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# Sequential Games with Perfect Information

Lecture 3, 29th April 2024 // Algorithmic Game Theory, SS 2024

# Previously ...

- A **mixed strategy** is a probability distribution on pure strategies.
- In a **mixed Nash equilibrium**, all players play best responses.
- **Nash's Theorem**: Mixed Nash equilibria always exist (for finite games).
- Nash equilibria for concrete games can be obtained via a translation to a **mixed integer program**:
  - Binary variables model the choices of pure strategies to put in the support;
  - real-valued variables model probabilities, utilities, and regret.
- Given a game and an equilibrium, it is FNP-complete to find *another* equilibrium for the game.

## Penalties

Two football players face off at a (simplified) single penalty kick. The kicker can kick left or right; the goal keeper can jump left or right. The kicker scores a goal iff they choose a different side than the keeper.

(Kicker, Keeper)	JumpL	JumpR
KickL	(-1,1)	(1,-1)
KickR	(1,-1)	(-1,1)

# Organisational Issue

Thursday, 9th May is a bank holiday (↪ no exercise sessions)

## Substitute:

Virtual session on Wednesday, 8th May 13:00 via Zoom:

<https://tu-dresden.zoom-x.de/j/69888281536?pwd=RmhndmFndGVOUT12Y1h5VUIwNjBsQT09>

# Overview

Motivation

Game Trees

Example: Media Streaming  
Backward Induction

Combinatorial Games

# Motivation

# Motivation

## Strategic Games in Normal Form:

- Only **one decision** is made by the players (choice of strategy)
- **Strategies** are seen as black boxes, no fine-grained analysis possible
- **Simultaneous** decisions: players do not know others' contribution to the final outcome

But many “real” games (Chess, Checkers, Go) are not like that. Instead:

- **Several decisions** are being made throughout the game
- **Strategies** can be seen as advice on which move to pick, **for every possible situation**
- Players move **sequentially** (typically taking turns)

In both models, there is **complete information** about the game:  
All players know possible strategies as well as utilities of all players.

# Sequential Games with Perfect Information

## Definition

A **sequential game with perfect information** consists of the following:

1. A set  $P = \{1, \dots, n\}$  of at least two players.
2. An  $n$ -tuple  $\mathbf{M} = (M_1, \dots, M_n)$  of sets  $M_i$  of **moves** for each player  $i$ .
3. A set  $H$  of **histories**, sequences  $[m_1, \dots, m_k]$  of moves  $m_j \in M_1 \cup \dots \cup M_n$ .
4. A subset  $Z \subseteq H$  of **terminal** histories.
5. A **player function**  $p: H \setminus Z \rightarrow P$  (indicating whose turn it is).
6. An  $n$ -tuple  $\mathbf{u} = (u_1, \dots, u_n)$  of utility functions  $u_i: Z \rightarrow \mathbb{R}$ .

Starting with the **empty history**  $[],$  in each history  $h = [m_1, \dots, m_k] \in H \setminus Z,$  player  $i = p(h)$  chooses a move  $m \in M_i,$  leading to  $[h; m] := [m_1, \dots, m_k, m].$

- **Perfect information:** All players know all previous moves of all players.
- Move  $m$  is **possible** in history  $h \in H$  iff  $[h; m] \in H.$

# Game Trees



# Background: Directed Trees

## Definition

1. A **directed graph** is a pair  $G = (V, E)$  with  $V$  a set of nodes and  $E \subseteq V \times V$  a set of edges (ordered pairs of nodes).
2. For  $u, v \in V$ , a **path** from  $u$  to  $v$  is a sequence  $\langle v_0, v_1, \dots, v_m \rangle$  of nodes that are all distinct and where  $u = v_0$ ,  $v_m = v$ , and  $(v_{i-1}, v_i) \in E$  for  $1 \leq i \leq m$ .
3. A (directed) **tree** is a directed graph  $G = (V, E)$  with one distinguished node  $r \in V$ , the **root**, such that for every node  $v \in V$  there is a unique path from  $r$  to  $v$ .
4. For an edge  $(u, v) \in E$  we say that  $u$  is the **parent** of  $v$ , resp. that  $v$  is the **child** of  $u$ . A node  $v \in V$  without children is a **leaf**.

## Observation

Every node in a tree is either the root or it has a unique parent.

# Representation via Game Trees

## Definition

A sequential game with perfect information is represented as a **game tree**:

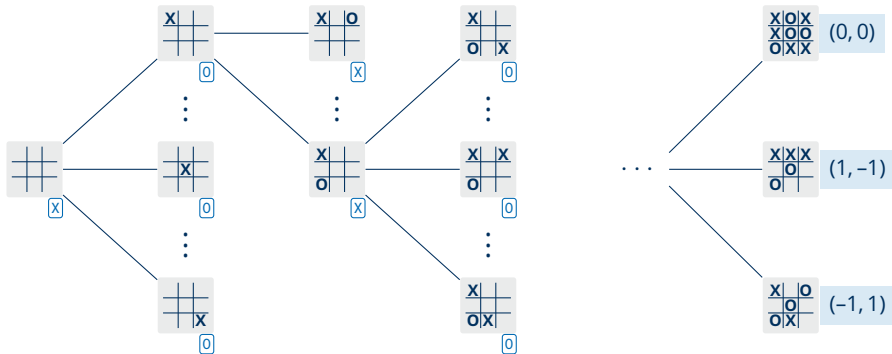
1. Each node of the game tree corresponds to a history  $h \in H$ , where
  - the root corresponds to the empty history  $[],$  and
  - leaves (and only leaves) correspond to terminal histories  $t \in Z.$
2. Each non-leaf node (**decision node**)  $h$  is labelled by  $p(h).$
3. The children of a node  $h$  are given by the possible moves of  $p(h).$
4. Each leaf node  $z$  is labelled by its utility value  $\mathbf{u}(z) = (u_1(z), \dots, u_n(z)).$

- Possible (i.e. legal) moves are encoded implicitly in the tree.
- Other forms of modelling legality are possible (e.g. another function).
- Representation of (sequential) games by trees is sometimes referred to as **extensive form** representation (as opposed to the normal form).

# Example: Tic-Tac-Toe

## Tic-Tac-Toe

Two players,  $X$  and  $O$ , take turns marking initially empty fields of a  $3 \times 3$ -table with their symbols  $X$  and  $O$ , respectively. Whoever manages to mark a full row, column, or diagonal with their symbol wins.  $X$  moves first.



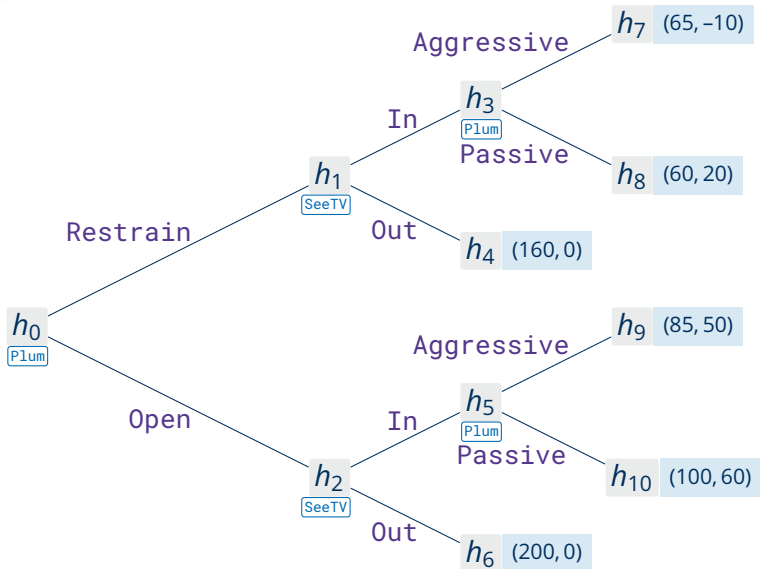
# Example: Media Streaming (Story)

## Media Streaming

(Gillman & Housman, 2019)

Tech company Plum plans the marketing campaign for their new device. Aggressive marketing is more expensive (and successful) than Passive marketing. Competitor SeeTV is working on a clone which will need some more time to market. SeeTV must decide whether to continue this work to be In on the market (or discontinue and be Out). SeeTV can clone the device better and more cheaply if they recruit some of Plum's engineers to work for SeeTV. In anticipation, Plum can put Restraining clauses into their engineers' contracts (barring them from working for competitors if they leave). However, such clauses require Plum to pay higher salaries than for Open contracts. The market is expected to support sales sufficient to generate 200m in net revenue. Restraining clauses cost additional 40m in salaries; aggressive marketing costs additional 25m and yields a 65% market share with restraining clauses and a 55% share without; passive marketing yields a 50% share. It costs SeeTV 40m to develop the clone with Plum engineers, and 80m without.

# Example: Media Streaming (Game Tree)



# Strategies in Game Trees

A **strategy** for a player in a game tree must assign to each decision node of that player exactly one (legal/possible) move.

## Example

- In media streaming, player **Plum** has the decision nodes  $h_0 = []$ ,  $h_3 = [\text{Restrained}, \text{In}]$ , and  $h_5 = [\text{Open}, \text{In}]$  (with two moves each).
- Player **SeeTV** has the decision nodes  $h_1 = [\text{Restrained}]$  and  $h_2 = [\text{Open}]$  (with two possible moves each).
- Thus player **Plum** has eight different strategies, among them  $\{h_0 \mapsto \text{Open}, h_3 \mapsto \text{Passive}, h_5 \mapsto \text{Aggressive}\}$  (“use open contracts; do passive [aggressive] marketing for restrained [open] contracts”).
- Player **SeeTV** has four different strategies, among them  $\{h_1 \mapsto \text{In}, h_2 \mapsto \text{In}\}$  written as  $(\text{In}, \text{In})$  (“enter the market in any case”) and  $(\text{Out}, \text{In})$  (“enter the market if (and only if) **Plum** uses open contracts”).

# From Game Trees to Strategic Games

## Definition

Let  $G$  be a finite sequential game with perfect information and  $P$  its players. The **associated normal-form game**  $G' = (P, \mathbf{S}, \mathbf{u}')$  is as follows:

- $\mathbf{S} = (S_1, \dots, S_n)$  with each  $S_i$  the set of all functions

$$s_i: \{h \in H \setminus Z \mid p(h) = i\} \rightarrow M_i \quad \text{with} \quad [h; s_i(h)] \in H \text{ for all } h \in p^{-1}(i)$$

- $\mathbf{u}' = (u'_1, \dots, u'_n)$  with each  $u'_i$  mapping the profile  $\mathbf{s} = (s_1, \dots, s_n)$  to

$$u_i(T(\mathbf{s}))$$

where  $T(\mathbf{s}) := T(\mathbf{s})([])$  and inductively

$$T(\mathbf{s})(h) := \begin{cases} h & \text{if } h \in Z, \\ T(\mathbf{s})([h; s_i(h)]) \text{ for } i = p(h) & \text{otherwise.} \end{cases}$$

Strategies  $s_i$  can be **reduced** by removing all move assignments to histories the strategy cannot reach.

# Solving Media Streaming (1)

Using the possible strategies of Plum and SeeTV, we can formulate media streaming as strategic game in normal form:

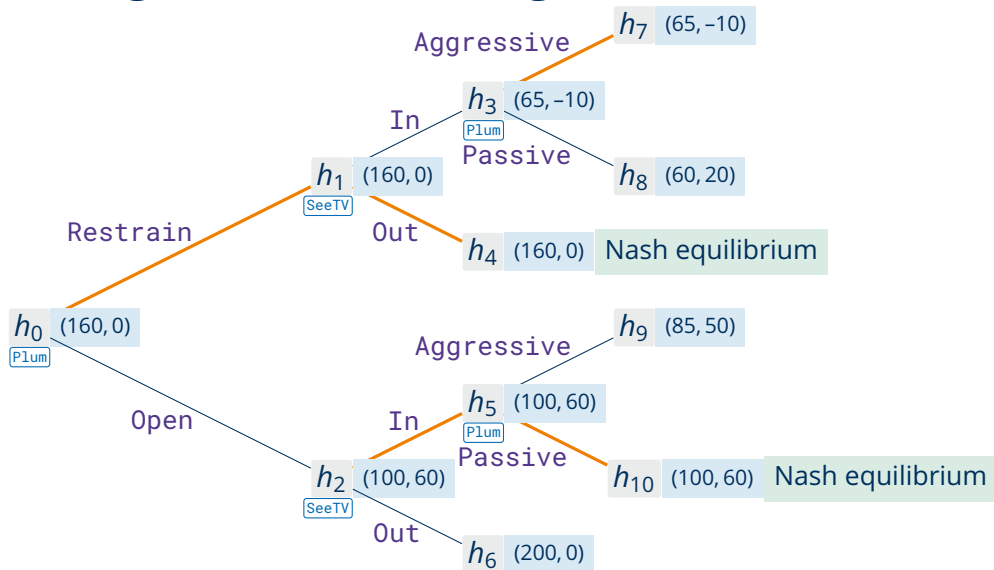
(Plum, SeeTV)	(In, In)	(In, Out)	(Out, In)	(Out, Out)
(Restrain, Aggressive, Aggressive)	(65, -10)	(65, -10)	(160, 0)	(160, 0)
(Restrain, Aggressive, Passive)	(65, -10)	(65, -10)	(160, 0)	(160, 0)
(Restrain, Passive, Aggressive)	(60, 20)	(60, 20)	(160, 0)	(160, 0)
(Restrain, Passive, Passive)	(60, 20)	(60, 20)	(160, 0)	(160, 0)
(Open, Aggressive, Aggressive)	(85, 50)	(200, 0)	(85, 50)	(200, 0)
(Open, Aggressive, Passive)	(100, 60)	(200, 0)	(100, 60)	(200, 0)
(Open, Passive, Aggressive)	(85, 50)	(200, 0)	(85, 50)	(200, 0)
(Open, Passive, Passive)	(100, 60)	(200, 0)	(100, 60)	(200, 0)

There are **four Nash equilibria in pure strategies** (and inessential further equilibria in mixed strategies).

Is this the “right” way to model and solve this game?



# Solving Media Streaming (2)



# Backward Induction

The approach used for solving the example can be generalised:

```
function backward-induction() { label(u, {}, []) } // start recursion at root
```

```
function label(u, s, h) {  
    if  $h \in Z$  then return // terminal nodes are labelled by u  
     $i := p(h)$  // obtain player to move  
     $v^* = (v_1, \dots, v_n) := (-\infty, \dots, -\infty)$  // initialise current best (for  $i$ ) payoff  
    for each  $m \in M_i$  with  $[h; m] \in H$  { // for each possible move  
        label(u, s,  $[h; m]$ ) // recursively label child  
        if  $u_i([h; m]) > v_i^*$  then { // better move found  
             $m^* := m$  // update current best move  
             $v^* := \mathbf{u}([h; m])$  // update current best value  
        }  
    }  
     $\mathbf{u}(h) := v^*$  // set utility value of this history  
     $s_i(h) := m^*$  // set best move for this history  
}
```

# Zermelo's Theorem

Theorem (Zermelo, 1913)

Every sequential game tree with a finite number of nodes has a backward induction solution, that is, a strategy profile and (utility) outcome.

Proof.

By induction on the length  $n$  of the game's longest terminal history:

- $n = 1$ : Pick the action with the highest payoff.
- $n \rightsquigarrow n + 1$ :
  - Consider a game  $G$  with longest terminal history of length  $n + 1$ .
  - Label all parents  $h$  of terminal nodes with the payoff vector of the terminal node of their best move  $m_h^*$ .
  - Define game  $G'$  by taking these  $h$  to be terminal (removing their children).
  - Game  $G'$  has a longest terminal history of length  $n$ .
  - Thus by induction hypothesis,  $G'$  can be solved by backward induction.
  - To obtain the strategy profile for  $G$ , append the moves  $m_h^*$  to that of  $G'$ . □

# Subgames

## Definition

Let  $G$  be a sequential perfect-information game and  $h$  be a history therein. The **subgame**  $G(h)$  of  $G$  beginning at  $h$  is as follows:

1. Players  $P$  and moves  $\mathbf{M}$  of  $G(h)$  are as in  $G$ .
2. Histories of  $G(h)$ :  $H_h := \{h' \mid [h; h'] \in H\}$
3. Terminal histories of  $G(h)$ :  $Z_h := H_h \cap Z$
4. Player function and utilities of  $G(h)$  are those of  $G$ .

## Example

- Tic-Tac-Toe has a subgame starting at  $\begin{array}{|c|c|c|} \hline X & & X \\ \hline O & & \\ \hline X & O & \\ \hline \end{array}$  where  $X$  can force a win.
- In media streaming's subgame at  $h_2$ , [SeeTV](#) can guarantee payoff 60.
- Every sequential perfect-information game  $G$  is a subgame of itself.

# Subgame Perfect Equilibria

## Definition

Let  $G$  be a sequential perfect-information game with players  $P = \{1, \dots, n\}$ . A strategy profile  $\mathbf{s} = (s_1, \dots, s_n)$  is a **subgame perfect equilibrium** iff for every non-terminal history  $h \in H$ , for every player  $i \in P$  and for every  $s'_i \in S_i$ :

$$u_i(Z^h(\mathbf{s})) \geq u_i(Z^h(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n))$$

where  $Z^h(\mathbf{t})$  denotes the terminal history of playing according to  $\mathbf{t}$  in  $G(h)$ .

## Theorem

In a sequential perfect-information game, subgame perfect equilibria coincide with the strategy profiles obtained by backward induction.

## Proof Idea.

In each step of backward induction, we choose a best response. □

# Combinatorial Games

# Combinatorial Games

## Definition

A **combinatorial game** is a sequential perfect-information game with two players  $P = \{1, 2\}$  where in each terminal history  $z \in Z$ , either:

- player  $i$  gets  $u_i(z) = 1$  and the other player  $j \neq i$  gets  $u_j(z) = -1$ ; or
- both players get  $u_1(z) = u_2(z) = 0$ .

Since for all  $z \in Z$  we have  $u_1(z) + u_2(z) = 0$ , those games are **zero-sum**.

**Examples:** Tic-Tac-Toe, Checkers, Chess, Go

## Observation

For each combinatorial game, either:

- Exactly one player has a winning strategy, or
- both players can guarantee a draw.

# Solving Games

## Note

Backward induction can **in principle** be used to solve finite games.

In practice, the **size** of the game (number of positions/histories) is key:

- Tic-Tac-Toe is (trivially) **strongly solved**, i.e. the optimal moves for every history of the game are known.
- Connect-Four is **strongly solved** (the first player can force a win).
- Checkers has been **weakly solved** in 2007, i.e. it is known that from the initial position, perfect play by both sides leads to a draw ( $\approx 5 \cdot 10^{20}$  pos.).
- Othello has recently been claimed to be solved (draw,  $\approx 10^{28}$  positions).
- Chess remains unsolved (6-piece endgame is strongly solved).
- Go remains unsolved (it is weakly solved on a  $7 \times 7$ -board).

From the perspective of computational complexity, however, all of these games can be solved in **constant time** (due to their finite size).

$\rightsquigarrow$  Analyse arbitrary-size **generalisations** of games.



# A Game on Maps

## Geography

Two players take turns naming countries (from an English dictionary). The first country can start with an arbitrary letter, but each next country (where repetition is disallowed) must start with the last letter of the previous country.

### Example

Ann and Bob play geography. Ann starts with **germany**, to which Bob responds with **yemen**. Then Ann says **norway** and the game is over with Ann winning.

## Geography

For a fixed directed graph  $G = (V, E)$  and vertex  $s \in V$ , two players, **exists** and **forall**, take turns in marking vertices in  $G$ . Player **exists** starts and has to mark a successor  $v_1$  of  $s$ , and from then on each player has to mark a vertex  $v_{i+1}$  that is a successor of the node  $v_i$  marked by the opponent in the previous round. No vertex may be marked twice, and a player unable to move loses.

# Winning Strategies in Geography

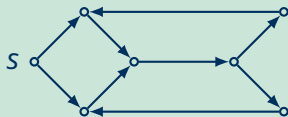
## Geography

**Given:** A directed graph  $G = (V, E)$  and a designated vertex  $s \in V$ .

**Question:** Is there a winning strategy for player `exists` in geography based on  $G$  with starting vertex  $s$ ?

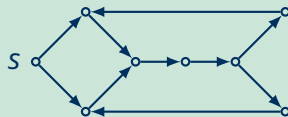
### Examples

Consider  $G_1 = (V_1, E_1)$ :



`exists` has **no** winning strategy,  
thus  $G_1 \notin$  **Geography**.

Consider  $G_2 = (V_2, E_2)$ :



`exists` **has** a winning strategy,  
thus  $G_2 \in$  **Geography**.

# Geography and Computational Complexity

## Recall: Computational Complexity

- Complexity class PSpace comprises all decision problems (languages) that can be decided by polynomially space-bounded Turing machines.
- A decision problem is PSpace-hard iff every problem in PSpace can be reduced to it in polynomial time.
- A problem is PSpace-complete iff it is in PSpace and PSpace-hard.

## Theorem

**Geography** is PSpace-complete.

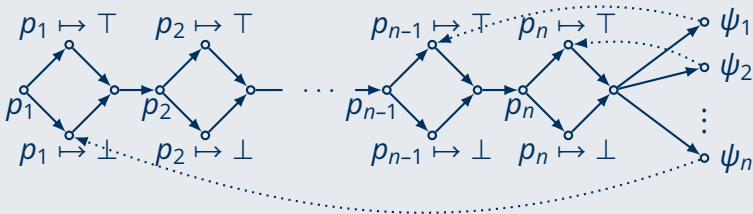
Proof (containment).

We use depth-first-search to traverse the game tree, where we only ever keep one path from the root to some decision node in memory, along with a list of children already considered for each node along that path.

# Geography is PSpace-complete

Proof (hardness).

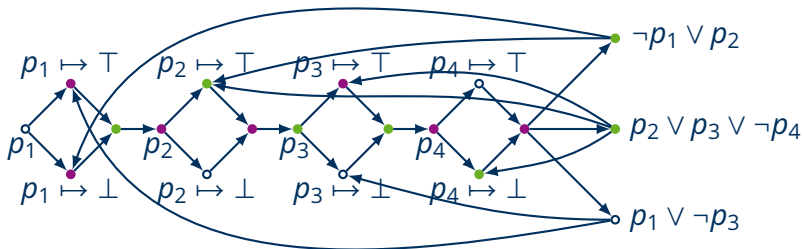
We reduce from the PSpace-hard problem **QBF-Truth**. Consider a quantified Boolean formula  $\Phi = \exists p_1. \forall p_2. \dots \exists p_{n-1}. \forall p_n. \Psi$  where  $\Psi = \psi_1 \wedge \dots \wedge \psi_m$  is a quantifier-free conjunction of clauses (disjunctions of literals). We create the following instance of **Geography**:



where for all  $1 \leq j \leq m$  we add an edge from  $\varphi_j$  to  $p_i = \top$  (resp.  $p_i = \perp$ ) iff  $p_i$  occurs in  $\psi_j$  (resp.  $\neg p_i$  occurs in  $\psi_j$ ). Then **exists** has a winning strategy iff  $\Phi$  is true: **exists** chooses  $p_i \mapsto \top$  iff  $p_i \mapsto \perp$  witnesses  $\Phi$  being true.  $\square$

# Hardness of Geography: Example

Consider QBF  $\Psi = \exists p_1. \forall p_2. \exists p_3. \forall p_4. ((\neg p_1 \vee p_2) \wedge (p_2 \vee p_3 \vee \neg p_4) \wedge (p_1 \vee \neg p_3))$ .  
The construction yields the following **Geography** instance (start node  $p_1$ ):



In this game of geography, **exists** has no winning strategy (**forall** has one), which is expected as  $\Psi$  is not true.

# Conclusion

## Summary

- **Game trees** are used to represent sequential (**extensive form**) games.
- Sequential games give rise to (different) strategic (normal form) games.
- In a game tree, a **strategy** assigns a move to each decision node.
- **Backward induction** can be used to solve sequential games.
- The **subgame perfect equilibrium** of a sequential game coincides with its backward induction solution.
- Geography is a game on graphs for which deciding existence of winning strategies is PSpace-complete.

## Action Points

- Implement backward induction and solve Tic-Tac-Toe.

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