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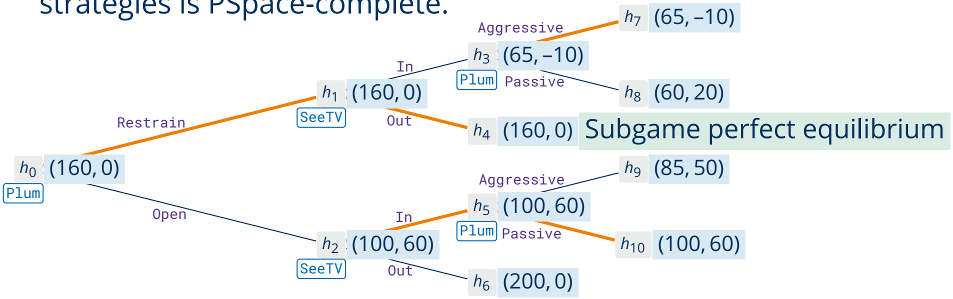
Playing Games: Alpha-Beta Tree Search

Lecture 4, 6th May 2024 // Algorithmic Game Theory, SS 2024

Previously . . .

- **Game trees** are used to represent sequential (**extensive form**) games.
- Sequential games give rise to (different) strategic (normal form) games.
- In a game tree, a **strategy** assigns a move to each decision node.
- **Backward induction** can be used to solve sequential games.
- The **subgame perfect equilibrium** of a sequential game coincides with its backward induction solution.
- Geography is a game on graphs for which deciding existence of winning strategies is PSpace-complete.



Overview

Two-Player Zero-Sum Games

Alpha-Beta Pruning

Heuristics

Two-Player Zero-Sum Games

Zero-Sum Games

Definition

A game with players P is **zero-sum** iff for all outcomes $z \in Z$, $\sum_{i \in P} u_i(z) = 0$.

Note: Every combinatorial game is zero-sum, but not vice versa.

Examples: Penalties, Rock-Paper-Scissors, Chess, Go

In what follows, we will focus on two-player zero-sum games.

Observation

For a two-player zero-sum game (with $P = \{1, 2\}$), the payoffs $\mathbf{u} = (u_1, u_2)$ are fully specified by giving u_1, as for every $z \in Z$ we have $u_2(z) = -u_1(z)$.

Two-Player Zero-Sum Sequential Games

We thus adapt our definition of sequential games with perfect information:

Definition

A **two-player zero-sum sequential game with perfect information** has:

1. The set $P = \{\texttt{max}, \texttt{min}\}$ of two (named) players.
2. A tuple $(M_{\texttt{max}}, M_{\texttt{min}})$ of sets of moves for each player; $M := M_{\texttt{max}} \cup M_{\texttt{min}}$.
3. A set H of histories, sequences $[m_1, \dots, m_k]$ of moves $m_j \in M$.
4. A subset $Z \subseteq H$ of terminal histories.
5. A player function $p \colon H \setminus Z \to P$ (indicating whose turn it is).
6. A utility function $u_{\texttt{max}} \colon Z \to \mathbb{R}$ for player $\texttt{max}$.

Starting with the empty history $[]$, in each history $h = [m_1, \dots, m_k] \in H \setminus Z$, player $i = p(h)$ chooses a move $m \in M_i$, leading to the history $[m_1, \dots, m_k, m]$.

Histories and States

Typically, it is more useful to describe a game other than through histories:

Definition

A **state-based game model** consists of the following:

- A set S of **states** of the game, with **initial state** $S_0 \in S$, and functions:
- **TURN** : $S \to P$ saying whose turn it is in a state.
- **MOVES** : $S \to 2^M$ yielding the legal moves in a state.
- **RESULT** : $S \times M \to S$ yielding the result of a move in a state (the next state).
- **IS-TERMINAL** : $S \to \{\top, \bot\}$ indicating whether a state is terminal.
- **UTILITY** : $S \to \mathbb{R}$ giving a terminal state's payoff for `max` (else undefined).

- Each history leads to exactly one state. ($[]$ leads to S_0.)
- One state may be reached through different histories.

Example: A state in Chess is given by the locations of the pieces on the board.

State Spaces and Their Representation

Definition

The **state space graph** associated with a state-based game model is the edge-labelled directed graph (V, E) with $E \subseteq V \times M \times V$, where

- $V \subseteq S$ is the $\subseteq$-least set such that $S_0 \in V$, and:

$$\text{if } s \in V \text{ and } m \in \textbf{MOVES}(s), \text{ then } \textbf{RESULT}(s, m) \in V;$$

- $(s_1, m, s_2) \in E$ iff $\textbf{RESULT}(s_1, m) = s_2$.

- The state space contains all states that are reachable from the initial state by sequences of legal moves.
- The state space can be huge: for Chess, there are at least 10^{40} positions (states).
- We thus typically only search parts of the state space (game tree).

Representing Games for Search

We will assume that the game tree is not explicitly given, but implicitly specified by a state-based game model that is parsimoniously represented (e.g. using a game description language like Stanford University's GDL).

Assumption: Game Representation

A state-based game model can be represented such that:

- The set S of states is described as an efficiently decidable formal language.
- The functions **TURN**, **MOVES**, **RESULT**, **IS-TERMINAL**, and **UTILITY** can all be computed efficiently.
- The full description of the game model has a practical size.

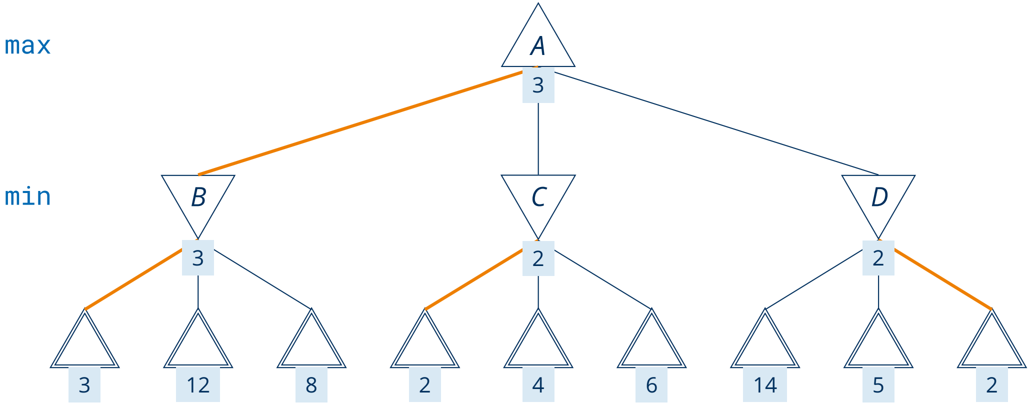
This assumption is especially relevant for games like Chess and Go, whose state-based models can be formalised (logically or through executable code), but whose game trees are too large to be explicitly represented.

Search in Game Trees

Recall: For combinatorial games, we used backward induction to solve them.

- For (general) zero-sum games, we also have to distinguish different utilities for the same player: Winning with 9 is better than winning with 1.
- This leads to a slightly more general algorithm: minimax search.
- Player `max` maximises their payoff $u_{\mathtt{max}}$ (also called the **value** of the game).
- Player `min` maximises their payoff $u_{\mathtt{min}} = -u_{\mathtt{max}}$, thus minimises $u_{\mathtt{max}}$.
- Each player knows that the other player maximises/minimises and takes this into account accordingly.

Minimax Tree Search: Example



Minimax Value of a Game

Definition

For a (state-based model of a) game, the **minimax value** of a state $s \in S$ is

$$\textbf{minimax}(s) := \begin{cases} \textbf{UTILITY}(s) & \text{if } \textbf{IS-TERMINAL}(s), \\ \max_{m \in \textbf{MOVES}(s)} \textbf{minimax}(\textbf{RESULT}(s, m)) & \text{if } \textbf{TURN}(s) = \texttt{max}, \\ \min_{m \in \textbf{MOVES}(s)} \textbf{minimax}(\textbf{RESULT}(s, m)) & \text{if } \textbf{TURN}(s) = \texttt{min}. \end{cases}$$

The **minimax value of the game** is **minimax**(S_0) for S_0 the initial state.

- The **minimax decision** at each node is the move leading to the maximal (resp. minimal) payoff in the next node.
- This definition of the optimal game value yields optimal responses of each player given that the respective other player also plays optimally.

Minimax Tree Search: Algorithm

```
function minimax-search(s: state) {                // allows to start search in an arbitrary state s
    if TURN(s) = max then { (v, m) := max-value(s) } else { (v, m) := min-value(s) }
    return m }                                      // return best move in s
```

```
function max-value(s: state) {
    if IS-TERMINAL(s) then return (UTILITY(s), null)     // base case: terminal state
    (v*, m*) := (−∞, null)                               // initialise current maximum
    foreach m ∈ MOVES(s) do {                            // try all moves
        (v′, m′) := min-value(RESULT(s, m))              // simulate move
        if v′ > v* then (v*, m*) := (v′, m) }            // update current maximum
    return (v*, m*) }                                    // return maximum
```

```
function min-value(s: state) {
    if IS-TERMINAL(s) then return (UTILITY(s), null)
    (v*, m*) := (+∞, null)
    foreach m ∈ MOVES(s) do {
        (v′, m′) := max-value(RESULT(s, m))
        if v′ < v* then (v*, m*) := (v′, m) }
    return (v*, m*) }
```

Minimax Tree Search: Complexity

Proposition

For a branching factor of b (maximal number of moves) and a depth of d (maximal length of histories), minimax search visits $O(b^d)$ terminal nodes.

$\leadsto$ Minimax tree search is impractical for complex games.

Example

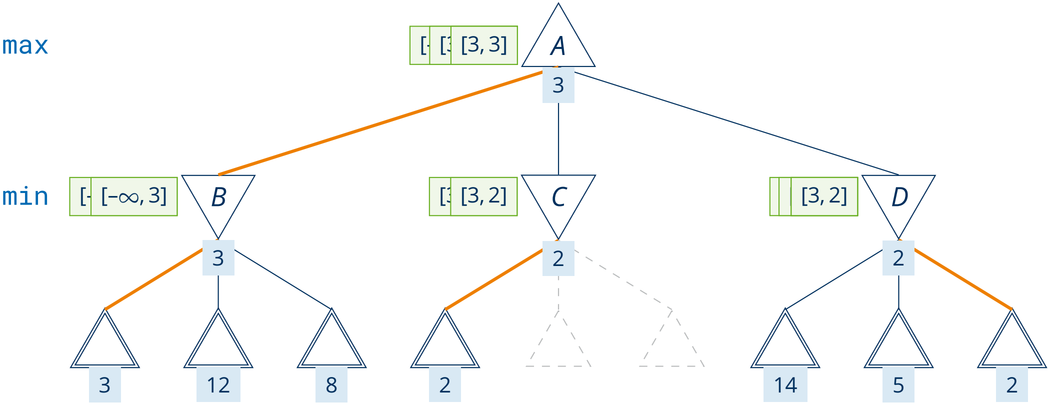
Chess has a branching factor of about 35 and average game length of about 80 ply (moves of a single player), so running minimax search to the leaves would need to expand $35^{80} \approx 10^{123}$ nodes.

There are at least two possible ways of reducing b^d:

- Reducing b: Do we really have to try out all possible moves?
  $\leadsto$ alpha-beta pruning
- Reducing d: Do we really have to play the game until the end?
  $\leadsto$ heuristic evaluation of states

Alpha-Beta Pruning

Alpha-Beta Pruning: Example



Alpha-Beta Tree Search: Algorithm

```
function alpha-beta-search(s: state) { if TURN(s) = max then
    (v, m) := max-value(s, −∞, +∞) else (v, m) := min-value(s, −∞, +∞) ;  return m }
```

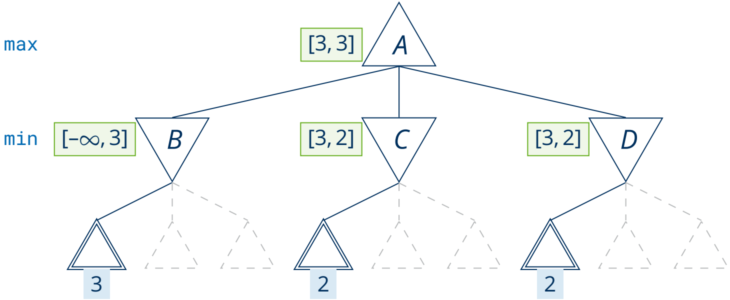
```
function max-value(s: state, α: ℝ±∞, β: ℝ±∞) {
    if IS-TERMINAL(s) then return (UTILITY(s), null)        // base case: terminal state
    (v*, m*) := (−∞, null)                                   // initialise current maximum
    foreach m ∈ MOVES(s) do {                                // try all moves
        (v′, m′) := min-value(RESULT(s, m), α, β)            // simulate move
        if v′ > v* then { (v*, m*) := (v′, m) ;  α := max(α, v*) }   // update maximum and α
        if v* ≥ β then return (v*, m*) }                     // prune irrelevant subtree
    return (v*, m*) }                                        // return maximum
```

```
function min-value(s: state, α: ℝ±∞, β: ℝ±∞) {
    if IS-TERMINAL(s) then return (UTILITY(s), null)
    (v*, m*) := (+∞, null)
    foreach m ∈ MOVES(s) do {
        (v′, m′) := max-value(RESULT(s, m), α, β)
        if v′ < v* then { (v*, m*) := (v′, m) ;  β := min(β, v*) }
        if v* ≤ α then return (v*, m*) }
    return (v*, m*) }
```

Alpha-Beta Tree Search: Complexity

The order in which nodes are expanded matters!

- In the worst case, $O(b^d)$ terminal nodes will be visited, even with pruning.
- In the best case, only $O(b^{\frac{d}{2}}) = O\left(\sqrt{b}^d\right)$ terminal nodes will be visited:



(Witnessing a winning strategy requires at least $b \cdot 1 \cdot \ldots \cdot b \cdot 1 = b^{\frac{d}{2}}$ leaves.)

- However, finding a perfect move ordering amounts to solving the game.
- In practice, earlier evaluations (history) or expert knowledge can be used.

Heuristics

Heuristic Evaluation

Recall: There are at least two possible ways of reducing b^d:

- Reducing b: Do we really have to try out all possible moves?
  $\rightsquigarrow$ alpha-beta pruning
- Reducing d: Do we really have to play the game until the end?
  $\rightsquigarrow$ heuristic evaluation of states

Terminology

A **heuristic** aims at reducing the search space of a given problem, typically trading this off for at least one of optimality, completeness, or computation.

Main Idea: Treat non-terminal states as if they were terminal, estimate value.

- Replace function $\textbf{IS-TERMINAL} : S \to \{\top, \bot\}$ by $\textbf{IS-CUTOFF} : S \times \mathbb{N} \to \{\top, \bot\}$,

  $\textbf{IS-CUTOFF}(s, d)$ ... "cut off search below state s in search depth d,"
- and function $\textbf{UTILITY} : S \to \mathbb{R}$ by $\textbf{EVAL} : S \to \mathbb{R}$,

  $\textbf{EVAL}(s)$ ... "estimate the prospective utility of state s (for player `max`)."

Restricting Depth: Heuristic Minimax Value

Heuristic Function **EVAL**: Technical Requirements

For all $s \in S$:

1. If **IS-TERMINAL**(s), then **EVAL**$(s) = $ **UTILITY**(s), otherwise
2. $\min_{s \in S_T}$ **UTILITY**$(s) \leq$ **EVAL**$(s) \leq \max_{s \in S_T}$ **UTILITY**(s)
   for $S_T := \{s \in S \mid$ **IS-TERMINAL**$(s)\}$.

- In practice, the heuristic function **EVAL** should be computable efficiently.
- **EVAL**(s) should strongly correlate with `max`'s "chances of winning" in s.

Definition

The **heuristic minimax value** of a state $s \in S$ (w.r.t. d, **IS-CUTOFF**, and **EVAL**) is

$$\mathbf{hmm}(s, d) := \begin{cases} \textbf{EVAL}(s) & \text{if } \textbf{IS-CUTOFF}(s, d), \\ \max_{m \in \textbf{MOVES}(s)} \mathbf{hmm}(\textbf{RESULT}(s, m), d + 1) & \text{if } \textbf{TURN}(s) = \texttt{max}, \\ \min_{m \in \textbf{MOVES}(s)} \mathbf{hmm}(\textbf{RESULT}(s, m), d + 1) & \text{if } \textbf{TURN}(s) = \texttt{min}. \end{cases}$$

Heuristic Evaluation Functions

- Typically require experience with or expert knowledge about the game.
- Often combine various features f_i of the state into one numerical value:

$$\textbf{EVAL}(s) = w_1 \cdot f_1(s) + \ldots + w_m \cdot f_m(s)$$

- Possible features can be:
  - Mobility: Measure the number of things a player can do (e.g. number of moves, number of reachable states within the next n moves, ...).
  - Goal proximity: How “close” (similar) is the current state to a final state?
  - Material: Count number (or “strength”) of pieces (if applicable and variable).
- Further features may exploit game-specific properties, e.g. persistence of markings in Tic-Tac-Toe or Connect-Four.

Heuristic Evaluation Functions: Examples

Example: Chess

- Add up “material values” of the player’s remaining pieces:
  pawn $\hat{=}$ 1, knight/bishop $\hat{=}$ 3, rook $\hat{=}$ 5, queen $\hat{=}$ 9.
- Assess board control (centre is better than edges or corners).

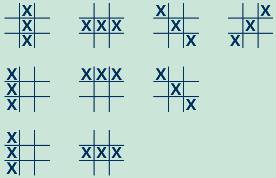
Example: Tic-Tac-Toe, Goal proximity

- There are 9 possible first moves for x: 1 centre, 4 sides, 4 corners.
- We can e.g. estimate in how many winning final positions they occur:

centre:

corner:

side:



Heuristic Alpha-Beta Tree Search: Algorithm

Algorithm:

In the pseudocode on Slide 17, replace the lines mentioning **IS-TERMINAL** by:

if **IS-CUTOFF**(s, d) **then return** (**EVAL**(*s*), **null**)

and keep track of the search depth d as for the heuristic minimax value.

When to cut off search?

- At a fixed depth d_{max}.
- After a fixed time, using iterative deepening and keeping track of best moves (to also improve move ordering in subsequent iterations).

When not to cut off search?

- Quiescence: Apply heuristic evaluation only to quiescent positions, those not facing pending moves that would significantly affect the evaluation.
- Horizon effect: An ultimately unavoidable opponent move is pushed beyond the horizon by delay tactics and thus seemingly avoided.

Improvements of Alpha-Beta Tree Search

- Move Ordering:
  - Static: Use human (expert) knowledge about the game.
  - Dynamic: Use iterative deepening and the history heuristic (moves that were useful in previous search iterations will probably be useful in later ones).
- Transposition Tables:
  - The same game state can be reached by different histories.
  - Recognising game states that have been visited before avoids re-searching.
- Variable Depth:
  - Strong moves are worth searching more deeply, weak moves (e.g. those expanded later with good move ordering) less so.
- Endgame Tables:
  - Endgames can be completely solved (doing bottom-up search with reverse moves) whenever the number of positions can be handled in practice.
  - The resulting strategies can be put into lookup tables and consulted in search.

Conclusion

Minimax Tree Search can be extended to more than two (say n) players:

- The **UTILITY** function returns an n-tuple $(v_1, \ldots, v_n)$ of utilities.
- Every player i only maximises v_i when it is their turn to move.

Summary

- Game trees can be succinctly represented by **state-based game models**.
- **Minimax Tree Search** can be used to solve sequential (two-player zero-sum) games with perfect information.
- **Alpha-Beta Pruning** allows to reduce the search space without sacrificing solutions.
- **Heuristic Evaluation** of states can be used to reduce search depth.
- Further heuristics may reduce the search space (typically with sacrifices).