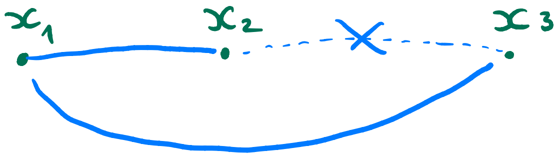
FMT Course : Lecture 3

The missing proofs on zero-one law for FO

E-F Games (but not too much)

Def. k-atomic type

= a formula with variables $x_1, \dots, x_k$ s.t.
for all $i \neq j$ we have $x_i \neq x_j$
and either $E(x_i, x_j)$ or $\neg E(x_i, x_j)$



$$= x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_1 \wedge E(x_1, x_2) \wedge \neg E(x_2, x_3) \wedge E(x_1, x_3)$$

! We often think about a k-type as the set of its conjuncts.

Def. A $(k+1)$-type t extends a k-type s iff $s \subseteq t$.

Def. An (s,t)-extension axiom is the formula

$$\forall x_1 \forall x_2 \dots \forall x_k \; s(x_1, \dots, x_k) \rightarrow \exists x_{k+1} \; t(x_1, \dots, x_k, x_{k+1})$$

$$\mathbb{EA} := \left\{ \begin{array}{l} \forall x \; \neg E(x,x), \\ \forall x \forall y \; E(x,y) \rightarrow E(y,x) \end{array}, \quad \sigma_{s,t} \;\middle|\; \begin{array}{c} k\text{-type } s, \\ k+1\text{-type } t, \\ s \subseteq t \end{array} \right\}$$

EA

Every extension axiom is almost surely true , i.e.

$$\mu_\infty(\sigma_{s,t}) = 1$$

for all $\sigma_{s,t} \in$ EA .

By compactness it implies that if

$EA \models \varphi$

then $\mu_\infty(\varphi) = 1$. !

EA is ω-categorical

(it has exactly one countable model up to isomorphism)

EA is complete

(for every φ either $EA \models \varphi$ or $EA \models \neg\varphi$)

$\mathbb{E}\mathbb{A}$

Every extension axiom is **almost surely true**, i.e.

$$\mu_\infty(\sigma_{s,t}) = 1$$

for all $\sigma_{s,t} \in \mathbb{E}\mathbb{A}$.

$\mathbb{E}\mathbb{A}$ is **ω-categorical** (it has exactly one countable model up to isomorphism)

$\downarrow$

$\mathbb{E}\mathbb{A}$ is **complete** (for every φ either $\mathbb{E}\mathbb{A} \vDash \varphi$ or $\mathbb{E}\mathbb{A} \vDash \neg\varphi$)

Theorem (Glebskii et al, Fagin)

FO has zero-one law, i.e. for all $\varphi \in FO$ we have $\mu_\infty(\varphi) = 0$ or $\mu_\infty(\varphi) = 1$.

Proof: By completness of $\mathbb{A}\mathbb{E}$ we know that $\mathbb{E}\mathbb{A} \vDash \varphi$ or $\mathbb{E}\mathbb{A} \vDash \neg\varphi$.

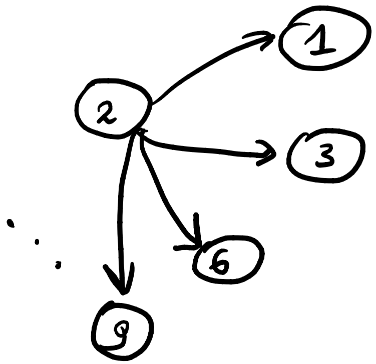
1° If $\mathbb{E}\mathbb{A} \vDash \varphi$ then $\mu_\infty(\varphi) = 1$.

2° Otherwise $\mathbb{E}\mathbb{A} \vDash \neg\varphi$, which implies $\mu_\infty(\neg\varphi) = 1$. So $\mu_\infty(\varphi) = 1 - \mu_\infty(\neg\varphi) = 0$.

$\square$

A **Rado graph** is a graph $G = (V, E)$ with $V := \mathbb{N}_+$ and $E(i, j)$ holds iff $p_i \mid j$ or $p_j \mid i$.

i-th prime number divides



Lemma: $G \models \sigma_{s,t}$ for all $\sigma_{s,t} \in \mathbb{AE}$.

Proof:

Let $\sigma_{s,t} := \forall x_1 \forall x_2 \dots \forall x_k \; s(x_1, \dots, x_k) \longrightarrow \exists x_{k+1} \; t(x_1, x_2, \dots, x_k, x_{k+1})$.

We divide $\{1, 2, 3, \dots, m\}$ into

$$\mathrm{Con} = \{ i \mid E(x_i, x_{k+1}) \in t \} \qquad \mathrm{Not\,Con} := \{ i \mid \neg E(x_i, x_{k+1}) \in t \}$$

Now take any $a_1, \dots, a_k \in V$ s.t. $G \models s(a_1, \dots, a_k)$.

We need to find $a_{k+1} \in V$ s.t. $G \models t(a_1, \dots, a_k, a_{k+1})$.

Take $a_{k+1} = \left(\prod_{i \in \mathrm{Con}} p_{a_i} \right) \cdot q$, where q is the smallest prime number greater than $p_{a_1} \cdot p_{a_2} \cdot \dots \cdot p_{a_k}$.

So for $i \in \mathrm{Con}$ (resp. $\mathrm{Not\,Con}$) we have $E(a_i, a_{k+1})$ (resp. $\neg E(a_i, a_{k+1})$).

Lemma: For any $\mathfrak{A}, \mathfrak{B}$ s.t. $\mathfrak{A} \models EA$ and $\mathfrak{B} \models EA$ we have $\mathfrak{A} \cong \mathfrak{B}$

countable

$A = \{ a_1, a_2, a_3, \dots \}$ $\qquad$ $B = \{ b_1, b_2, b_3, \dots \}$

By induction we will create partial isomorphism $p_0, p_1, p_2, \dots$

such that $p_0 \subseteq p_1, \; p_1 \subseteq p_2, \; p_2 \subseteq p_3, \dots$

$p = \bigcup_{n=0}^{\infty} p_n$ will turn out to be an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$

$p_0 = \emptyset$ $\qquad$ Assume that we already have p_k.

k is even $\qquad$ k is odd (similar)

$p_k = \{ (a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}), \dots, (a_{i_k}, b_{i_k}) \}$

Take $a_{i_{k+1}}$, s.t. i_{k+1} is the smallest index s.t. $a_{i_{k+1}}$ doesn't appear in p_k.

$p_k = \{(a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}), \ldots, (a_{i_k}, b_{i_k})\}$

Take $a_{i_{k+1}}$ s.t. i_{k+1} is the smallest index s.t. $a_{i_{k+1}}$ doesn't appear in p_k.

in $\mathfrak{A}$:

It means that there is a k-type s s.t $\mathfrak{A} \models s(a_{i_1}, \ldots, a_{i_k})$

Similarly we can find a unique $(k+1)$-type t. st. $\mathfrak{A} \models t(a_{i_1}, \ldots, a_{i_{k+1}})$

(since p_k is a partial isomorphism) we know that

$\mathfrak{B} \models s(b_{i_1}, \ldots, b_{i_k})$, Since $\mathfrak{B} \models EA$, we know $\mathfrak{B} \models \forall x_1 \ldots \forall x_k \; s(x_1, \ldots, x_k)$

$\rightarrow \exists x_{k+1} \; t(x_1, \ldots, x_{k+1})$

Hence there is $b_{i_{k+1}} \in B$ s.t $\mathfrak{B} \models t(b_{i_1}, \ldots, b_{i_{k+1}})$. So let $p_{k+1} = p_k \cup \{(a_{i_{k+1}}, b_{i_{k+1}}$

Lemma : EA is complete.

for all $\varphi \in FO$ we have $EA \models \varphi$ or $EA \models \neg\varphi$.

Proof:

Assume that it is not the case, which means that

$\mathfrak{A} \models EA$ and $\underline{\mathfrak{A} \models \varphi}$ and

$\mathfrak{B} \models EA$ and $\underline{\mathfrak{B} \models \neg\varphi}$.

Easy observation $|EA| = \aleph_0$, so also $|EA \cup \{\varphi\}| = |EA \cup \{\neg\varphi\}| = \aleph_0$.

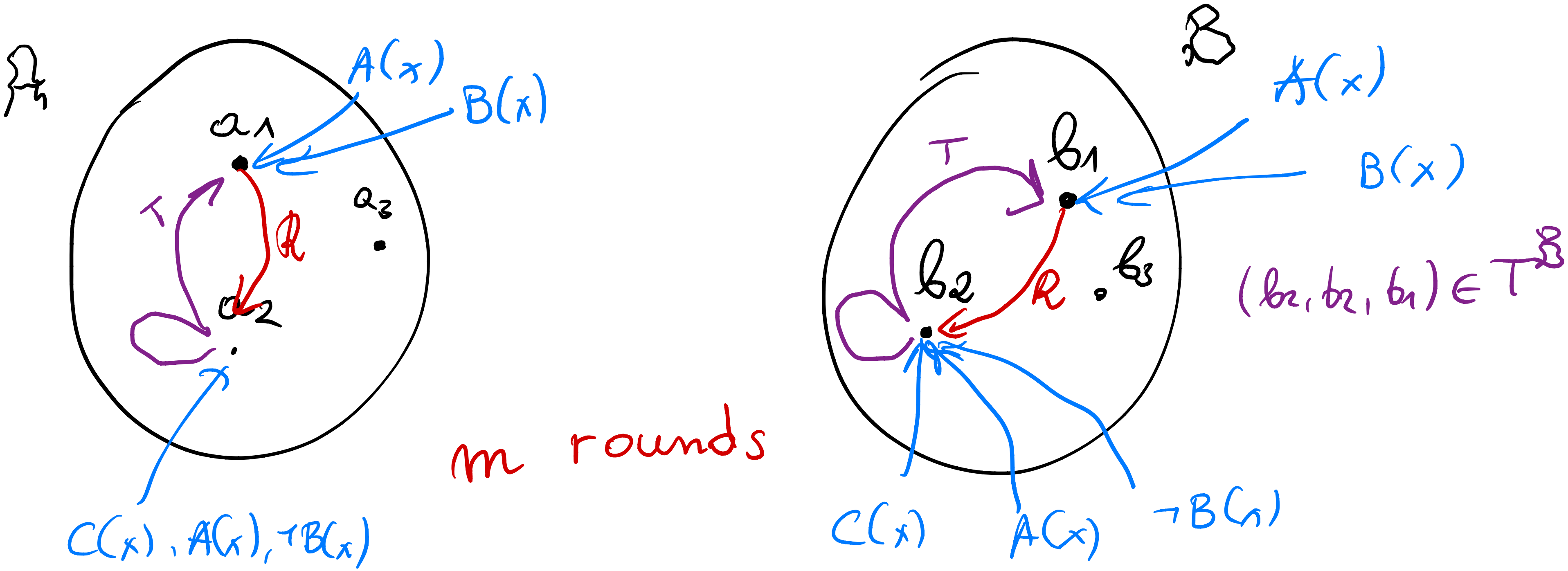
Hence, by Skolem Theorem we can assume that $\mathfrak{A}$ and $\mathfrak{B}$ are countable.

Because EA is ω-categorical, it has exactly one countable model.

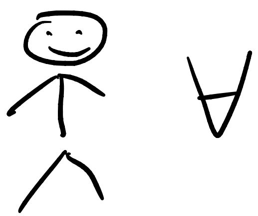
So $\mathfrak{A} \cong \mathfrak{B}$. Hence, they satisfy the same FO-formulae. It implies that $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \neg\varphi$. A contradiction. $\square$

Ehrenfeucht – Fraïssé Games

(E-F Games)

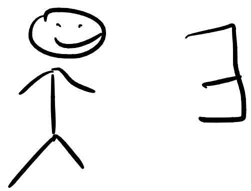


B∀art



Duplicator

S∃bastian



Spoiler