mathcal{O})$ since the rule (NN) always introduces new individual names. Hence, (i) and (T18) imply that $\pi(x) = \pi(y)$, and thus we can again merge these two nodes.

Finally, consider the rule (NN). If all its preconditions are satisfied by $({\geqslant} n\, r.C)(x)$ and y, then we know that it has not been applied to this number restriction at x (or any node that was merged into x) before. Hence, ${\geqslant} n\, r.C$ must occur in $\mathcal{O}$, and thus (i) and (T15) imply that there are exactly $m \leqslant n-1$ elements

$z'_1, \ldots, z'_m \in \Delta'$ for which $\mathcal{L}(x)$ entails $(\geqslant n\, r.C)(x) < \min\{r(f(x), z'_i), C(z'_i)\}$, $1 \leqslant i \leqslant m$. This shows that we can apply the rule and create m new nominal nodes $z_1, \ldots, z_m \in \Delta$, for which we set $f(z_i) := z'_i$, $1 \leqslant i \leqslant m$, without violating the conditions.

Using Lemma A.3, this shows that after finitely many steps we will have produced a complete and clash-free completion graph. $\square$