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A Note on Interaction and Incompleteness

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Abstract

The notion of interaction and interaction machines, developed by Peter Wegner, includes the comparison between incompleteness of interaction machines and Gödel incompleteness. However, this comparison is not adequate, because it combines different notions and different sources of incompleteness. In particular, it merges syntactic with two senses of semantic completeness, and results about truth (Tarski) with results about provability and their consequences (Gödel). The comparison also overlooks structural differences in the way diagonalization produces incompleteness. More generally, the comparison is unlikely because interaction incompleteness is supposed to come from a system's involvement with its environment, whereas Gödel incompleteness comes from a system's involvement with itself.

Keywords: interaction, incompleteness, Wegner, Gödel, Tarski, diagonalization

1 Introduction

In a series of papers, notably in [15] and [16], Peter Wegner has championed the paradigm of interaction (symbolic interaction). He typically begins by observing that interaction is not expressible by a finite initial input string [16, p. 315], and then introduces what he calls interaction machines, which

> extend Turing machines by adding dynamic input/output (read/write) actions that interact directly with an external environment [16, p. 316, definition 1].

Instead of 'finite initial input strings', these machines are supposed to have dynamically generated input streams, 'mathematically modeled by infinite sequences' [15, p. 89]. This characterization is the basis for what Wegner calls incompleteness of interaction machines, the most precise sense of which is that 'the set of computations of an interaction machine cannot be enumerated' [15, p. 89]. Wegner compares interaction incompleteness in this sense with Gödel's incompleteness result for formal arithmetic:

> Gödel's discovery that the integers cannot be described completely through logic [. . .] may be adapted to show that interaction machines cannot be completely described by first-order logic [15, p. 83].

This comparison has been previously examined by Ekdahl [3], who showed that it depends on confusing two different senses of completeness, and pointed out a certain contradiction in Wegner's treatment of completeness [3, p. 4]. The present note amplifies Ekdahl's conclusions by adding a third sense of completeness, present in Wegner's comparison. The main addition of the note to what Ekdahl has already noticed is the finding that the comparison between interaction incompleteness and Gödel incompleteness is also inadequate at the level of the sources of these sorts of incompleteness. Even before looking closer into them, it may be noted that the comparison seems strange, because interaction incompleteness is supposed to come from a system's involvement with its environment, whereas Gödel incompleteness comes from a system's involvement with itself.

The note first revisits briefly the incompleteness ambiguity noted by Ekdahl, as a way of introducing two basic concepts of completeness (section 2.1), and then points out a further, third sense of completeness in Wegner's comparison (section 2.2). Next, moving to the level of the sources of incompleteness (section 3), the note shows that Wegner does not present correctly the way in which Gödel incompleteness is obtained (section 3.1). The most detailed argument of the note, close to the actual technical level invoked in the comparison, then points out inadequacies in Wegner's treatment of the role of the specific mechanism of incompleteness (diagonalization) in his concept of interaction incompleteness (section 3.2.1) and its supposed connection with Gödel incompleteness (section 3.2.2). The discussion that follows (section 4) returns to more general questions about interaction, computation and incompleteness, attempting to address the intuitions which might have motivated Wegner's comparison.

2 Senses of completeness

One reason for the inadequacy of the comparison between interaction incompleteness and Gödel incompleteness is that, in the treatment of Gödel incompleteness, it conflates various senses of completeness. Ekdahl [3] has already noted two of them, which will be revisited here, using different textual examples, as a way of refreshing two basic concepts of completeness before drawing attention to a third one.

2.1 Semantic and syntactic completeness

In one of his more representative publications [16], Wegner introduces the notion of completeness by stating that 'a logic is complete if all tautologies [formulae true in all interpretations] are provable' [16, p. 343, definition 17] and then, without changing the definition, states that 'Gödel proved incompleteness [. . .]' [16, p. 344]. However, the sense of completeness in which Gödel proved incompleteness is different from the completeness which Wegner refers to. The completeness in his definition is basically a relation between a theory and what it can be about; more precisely, it is a property of the theory whose definition involves the interpretations of the theory, which is why it is called semantic completeness. By contrast, the completeness in Gödel's incompleteness result is a property of the theory alone, namely the provability or refutability of any statement (syntactic completeness). Moreover, and somewhat ironically, not only did Gödel *not* prove incompleteness in Wegner's sense, but he proved *completeness* in that sense [8, p. 67]. That is, Gödel did *not* prove what Wegner's statement implies, namely that some formulae true in all interpretations are not provable (semantic incompleteness); on the contrary, he proved that all such formulae *are* provable (semantic completeness). Gödel also proved, quite independently, that some statements are neither provable nor refutable in a sufficiently consistent system of arithmetic, and related theories (syntactic incompleteness), but that is a very different story. It might be added that this 'misunderstanding of the completeness and incompleteness results' noted by Ekdahl [3, p. 4] is frequent enough, and that textbooks rarely bother to 'decrease the likelihood that the reader will assume that Gödel's incompleteness theorem has something to do with semantic completeness', as McCawley's does [7, pp. 74–6].

2.2 A third sense of completeness

A further, third sense of completeness, present in Wegner's comparison, completes the confusion about completeness noted by Ekdahl [3]. He also mentioned this sense of completeness as a third possibility, but it was not actually present in the paper he examined [16]. The third sense of completeness concerns the relation between what is provable in a theory and what is true in a particular model. This sense of completeness is explicit in the following definition, from a later paper: 'a logic is [...] complete if all true assertions of the modeled domain are theorems' [17, p. 64]. Apart from the unusual choice of the term defined (logic does not model domains, theories do), the definition is unexceptional, but it does not capture the notion of completeness involved in Gödel's incompleteness. It might be added that Wegner also uses the notion of (in)completeness in a colloquial, non-technical way: 'Gödel showed incompleteness of the integers [...]' [15, p. 89]. Here, incompleteness is a property of the "integers" (natural numbers, more likely), not of a theory about them (or a logic). Disregarding this further sort of (in)completeness, the three basic senses of completeness between which Wegner moves concern

- the relation between provability and truth in *all models* (Gödel's completeness)
- the relation between provability and truth in a *particular model*
- the provability or refutability of *all statements* (Gödel's incompleteness)

The confusion noted by Ekdahl involves the first and the third sense of completeness, while the additional confusion in the definition above involves the last two senses. The first sense of completeness is involved in Gödel's completeness result, which is related to logic; the third sense of completeness is involved in Gödel's *in*completeness result, which is related to arithmetic.

3 Sources of incompleteness

The misunderstanding about the nature of Gödel incompleteness, presented above, is accompanied by a misunderstanding of the sources of incompleteness, both of Gödel incompleteness and interaction incompleteness. This misunderstanding, at the level of reasons for incompleteness, is central to the comparison between the two sorts of incompleteness, and is also interesting enough in itself to reconstruct in some detail.

3.1 Sources of Gödel incompleteness

Wegner says that

> if the logic is both sound and complete, [...] the number of true assertions expressible by theorems is recursively enumerable [...]
> Gödel proved incompleteness using a diagonalization argument to show that true statements were not recursively enumerable [16, p. 343–4].

In order to understand and evaluate this passage, it is not actually necessary to know what "recursively enumerable" means, and what a diagonalization argument is. These will be explained when they come up again later, but what matters here is only the top structure of Wegner's statements. If the first one is abbreviated as

$$\text{sound and complete} \rightarrow \text{truths are RE}$$

the second can be written as

Gödel proved incompleteness using D to show that truths are not RE

Wegner thus suggests that Gödel showed that the right-hand side of the implication above is false, so the left-hand side must be false too; assuming soundness (truth of theorems), this would indeed establish incompleteness. This interpretation of Wegner's idea of Gödel's proof is rather accommodating, because Wegner talks of 'the number of true assertions *expressible by theorems*', not the "number" (set, more likely) of true assertions. Thus, strictly speaking, the right-hand side of the implication above should be '*provable* truths are RE', which does not need the condition provided by the implication. But the point is that even if Wegner's statements are sympathetically interpreted to constitute a valid argument, this argument does not present correctly Gödel's proof of incompleteness. Gödel did not prove incompleteness in the way suggested, in the first place because this would be the wrong, semantic sort of incompleteness. This can be seen if the first deleted part of the quotation above is restored:

> if the logic is both sound and complete, then there is a one-to-one correspondence between syntactic theorems and semantically true assertions *for all models*, and the number of true assertions expressible by theorems is recursively enumerable [...] (emphasis mine)

The part restored here was initially deleted in order to bring out Wegner's idea of the basic logical (propositional) structure of Gödel's proof. However, the sense of completeness present in this part (semantic completeness) is not appropriate for the whole passage, since Gödel proved a different sort of incompleteness. This was already explained in section 2, but what should be added here is that Gödel proved incompleteness independently of the implication above. Gödel proved incompleteness directly, by constructing a statement which is neither provable nor refutable in a (sufficiently) consistent system (of arithmetic).

Similar objections apply to an earlier, somewhat more precise version of the claim quoted above:

> The set of true statements of a sound and complete logic can be enumerated as a set of theorems and is therefore recursively enumerable. Gödel showed incompleteness of the integers by showing that the set of true statements about integers was not recursively enumerable [15, p. 89].

The fact that truths about "integers" (natural numbers) are not recursively enumerable means that they cannot be listed using certain well-behaved combinations of simple ways of specifying things. Again, knowing precisely what these ways and combinations are is not necessary to understand and evaluate the claim that Gödel showed incompleteness of the integers *by* showing that truths about them are not enumerable in these ways and combinations. Incidentally, the previously quoted version of the claim shows that the claim itself is not an excessively literal reading of Wegner's statement. In any case, the fact that truths about "integers" are not enumerable in certain ways and combinations was not instrumental in Gödel's incompleteness proof. In that proof, Gödel actually avoided the concept of truth; as Feferman says, Gödel 'took pains to eliminate the concept of truth from the main results of 1931' [4, p. 106] (it is present only in his introductory sketch of these results, which will be used in the next section).

In his (first) incompleteness theorem, Gödel proved that a particular statement of arithmetic is not provable if the theory is consistent. He constructed this statement to be readable as a meta-theoretical statement too, saying precisely that about itself, namely that it is not provable. Assuming that arithmetic is consistent, the statement is true. This formulation is a little too simple (there are models of arithmetic in which the statement is false), but the point is that this is the path to the conclusion that some arithmetical truth is not a theorem. This conclusion can indeed also be reached in the way Wegner suggests:

Theorems are recursively enumerable
Arithmetical truths are not recursively enumerable
Therefore, some arithmetical truth is not a theorem

However, this argument only establishes a consequence of Gödel's incompleteness theorem, not the theorem itself. Perhaps more importantly, not only does the argument fall short of establishing Gödel's incompleteness result, but it also goes the wrong way about it, so to speak. The reason is that the argument raises the question of its second premise: where did that premise come from? The result that truths about natural numbers are not recursively enumerable was actually established after Gödel's incompleteness theorem, and is better associated with Tarski. It follows from two other results: the fact that recursively enumerable sets are definable by arithmetical formulas [10, p. 126], and the result that the set of true statements about natural numbers is not so definable [10, p. 122], which is closely related to Tarski's result proper. Both express the idea that the property of being a true sentence of arithmetic is not expressible in it (Tarski's theorem says, roughly, that if arithmetic is consistent, it does not have a truth predicate). The error in Wegner's idea of the source of Gödel incompleteness is now clear, and can be restated briefly: the logical path goes, not from Wegner's pseudo-Tarski to Gödel, but from Gödel to Tarski (Tarski's theorem can be proved from Gödel's (first) incompleteness theorem, by way of two other theorems).

3.2 Diagonalization

Interestingly, Wegner's misunderstanding of the reasons for Gödel incompleteness is matched by a misunderstanding of the reasons for what he calls interaction incompleteness. To refresh the notions involved, interaction machines

> extend Turing machines by adding dynamic input/output (read/write) actions that interact directly with an external environment [16, p. 316, definition 1].

These machines are supposed to be incomplete in the sense that 'the set of computations of an interaction machine cannot be enumerated' [15, p. 89]. Wegner compares this sort of incompleteness with Gödel's incompleteness and claims that

> the incompleteness proof for interaction machines is actually simpler than Gödel's, following directly from non-enumerability of infinite sequences [16, p. 344].

An earlier version of this claim was more explicit:

> interaction machine incompleteness follows from nonenumerability of infinite sequences over a finite alphabet *and does not require diagonalization* [15, p. 89] (emphasis mine)

This remark about diagonalization is worth going into a little deeper because Wegner's statements about it are the most precise of all his statements, closest to the level of formal proofs which should ground the comparison between interaction incompleteness and Gödel incompleteness. Wegner says that interaction incompleteness is 'a form of Gödel incompleteness' [16, p. 315], and that

> Gödel's discovery that the integers cannot be described completely through logic, [...] may be adapted to show that interaction machines cannot be completely described by first order logic [15, p. 83].

These claims are another reason for looking closer at the background of proofs which Wegner's comparison invokes.

3.2.1 Diagonalization in interaction incompleteness

Interaction machine incompleteness as non-enumerability of computations indeed follows from non-enumerability of infinite input sequences. What is more problematic is the claim that interaction incompleteness does not require diagonalization, because non-enumerability of infinite sequences seems to require it. That is, diagonalization is precisely the method used to prove that infinite sequences over a finite alphabet are not enumerable [2, p. 17]. This is established by taking the sequences of any proposed enumeration to be the rows of a matrix such as

$$\begin{array}{cccc} a & b & c & \dots \\ b & c & a & \dots \\ a & c & d & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

and constructing a new sequence by going down the diagonal and stringing together elements different from the diagonal ones, producing a sequence such as

$$\begin{array}{cccc} b & a & c & \dots \end{array}$$

Since each element of this sequence was chosen to differ from the corresponding element in one of the sequences, this sequence is different from all the sequences in the proposed enumeration, so it could not have been included in it.

Returning now to the claim that 'interaction machine incompleteness [...] does not require diagonalization' [15, p. 89]: Wegner does not explain this claim, or offer an alternative proof that infinite sequences over a finite alphabet are not enumerable. As a matter of fact, such proofs which do not use diagonalization can be constructed, for example by what might be called parabolization: the sequence which falls outside any given enumeration can be defined by taking its first element to be different from the first element of the first sequence, its fourth element different from the fourth element of the second sequence, its ninth element different from the ninth element of the third sequence, and so on. No matter how the intervening elements are chosen, the resulting sequence will differ from all the sequences in the enumeration: from sequence number i in element number i^2. That is, instead of going down the diagonal of the matrix above, some other path such as a parabola can be taken. So, it should be conceded that "interaction machine incompleteness does not require diagonalization", even though alternative proofs are needlessly more complex, which is why they don't appear in the literature. However, what is more important is that the remark about diagonalization also holds for Gödel's incompleteness, to an even greater extent, so the remark looses its point.

Diagonalization and what was called parabolization use the same proof idea, namely the systematic construction of a sequence which differs from each of the enumerated sequences in a prescribed position, prescribed by a certain function (identity in the case of diagonalization, a quadratic function in the case of "parabolization"). On this basis, it might be said that diagonalization and "parabolization" are really the same thing, and should not be distinguished. However, this objection would provide no support for the claim that interaction machine incompleteness does not require diagonalization. On the contrary: the claim is true if diagonalization is understood in the usual, literal sense, referring to the diagonal in the matrix above. The case of "parabolization" supported the claim, and the objection would remove this support.

3.2.2 Diagonalization in Gödel incompleteness

In Gödel's introductory sketch of the incompleteness proof [5, pp. 7-8], the starting point is the matrix of statements:

$$\begin{array}{cccc} P_0(0) & P_0(1) & P_0(2) & \dots \\ P_1(0) & P_1(1) & P_1(2) & \dots \\ P_2(0) & P_2(1) & P_2(2) & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

where $P_i(x)$ is the i-th property of natural numbers in some enumeration of such properties. If the diagonal procedure in the proof of non-enumerability of infinite sequences is described by the term

$$\neg\, d_i \tag{3.1}$$

(abusing notation a little in order to bring out similarities), the procedure in Gödel's proof is described by the term

$$\neg\, provable(d_i) \tag{3.2}$$

That is, non-provability of the statements along the diagonal in the matrix above defines a certain property of natural numbers: a number i has this property if the statement $P_i(i)$ is not provable. This property has a certain number g in the enumeration of properties, and the incompleteness result follows from considering what happens at the corresponding place on the diagonal, with the statement $P_g(g)$. This statement says that the number g has the property $P_g(x)$, which means that the statement $P_g(g)$ is unprovable, by the definition of that property. The statement $P_g(g)$ is thus the statement mentioned in the previous section, namely the statement that says of itself that it is not provable. A short reflection on the definition of $P_g(x)$ then shows that $P_g(g)$ is neither provable nor refutable; the interested reader can find this reflection in the appendix.

Comparing the two incompleteness proofs, it is obvious that there are major structural differences in their use of diagonalization. In the proof of non-enumerability of infinite sequences, a new sequence was constructed for any enumeration of sequences; in Gödel's proof, the property constructed from a given enumeration of properties is a special member of that enumeration. This goes directly against Wegner's and Goldin's claim that

> Gödel proved his theorem by showing that arithmetic over the integers could not be expressed by an enumerable number of formulae, using diagonalization to prove nonenumerability [18, p. 14].

Another difference between the proofs is that the first one uses elements satisfying a certain property (that of being different from the diagonal element), whereas the second proof uses such a property itself (that of having an unprovable diagonal element). The first proof only walks down the diagonal to define a new row; the second proof walks down the diagonal to single out a certain row whose diagonal element is itself part of the walk. The term (3.1) describes a procedure, or the resulting new row of the matrix, at the meta-level, whereas (3.2) describes a special row, and does so in the same theory in which the other properties are formulated. This difference will be clearer if what it takes to describe the special row is indicated in some more detail:

$$\neg\, provable(\ subst(P_i(x), num(i))\ ) \tag{3.2}$$

This formula indicates the basic structure of the special property $P_g(x)$; the term $subst(P_i(x), num(i))$ describes the result of substituting, into the i-th property $P_i(x)$, the numeral of its number i for the free variable. This procedure, whose description is part of $P_g(x)$, is then used on $P_g(x)$ itself to produce $P_g(g)$. This should make obvious the considerable difference between the proofs of interaction incompleteness and Gödel incompleteness: the first proof only performs a procedure, whereas the second formalizes a procedure and then applies it to its own formalization. It should thus be clear that there is no basis for saying that interaction incompleteness is 'a form of Gödel incompleteness' [16, p. 315], or that 'Gödel's discovery [. . .] may be adapted to show that interaction machines cannot be completely described [. . .]' [15, p. 83]. The double use of diagonalization in Gödel's proof is a sophisticated elaboration of the use of diagonalization in interaction incompleteness, so it makes little sense to say that it can be "adapted" for such simple use.

Finally, taking up again Wegner's remark that interaction incompleteness does not require diagonalization [15, p. 89]: the same applies to Gödel incompleteness, only more so, so to speak. That is, more variations that do not use diagonalization are possible in Gödel incompleteness, because of its greater complexity. First, diagonalization as the special substitution of the numeral of a formula's Gödel number "back" into the formula, in place of the free variable, can be replaced by some other substitution, corresponding to a different path through the rows of the matrix above, for example along lines parallel to the diagonal [1], or along other paths that can be described in the formal system, such as the parabola in the interaction incompleteness example. The condition that the alternative path can be described in the formal system is necessary in order to replace the term $subst(P_i(x), num(i))$, which describes the diagonal substitution in (3.2), with a description of the alternative path. A further variation that does not use diagonalization is the use of some other operation instead of substitution, such as concatenation [11]. Finally, it is also possible to eliminate diagonalization from the surface of the proof and push it down into its presupposition, namely into the system of numbering properties of natural numbers (non-standard or Kripke codes) [12, p. 628]. Such "pre-diagonalization" makes it possible to construct sentences that refer to themselves directly, through the numeral of their own number, instead of using operations such as substitution or concatenation.

4 Discussion

The claim that there is a connection between interaction incompleteness and Gödel incompleteness seems implausible even without going into the details of their proofs and the role of diagonalization in these proofs. The two kinds of incompleteness have basically different sources: Gödel incompleteness comes, so to speak, from a system's involvement with itself, not from involvement, 'interaction with an external environment', because it depends on reflecting the relation of provability for a system within the system itself. Simplifying considerably in order to address the intuition which might motivate Wegner's comparison, it could be said that interaction incompleteness is computation overwhelmed by input, whereas Gödel incompleteness is computation overwhelmed by becoming its own input. It might even be said that Gödel incompleteness comes from self-interaction, which would at least explain why Wegner calls diagonalization an interactive process [16, p. 318]. However, this would invert Wegner's statement of the relation between interaction incompleteness and Gödel incompleteness. If self-interaction is a form of interaction, it could only be the case that Gödel incompleteness is a form of interaction incompleteness, not the other way around. But saying this would amount to little more than stretching meanings in order to meet Wegner's intuitions and correct them. It is much better to say that the two forms of incompleteness are independent, and only share, in a general way, a certain method of proof.

More generally, interaction itself, human or otherwise, does not seem to have any feature of Gödel incompleteness, though some kinds of interaction might recall a Gödel-type situation. A conceivable sort of example might be social situations whose framework, not to say consistency, is defined by obvious but unspeakable or unutterable truth ("elephant in the room" or "naked emperor" phenomena). A likely place to look for better connections between interaction and Gödel incompleteness might be the field of pragmatic paradoxes ("paradoxical interaction") [13].

If incompleteness is supposed to be the basic feature of interaction, 'the essential ingredient distinguishing interactive from algorithmic models of computing' [15, p. 88], this is not Gödel incompleteness. A better case can actually be made for connecting Gödel incompleteness not to interaction machines but to classical Turing machines. Webb [14] has indicated in some detail how Gödel incompleteness brings the richness of behavior and unpredictability, supposedly characteristic of interaction machines, to classical Turing machines [14, ch. IV, esp. p. 193]. Other aspects of the significance of Wegner's notion of interaction for the theory of computation have been examined elsewhere [9], but what can be added here is that this significance is limited, in Wegner's own papers and in some collaborative efforts [18], by the fact that the characterization of interaction machines mainly concerns the input side of computations. More precise implications for computational architectures servicing such input may be drawn in papers inspired by Wegner's emphasis on interaction [6], but these papers don't suggest connections with Gödel's incompleteness.

A Appendix

The proof of Gödel incompleteness from the statement $P_g(g)$ in section 3.2.2 goes as follows: if the statement were provable, this would mean that the number g has the property $P_g(x)$, which would in turn mean that $P_g(g)$ would not be provable, by the definition of that property. Since the assumption that $P_g(g)$ is provable would thus lead to contradiction, the statement cannot be provable. On the other hand, if the statement $P_g(g)$ was refutable, this would mean that the number g does not have the property $P_g(x)$, which would in turn mean that $P_g(g)$ is provable, by the definition of that property; since the assumption that $P_g(g)$ is refutable would lead to contradiction, it cannot be refutable either.

B Acknowledgment

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Two Restrictions on Contraction

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Abstract

I show two simple limitations of sequent systems with multiplicative context treatment: contraction can be restricted neither to atoms nor to the bottom of a proof tree.

Keywords: sequent calculus, contraction, deep inference

1 Motivation

The motivation for the present work is to find out whether there is a sequent system that possesses certain properties of system SKS, a set of rules for classical propositional logic introduced in [1]. System SKS is not a sequent system, but is presented in a more general formalism, the *calculus of structures* [2]. In this formalism, an inference rule has only one premise: derivations are sequences of rule instances, not trees as in the sequent calculus. While the sequent calculus restricts the application of rules to the main connective of a formula, the calculus of structures is more expressive by admitting *deep inference*, meaning that rules can be applied anywhere inside formulae.

Similarly to sequent systems, system SKS has a contraction rule which, when seen bottom-up, duplicates a formula. This contraction rule can be restricted 1) to atoms and 2) to the bottom of a proof. Apart from contraction, no other rule duplicates formulae. The two restrictions on contraction thus respectively entail the following two interesting properties [1]:

1. Applying a rule may involve duplicating atoms, but not duplicating arbitrarily large non-atomic formulae.
2. Proofs can be separated into two phases (seen bottom-up): The lower phase only contains instances of contraction. The upper phase contains instances of the other rules, but no contraction. No formulae are duplicated in the upper phase.

The question is whether the extra expressive power of the calculus of structures is needed for these properties, or whether they can be obtained in sequent systems as well. In system G3cp [3], for example, contraction is admissible and can thus trivially be restricted to atoms or to the bottom of a proof. However, G3cp has an additive (or context-sharing) R∧-rule, so these restrictions on contraction do not entail the above mentioned interesting properties. Contraction is admissible, but additive rules such as R∧ implicitly duplicate formulae which may be non-atomic. Of course, R∧ is not eliminable. To obtain a proof separation similar to the one for system SKS, one would have to restrict contraction *and* R$\wedge$ to the bottom of a proof tree, which is not possible. Other sequent systems with an additive R$\wedge$-rule share these problems.

To answer the question whether there is a sequent system with the properties of system SKS, I thus consider systems with a multiplicative (or context-splitting) R$\wedge$-rule, exemplified by system GS1p with multiplicative context treatment [3], shown in Fig. 1.

$$\mathsf{Ax}\,\frac{}{\vdash A, \bar{A}} \qquad \mathsf{RC}\,\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \mathsf{RW}\,\frac{\vdash \Gamma}{\vdash \Gamma, A}$$

$$\mathsf{R}\vee\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \qquad \mathsf{R}\wedge\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}$$

FIG. 1. GS1p with multiplicative context treatment

The rules Ax, RC and RW are respectively called *axiom*, *contraction* and *weakening.* Propositional variables p and their negations $\bar{p}$ are *atoms*, with the negation of the atom $\bar{p}$ defined to be p. Atoms are denoted by $a, b, \ldots$. Formulae, denoted by $A, B, \ldots$, are in *negation normal form*, meaning that they contain negation only on atoms. $\bar{A}$ denotes the negation normal form of the negation of formula A. A *derivation* (also called partial proof) is a tree of rule instances. A *proof* is a derivation where all leaves are axioms. In a derivation, *all contractions are at the bottom* if no contraction is applied above a rule different from contraction. An application of the contraction rule is said to be *atomic* if its principal formula is an atom. The *endsequent* of a derivation is the sequent at the root. The system GS1p *with atomic axiom* is GS1p with the formulas A and $\bar{A}$ in the axiom required to be atoms.

2 Restricting Contraction in the Sequent Calculus is Impossible

In the following, I will show that GS1p does not possess the properties of SKS.

Proposition 2.1 There is a valid sequent that has no proof in multiplicative GS1p in which all contractions are atomic.

PROOF. Consider the following sequent:

$$\vdash a \wedge b, (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}) \quad . \tag{2.1}$$

There are no single atoms, so contraction cannot be applied. Each applicable rule leads to a premise that is not valid. ∎

Proposition 2.2 There is a valid sequent that has no proof in multiplicative GS1p in which all contractions are at the bottom.

PROOF. Consider the following sequent:

$$\vdash a \wedge a, \bar{a} \wedge \bar{a} \quad . \tag{2.2}$$

It suffices to show that, for any number of occurrences of the formulae $a \wedge a$ and $\bar{a} \wedge \bar{a}$, the sequent

$$\vdash a \wedge a, \ldots, a \wedge a, \bar{a} \wedge \bar{a}, \ldots, \bar{a} \wedge \bar{a} \tag{2.3}$$

is not provable in GS1p without contraction. Since the connective $\vee$ does not occur in this sequent, the only rules that can appear in contraction-free derivations with this endsequent are Ax, R$\wedge$ and RW. The only formulae that can appear in such derivations are $a \wedge a, \bar{a} \wedge \bar{a}, a$ and $\bar{a}$. Consequently, the only formulae that can appear in an axiom are the atoms a and $\bar{a}$. A leaf can thus be closed with an axiom only if it contains exactly two single atoms (as opposed to two non-atomic formulae).

We prove by induction on the size of the derivation that each such derivation has a leaf which contains at most one single atom. The base case is trivial. For the inductive case, consider a derivation $\mathcal{D}$. Remove a rule instance ρ from the top of $\mathcal{D}$, to obtain a derivation $\mathcal{D}'$. Let l be the leaf with the conclusion of ρ. By inductive hypothesis, $\mathcal{D}'$ has a leaf with at most one single atom. Assume that this leaf is l, otherwise the inductive step is trivial. The rule instance ρ can not be an axiom, because there is at most one single atom in l. If ρ is a weakening then the premise of ρ contains at most one single atom. If ρ is an instance of R$\wedge$ then the only single atom that may occur in the conclusion goes to one premise. The other premise contains at most one (i.e. exactly one) single atom. ∎

A referee found a simpler proof of Proposition 2.2 by using the following fact:

Fact 2.3 If a sequent has a contraction-free proof in GS1p then it has a contraction-free proof in GS1p with atomic axiom.

This proof is as follows: consider proofs in GS1p with atomic axiom. By a trivial induction on the structure of the proof it follows that every contraction-free proof has an endsequent with at least two single atoms or at least one occurence of the connective $\vee$. Thus, sequent (2.3) has no contraction-free proof in GS1p with atomic axiom. By contrapositive of the above fact, it has no contraction-free proof in GS1p.

The reason for presenting the more complex proof is that it is more general: it applies to systems for which the above fact does not hold, e.g. GS1p with multiplicative R$\wedge$ and additive R$\vee$. In fact, the proofs of Propositions 2.1 and 2.2 rely on the multiplicative context treatment in the R$\wedge$-rule, but work regardless of whether the system in question is for propositional or for first-order predicate logic, whether it is two- or one-sided, whether or not rules for implication are in the system, whether it is related to G1 (explicit weakening) or G2 (weakening built into the axiom) and whether a multiplicative or additive version of the R$\vee$-rule is used. In those sequent systems, contraction can thus neither be restricted to atoms nor to the bottom of a proof. Consequently, those systems do not have the interesting properties of system SKS.

$$
\mathsf{c}\,\dfrac{\mathsf{c}\,\dfrac{\mathsf{m}\,\dfrac{\mathsf{R}\wedge\,\dfrac{\mathsf{R}\vee\,\dfrac{\mathsf{R}\wedge\,\dfrac{\vdash a,\bar a \quad \vdash b,\bar b}{\vdash a \wedge b, \bar a, \bar b}}{\vdash a \wedge b, \bar a \vee \bar b} \qquad \mathsf{R}\vee\,\dfrac{\mathsf{R}\wedge\,\dfrac{\vdash a,\bar a \quad \vdash b,\bar b}{\vdash a \wedge b, \bar a, \bar b}}{\vdash a \wedge b, \bar a \vee \bar b}}{\vdash a \wedge b, a \wedge b, (\bar a \vee \bar b) \wedge (\bar a \vee \bar b)}}{\vdash (a \vee a) \wedge (b \vee b), (\bar a \vee \bar b) \wedge (\bar a \vee \bar b)}}{\vdash (a \vee a) \wedge b, (\bar a \vee \bar b) \wedge (\bar a \vee \bar b)}}{\vdash a \wedge b, (\bar a \vee \bar b) \wedge (\bar a \vee \bar b)}
$$

$$
\mathsf{c}\,\dfrac{\mathsf{c}\,\dfrac{\mathsf{c}\,\dfrac{\mathsf{c}\,\dfrac{\mathsf{m}\,\dfrac{\mathsf{R}\wedge\,\dfrac{\mathsf{R}\vee\,\dfrac{\mathsf{R}\wedge\,\dfrac{\vdash a,\bar a \quad \vdash a,\bar a}{\vdash a, a, \bar a \wedge \bar a}}{\vdash a \vee a, \bar a \wedge \bar a} \qquad \mathsf{R}\vee\,\dfrac{\mathsf{R}\wedge\,\dfrac{\vdash a,\bar a \quad \vdash a,\bar a}{\vdash a, a, \bar a \wedge \bar a}}{\vdash a \vee a, \bar a \wedge \bar a}}{\vdash (a \vee a) \wedge (a \vee a), \bar a \wedge \bar a, \bar a \wedge \bar a}}{\vdash (a \vee a) \wedge (a \vee a), (\bar a \vee \bar a) \wedge (\bar a \vee \bar a)}}{\vdash (a \vee a) \wedge (a \vee a), (\bar a \vee \bar a) \wedge \bar a}}{\vdash (a \vee a) \wedge (a \vee a), \bar a \wedge \bar a}}{\vdash (a \vee a) \wedge a, \bar a \wedge \bar a}}{\vdash a \wedge a, \bar a \wedge \bar a}
$$

FIG. 2. Proofs using deep inference and medial

3 Restricting Contraction by Using Deep Inference

To complete this exposition, I want to give an idea on how the sequents (2.1) and (2.2) are proved in SKS. Contraction can be restricted to the bottom of a proof, because it applies anywhere deep inside a formula. A corresponding rule in sequent calculus notation might look like

$$
\mathsf{c}\,\dfrac{\vdash \Gamma, F\{A \vee A\}}{\vdash \Gamma, F\{A\}} \quad ,
$$

where $F\{\ \}$ is a formula context. Contraction can be restricted to atoms because of deep inference and a rule which is called *medial*. A corresponding rule in sequent calculus notation might look like

$$
\mathsf{m}\,\dfrac{\vdash \Gamma, A \wedge C, B \wedge D}{\vdash \Gamma, (A \vee B) \wedge (C \vee D)} \quad .
$$

I do not want to suggest that those rules should be added to sequent systems, I just present them as sequent calculus rules to avoid going into technical details of system SKS, which can be found in [1]. Using deep inference and medial, we can prove the sequents (2.1) and (2.2) as shown in Fig. 2. Note that in both proofs all contractions are atomic and at the bottom.

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On Indistinguishability and Prototypes

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Abstract

Tolerance spaces are sets equipped with a reflexive, symmetric, but not necessarily transitive, relation of indistinguishability, and are useful for describing vagueness based on error-prone measurements. We show that any tolerance space can be embedded in one generated by comparisons using prototypical objects. As a result propositions, definable on a tolerance space can be translated into propositions behaving classically.

Keywords: distinguishability, discriminability, tolerance space, orthologic, prototypes, measurement

1 Introduction

The part of human knowledge that builds on observation employs the concept of similarity to organize itself. The reason is twofold. First, similarity allows categorization by grouping objects that share the same attributes. Second, similarity allows compactness of knowledge, for we need neither examine nor memorize the particular details of individual objects, as long as we have experienced similar ones.

In [8], we argued that similarity is not a primitive concept, but rather it is based on the notion of indistinguishability. According to our view: two objects are similar if they cannot be distinguished under a certain tolerance. By tolerance we mean the power of discrimination. For example, although two similar houses might appear different in various details when we stand in front of them, they might appear identical if we observe them from an appropriate distance x. Thus, indistinguishability at distance x implies similarity. The smaller the distance x, the more similar the objects are.[1]

The aim of this paper is to explore mathematically the relation of indistinguishability as it arises from measurements, or simply comparisons, prone to error. In the next section, we argue that each indistinguishability criterion based on measurement gives rise to an exact reflexive and symmetric indistinguishability relation whose logic is orthologic. In Section 3 we show the converse, namely, for each reflexive and symmetric indistinguishability relation there is a criterion for it based on a measurement valuation. In Section 4, we explore indistinguishability relations whose criterion is a comparison with a chosen set of prototypical objects. The logic of those relations is classical, and every indistinguishability relation can be embedded in one of them,

[1] Although this example depends on an explicit distance function, such a function might not be available. Power of discrimination depends on the information available. For instance, two houses are indistinguishable, and therefore similar, if we know *only* that they are in they same neighborhood, have the same number of rooms, and are both near public transportation.

translating in essence, orthologic to classical logic.

2 The Indistinguishability Relation

We begin by giving the definition of the mathematical structure where this paper is based, and which formalizes the notion of the indistinguishability that we assume:

Definition 2.1 A *tolerance space* is a pair

$$(X, \sim)$$

where X is a set of objects, and $\sim$ is a binary relation between members of X called the *indistinguishability relation*. We shall assume the following properties for $\sim$:

1. $x \sim x$ (Reflexivity)
2. $x \sim y$ implies $y \sim x$ (Symmetry)

for all $x, y \in X$. The complement of the indistinguishability relation will be called *distinguishability* and denoted by $\not\sim$.

Neither the concept of tolerance space nor the idea of using a reflexive and symmetric relation in order to express indistinguishability is new. Although many authors have argued that indistinguishability is better expressed through equivalence ([1, 12, 7]), many have also dropped transitivity as early as [17] (see also [10] for similarity), while others have weakened transitivity, the most notable example being the t-norm transitivity of the similarity relation in Fuzzy sets ([22]). A set equipped with a reflexive symmetric relation has been called a tolerance space in [23], *proximity* space in [3] and resolution space in [8]. In the framework of Rough Sets [16], a reflexive, symmetric, and transitive relation of indistinguishability is called *indiscernibility*, and if it is only reflexive and symmetric, it is called a *tolerance* relation ([23],[13]).

The class of tolerance spaces is a very general and intuitive framework for studying vagueness, for it makes the fewest assumptions possible. Examples are easily modeled by a tolerance space, as the indistinguishability relation needs no quantitative information. Metrics are perhaps the easiest way to generate distinguishability, for example, let $x \not\sim y$ if and only if $d(x, y) \geq \epsilon$ for some appropriate fixed metric d and non-negative real number ϵ. Other examples follow:

Example 2.2 Let X be the set $\{a, b, c, d\}$ with the smallest indistinguishability relation containing $a \sim b$, $b \sim c$, $c \sim d$, and $d \sim a$ (see Figure 1 where double arrows denote the indistinguishability relation among distinct elements).

Example 2.3 Let S be the set of finite binary strings of finite length n. We can say that two strings are indistinguishable when they have the same length and differ in at most one digit. Otherwise, they are distinguishable.

Example 2.4 Let D be a set of documents (sets of terms) and n a positive integer. Two documents d_1 and d_2 are indistinguishable when they have at least n common terms. That is,

$$d_1 \sim d_2 \quad \text{iff} \quad |d_1 \cap d_2| \geq n.$$

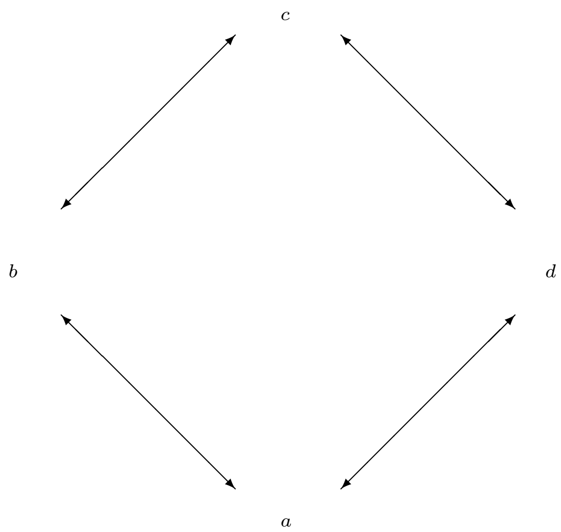

FIG. 1. The tolerance space of Example 2.2.

Example 2.5 Suppose we are given an instrument that can measure the velocity of passing cars. The instrument is digital; therefore, our visual ability to discriminate between measurements is irrelevant. Let us suppose that the instrument displays integers of some velocity measure and the manufacturer of the instrument guarantees an error of less than 0.6.

What is the tolerance space that corresponds to the above example? Let us denote the physical state of the car at a certain moment with s and the result of measurement of its velocity with $v(s)$. If we measure a passing car and read 60, then we are sure that its velocity is not 59. However, if we measure a second car and read 59, then the velocities of the two cars might be the same, for they could both assume the same value in the interval $(59.4, 59.6)$. To ensure distinctness of the two car states with respect to their velocity on the basis of the manufacturer's guarantee on error, we need to have two measurements that differ at least by a number greater than 1.2. So the tolerance space is $(S, \sim)$ where S is the set of car states we measured and the indistinguishability relation is $s_i \sim s_j$ if and only if $|v(s_i) - v(s_j)| < 1.2$.

The above examples illustrate how a reflexive, symmetric but not necessarily transitive, indistinguishability relation is generated by simple, widely practiced indistinguishability criteria. It is worth pointing out that the above examples also impose the law of the excluded middle to indistinguishability. Indistinguishability arises as *failure to distinguish*. This is akin to negation as failure in common sense reasoning. Applications need criteria that give rise to decisions, and as a consequence, the excluded middle is the natural choice. It is important to note that the classification of two objects as indistinguishable must be considered defeasible. A more stringent indistinguishability criterion may distinguish two objects that were previously considered indistinguishable. In example 2.5, a new measuring device with smaller error will distinguish more car states with respect to velocity.

Indistinguishability by no means confines itself to contexts where measuring device criteria impose exactness or exclude transitivity (see [14] and [11]).

We will now state some well-known results (up to Theorem 2.10) on tolerance spaces and propositions that can be defined on them ([4],[9]).

The study of a space equipped with a reflexive and symmetric relation gives rise to a Galois Connection that in turn generates a complete ortholattice of subsets (see [4]).

Definition 2.6 Let $X \subseteq D$. The *discriminant* operator $^{\dashv} : \wp(D) \rightarrow \wp(D)$ is defined by

$$X^{\dashv} = \{y | \forall x \in X, x \not\sim y\}.$$

The complement of $X^{\dashv}$ is

$$X^{\sim} = \{y | \exists x \in X, x \sim y\}.$$

We have the following

Proposition 2.7 1. $X \subseteq Y$ implies $Y^{\dashv} \subseteq X^{\dashv}$ (that is the discriminant operator is antitone).

2. $X \subseteq X^{\dashv\dashv}$, $X^{\dashv} = X^{\dashv\dashv\dashv}$

3. $^{\dashv\dashv} : \wp(D) \rightarrow \wp(D)$ is a closure operator.

In fact, it is well known that an antitone operator on a lattice induces a pair of Galois connections (see [5]) (here the lattice is the lattice of the powerset of D). Closure operators in this case can be generated as a composition of the maps that form the Galois connection.

The closed subsets of the closure operator will be called *stable.* Stable subsets have the form $X^{\dashv\dashv}$.

The proposition below follows from a more general result about the closed subsets of a closure operator (see [4]).

Proposition 2.8 The stable subsets of D form a complete lattice under $\subseteq$. If $\{A_i\}_{i \in I}$ is a family of stable sets then

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i \qquad \bigvee_{i \in I} A_i = (\bigcup_{i \in I} A_i)^{\dashv\dashv}.$$

The algebra of stable propositions on a tolerance space forms an ortholattice ([8]). The proof of the completeness runs along the lines of Goldblatt's representation of ortholattice with its filters ([9]). The equational theory of ortholattices appears in Table 1.

Theorem 2.9 *(Soundness)* The set of stable propositions on a tolerance space forms an ortholattice.

Theorem 2.10 *(Completeness)* For every ortholattice T there is a tolerance space R_T such that T can embedded in the complete ortholattice of stable subsets of R_T.

TABLE 1. Ortholattice Theory

$x \wedge x = x$	$x \vee x = x$	(Idempotence)
$x \wedge y = y \wedge x$	$x \vee y = y \vee x$	(Commutativity)
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	$x \vee (y \vee z) = (x \vee y) \vee z$	(Associativity)
$x \wedge (x \vee y) = x \vee (x \wedge y) = x$		(Absorption)
$x \wedge x^{\dashv} = \bot$	$x \vee x^{\dashv} = \top$	
$(x \wedge y)^{\dashv} = x^{\dashv} \vee y^{\dashv}$	$(x \vee y)^{\dashv} = x^{\dashv} \wedge y^{\dashv}$	(de Morgan)
$(x^{\dashv})^{\dashv} = x$		(Involution)
$x \vdash y$ is equivalent to either $x \wedge y = x$ or $x \vee y = y$		

Stable subsets determine distinguishability in the following sense:

Proposition 2.11 Let $(X, \sim)$ be a tolerance space and $x, y \in X$. Then, x and y are distinguishable if and only if there exists a stable subset A of X such that $x \in A$ and $y \in A^{\dashv}$.

PROOF. If there exists a stable subset A of X, such that $x \in A$ and $y \in A^{\dashv}$, we have that $x \not\sim z$ for all $z \in A^{\dashv}$ and in particular $x \not\sim y$. For the other direction it suffices to notice that if $x \not\sim y$ then $y \in \{x\}^{\dashv}$. ∎

The space of stable subsets characterize a natural notion of specificity among the objects of a tolerance space. Suppose x and y are two elements of a tolerance space $(X, \sim)$ such that, for all z in X, if z is indistinguishable from x then z is indistinguishable from y. In this case, x carries more information as it is able to isolate more elements through comparison than y. Let us call x *more specific* than y and denote this relation with $\leq$, that is $x \leq y$. It turns out that specificity can also be expressed in terms of the stable subsets.

Proposition 2.12 Let $(X, \sim)$ be a tolerance space and $x, y \in X$. Then, for all stable subsets A of $(X, \sim)$, $y \in A$ implies $x \in A$ if and only if $x \leq y$.

PROOF. Suppose, towards a contradiction, that there exists $z \in R$ such that $z \not\sim y$ but $z \sim x$. Then $y \in \{z\}^{\dashv}$ but $x \notin \{z\}^{\dashv}$ and $\{z\}^{\dashv}$ is stable. For the other direction, suppose there exists a stable subset A of X such that $y \in A$ but $x \notin A$. Since $x \notin A$, there exists $z \in A^{\dashv}$ such that $z \sim x$. But $z \not\sim y$. ∎

It is straightforward to show that the specificity is a preorder relation, i.e. reflexive and transitive. Let us define a new relation $\approx$ with $x \approx y$ if $x \leq y$ and $y \leq x$. This relation is an equivalence relation, also known as indirect indiscriminability (page 238 in [21]). Indirect indiscriminability was used by Dummett ([6]) to show that an indistinguishability relation that is exact, reflexive, and transitive is paradoxical: Suppose we observe a minute-hand at five different moments, as it continuously revolves. Continuity of the movement is not necessary as long as the positions of the minute hand at each instant are close enough so that we cannot tell whether the minute-hand has moved in the first four instants. At the fifth instant, we are sure that the minute hand is no longer in the first instant, but we cannot tell the difference between the positions at the fourth and fifth instant. What we have constructed are three different positions n_1, n_4, and n_5 such that $n_1 \sim n_4$, $n_4 \sim n_5$ but $n_1 \not\sim n_5$. So positions n_4 and n_1 are distinguishable *when compared with* n_5, i.e. positions n_4 and n_1 are distinguished in an indirect manner. This way, Dummett argues, starting with a set of distinguishability assertions there is a possibility of generating more such assertions by employing indirect discriminability. In the next section, we will see that we may always develop a criterion for indistinguishability which indirect discriminability does not satisfy.

3 Measurements with Error

In this section, we will show how a reflexive symmetric indistinguishability relation can be generated simply by a set-theoretic intersection.

We will first give a simple formalization of what we mean by measurement with error.

Definition 3.1 A *measurement system* is a triple (X, V, m) where X is a domain of objects, V is a set of *values* and $m : X \to \wp(V) \setminus \emptyset$ is the *assignment map.*

A measurement system not only records the outcome of the measurement, but also the context under which it is achieved. The assignment map should account for any error introduced during measurement. Therefore, m assigns to each $x \in X$ a subset of values from V. Note that our definition of measurement is more permissive than the standard one, where values are real numbers or belong to some partially or totally ordered set ([19],[18]).

Also a measurement system is not assumed to be objective. A different context or a different method of measurement is meant to induce different measurement systems.

Example 3.2 Suppose that a voting machine is counting votes automatically in an election and the manufacturer guarantees an error of 0.01%. Let P be the set of candidate parties and v_p be the number of votes that the voting machine reports for each party $p \in P$. Then the measurement system is (P, N, m) where $m(p) = \{n \in N \mid v_p - \lceil v_p \times 10^{-4} \rceil \leq n \leq v_p + \lceil v_p \times 10^{-4} \rceil\}$.

Now we will show that there is a unique tolerance space corresponding to a measurement system. For this, we need to answer first what distinguishability relation we should employ. Indirect indistinguishability, as defined in the previous section, will distinguish two objects x and y when one of them, say x, is assigned a value v that y is not. This form of indistinguishability is akin to the T_0-separability in topological spaces but it is too weak for our purposes. We shall impose a stronger form of indirect distinguishability that takes into account all values. Call two objects x and y *M-distinguishable* if there is no value that is assigned to both. Formally, $x \sim_M y$ if and only if $m(x) \cap m(y) \neq \emptyset$. Then it easily follows that

Proposition 3.3 Let $M = (X, P, m)$ be a measurement system. Then $(X, \sim_M)$, where $\sim_M$ is defined as above, is a tolerance space.

For the other direction, i.e to construct a measurement system out of a tolerance space, we need a set of values. We will follow a procedure reminiscent of the construction of equivalence classes in a set equipped with an equivalence relation.

Definition 3.4 Let $R = (X, \sim)$ be a tolerance space. A *cluster* C for R is a subset of X such that $x \sim y$ for all $x, y \in C$. An *indistinguishability class* is a cluster which is maximal, i.e. is not a proper subset of another cluster.

Note that every singleton is a cluster and, therefore, there is a cluster C such that $x \in C$, for every $x \in X$. The existence of indistinguishability classes is guaranteed by Zorn's Lemma. This has been showed in [20]. All tolerance spaces can be induced by a measurement system, as the next theorem shows. In other words, a reflexive symmetric relation can be simulated by non-empty intersection on a set of subsets.

Theorem 3.5 Given a tolerance space $R = (X, \sim)$ there is a measurement system such that

$$R = (X, \sim_{M_R}).$$

Proof. Let V_R be the set of indistinguishability classes of R and m_R a map from X to V_R such that $m_R(x) = \{D \mid x \in D, D \in V_R\}$, for all $x \in X$. Note that $m_R(x) \neq \emptyset$ for all $x \in X$. That is, the map m_R assigns to every $x \in X$ the set of all maximal clusters containing it. Now let M_R be the measurement system (X, V_R, m_R). Suppose that $x \sim_{M_R} y$. Then $m_R(x) \cap m_R(y) \neq \emptyset$, so there exists a cluster D in $m_R(x) \cap m_R(y)$. This implies that both x and y must belong to D and therefore $x \sim y$. For the other direction, if $x \sim y$ then $\{x, y\}$ can be extended to a maximal cluster D and therefore $D \in m_R(x) \cap m_R(y)$. So $x \sim_{M_R} y$. ∎

The above theorem can be applied as follows to Dummett's paradox mentioned in the previous section. Recall that three positions n_1, n_4, and n_5 were constructed with $n_1 \sim n_4$, $n_4 \sim n_5$ but $n_1 \not\sim n_5$. Assuming that $\sim$ is exact, reflexive and symmetric, then, by Theorem 3.5, there is a set of values and an assignment map such that those assertions are generated by a measurement system. There exist values that are assigned to both n_1 and n_4 and, similarly, to n_4 and n_5. Positions n_1 and n_5 are assigned no common values. Two objects cannot be compared directly but only through their assigned values. So this setup does not render n_1 and n_4 indistinguishable, as they could both be assigned to a common value. Dummett's argument rests on visual inspection, so the above explanation may at best be limited if an explicit measurement is not involved. On the other hand, the human perceptual system according to sensory science is a measuring instrument ([2]).

Due to Theorem 3.5, indistinguishability assertions forming a tolerance space reduce to consistency assertions. Two objects are indistinguishable when there is a value that is assigned to both, much like two state descriptions that are consistent when there is a state that fits both. Therefore, indirect discriminability does not imply distinguishability because relative inconsistency does not imply inconsistency, i.e. if a is inconsistent with c and b is consistent with c then a and b are not necessarily inconsistent. This reading may help us to better understand vague predicates on a tolerance space. However, it cannot offer a comprehensive approach to vagueness and the reason is twofold. First, indistinguishability is one of many causes for vagueness. For example, two objects could be equally characterized as red although their color is distinguishable. Hence, apart from distinguishability we may rely on other factors for the application of vague predicates. Secondly, if a predicate F is true for an object a, and b is indistinguishable from a then one may infer that $F(b)$ is plausible but not that $F(b)$ is true. However, if b is distinguishable from the objects that are distinguishable from the objects where F is known to be true, then b will be included in the stable closure of F. In this sense, stable extensions are amenable to revision and therefore orthologic may not account for all vagueness phenomena in tolerance spaces.

4 Prototype Completeness

We will now investigate a class of tolerance spaces which incorporate the values of measurements. Then a measurement can be seen as an embedding of a sets of objects in an appropriate tolerance space rather than in a powerset of values. The upshot is that a measurement must be the result of a comparison, and measurement values must be of the same nature as the objects we want to measure. Tolerance spaces can be generated by a set of values P in the following sense:

Definition 4.1 Given a tolerance space $R = (X, \sim)$ and $P \subset X$ then call x and y *P-indistinguishable* ($x \sim_P y$) if there is $p \in P$ such that $x \sim p$ and $y \sim p$. In case, $\sim_P \equiv \sim$ we will say that *R is generated by P*.

Note that $P \subseteq P'$ implies $\sim_P \subseteq \sim_{P'}$. The proof of Theorem 3.5 makes use of indistinguishability classes to construct values so we will seek representatives of those within the tolerance space.

Definition 4.2 Given a tolerance space $R = (X, \sim)$, an element p of X will be called a *prototype for R* if it belongs to exactly one indistinguishability class (that is, it belongs to no other indistinguishability class).

Proposition 4.3 Let $R = (X, \sim)$ be a tolerance space. Then,

1. p is a prototype for R if and only if for all $x, y \in X$, $x \sim p$ and $y \sim p$ implies $x \sim y$,
2. if $P \subseteq X$ is a set of prototypes for R, then $\sim_P \subseteq \sim$,
3. if p and p' are prototypes for R, then $p \sim p'$, $p \sim x$ and $p \sim y$ implies $p' \sim x$ and $p' \sim y$, i.e. indistinguishability is an equivalence relation on prototypes.

Proof. For the right to left direction of 1, suppose that p belongs to two clusters C_1 and C_2 such that $C_1 \neq C_2$. This implies that there exist $x \in C_1$ and $y \in C_2$ such that $x \not\sim y$. However, $p \sim x$ and $p \sim y$ so using the hypothesis $x \sim y$, a contradiction. The other direction is similar.

For 2, let $x \sim_P y$. Then there exists $p \in P$ such that $x \sim p$ and $y \sim p$. From 1, it follows that $x \sim y$. ∎

In case a tolerance space R is generated by a set of prototypes P (see Definition 4.1), then R will be called *prototype complete* and P a *complete* set of prototypes. Note that not all tolerance spaces are prototype complete (see Example 2.2). By Proposition 4.3.3, the existence of a complete set of prototypes implies the existence of a complete set of pairwise distinguishable prototypes which is minimal.

Proposition 4.3.2 shows that one can use the prototypes available in a tolerance space to approximate the indistinguishability relation from below. Again, the proof of Theorem 3.5 indicates that in order to construct a complete set, it would be enough to add a prototype for each indistinguishability class.

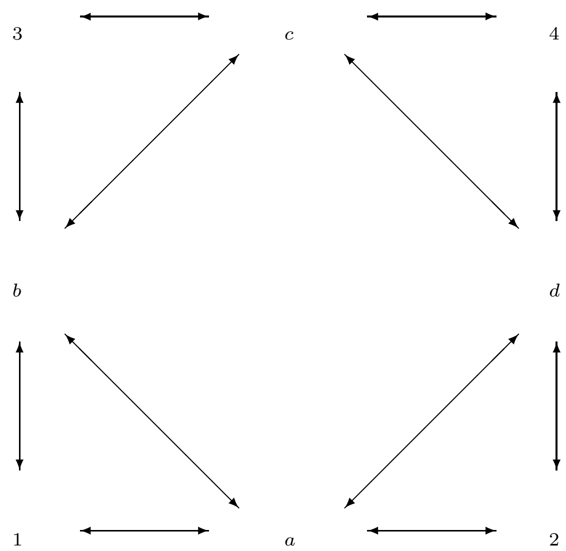

FIG. 2. Prototypes added in Figure 1.

Example 4.4 Consider the tolerance space of Example 2.2. If 1 is the prototype for the indistinguishability class $\{a, b\}$ then we have that $a \sim 1$, $b \sim 1$ and 1 is distinguishable from all other elements, including the rest of the prototypes. We add four prototypes $\{1, 2, 3, 4\}$ one for each indistinguishability class as shown in Figure 2.

However, we do not need a prototype from each indistinguishability class to generate the indistinguishability relation, as the following example shows.

Example 4.5 In the tolerance space of Figure 3, the indistinguishability class $\{a, b, c\}$ lacks a prototype, yet the set of prototypes $\{1, 2, 3\}$ is complete.

A prototype complete tolerance space has the following important property:

Theorem 4.6 The algebra of stable sets of a prototype complete space is a Boolean algebra.

To prove the above theorem, we will show the following three lemmas.

Lemma 4.7 Let $R = (X, \sim)$ be a prototype complete tolerance space. Then every stable subset of X has the form $S^{\dashv}$ for some $S \subseteq P$, where P is a complete set of pairwise distinguishable prototypes for R.

PROOF. Let A be a subset of X and $S = A^{\sim} \cap P$. We will show that $A^{\dashv} = S^{\dashv}$.

To show that $A^{\dashv} \subseteq S^{\dashv}$, suppose there exists $y \in A^{\dashv}$ such that $y \notin S^{\dashv}$ towards a contradiction. This implies that there is $s \in S$ such that $y \sim s$. Now $s \in A^{\sim}$ by the

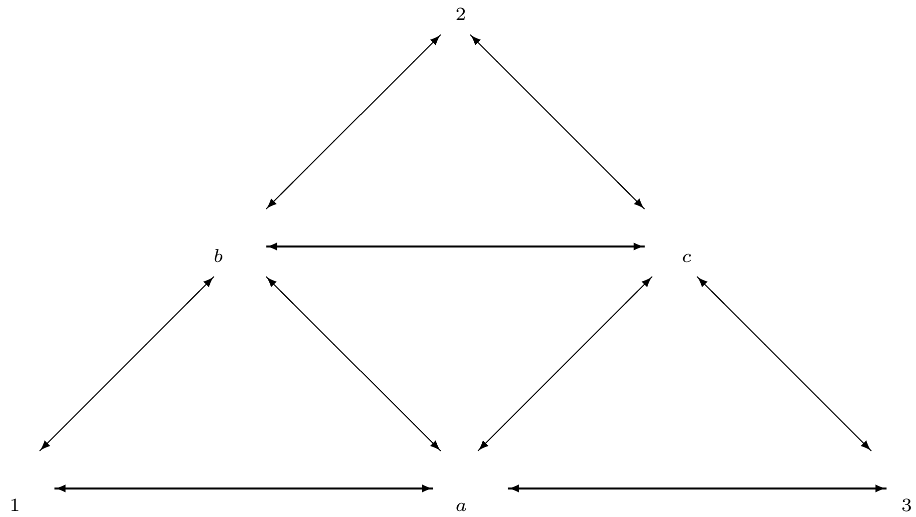

FIG. 3. Example 4.5.

definition of S, so there is $x \in A$ such that $s \sim x$. So we now have a prototype s which is indistinguishable from both x and y. This makes x and y indistinguishable which is a contradiction because $x \in A$ and $y \in A^{\dashv}$.

For the other direction, suppose there exists $y \in S^{\dashv}$ such that $y \notin A^{\dashv}$ towards a contradiction. This implies that there is $x \in A$ such that x is indistinguishable from y. This in turn implies that they have a common indistinguishable prototype, say p. Since $y \in S^{\dashv}$, p must belong to $P - S$. On the other hand, $x \in A$ implies that $p \in A^{\sim}$ and therefore $p \in S$, which is a contradiction. ∎

Lemma 4.8 Let $R = (X, \sim)$ be a prototype complete tolerance space. If P is a complete set of pairwise distinguishable prototypes for R, then $S^{\dashv\dashv} = (P - S)^{\dashv}$, for all $S \subseteq P$.

PROOF. Observe that $(P - S)^{\sim} \cap P = P - S$ and $(P - S)^{\dashv} \cap P = S$. Then the lemma follows from Lemma 4.7. ∎

Lemma 4.9 Let $R = (X, \sim)$ be a prototype complete tolerance space. If P is a complete set of pairwise distinguishable prototypes for R, then $(S^{\dashv} \vee T^{\dashv}) = (S \cap T)^{\dashv}$, for all $S, T \subseteq P$.

PROOF. Observe that,

$$\begin{array}{rcll} (S^{\dashv} \vee R^{\dashv})^{\dashv} & = & (S^{\dashv\dashv} \wedge R^{\dashv\dashv}) & \text{(ortholattice identity)} \\ & = & (P-S)^{\dashv} \wedge (P-R)^{\dashv} & \text{(Lemma 4.8)} \\ & = & (P-S)^{\dashv} \cap (P-R)^{\dashv} & (\wedge \equiv \cap \text{ for stable subsets}) \\ & = & ((P-S) \cup (P-R))^{\dashv} & \\ & = & (P-(S \cap R))^{\dashv} & \\ & = & (S \cap R)^{\dashv\dashv} & \text{(Lemma 4.8)} \end{array}$$

so $(S^{\dashv} \vee R^{\dashv})^{\dashv\dashv} = (S \cap R)^{\dashv\dashv\dashv}$ which implies $S^{\dashv} \vee R^{\dashv} = (S \cap R)^{\dashv}$. ■

Proof. [Proof of Theorem 4.6] Instead of verifying the Boolean algebra identities, we will construct an isomorphism from the lattice of the stable subsets of R to the boolean algebra of the subsets of a complete set P of prototypes. In particular, we will show that the map $F(S^{\dashv}) = P - S$ is an isomorphism. It is easy to see that F is well defined, injective and surjective because of Lemma 4.7. We will show that F is a homomorphism. Note the use of Lemma 4.9 in Case $\vee$.

Case $^{\dashv}$:

$$\begin{array}{rcl} F(S^{\dashv\dashv}) & = & F((P-S)^{\dashv}) \\ & = & P-(P-S) \\ & = & P - f(S^{\dashv}) \end{array}$$

Case $\wedge$:

$$\begin{array}{rcl} F(S^{\dashv} \wedge T^{\dashv}) & = & F((S \cup T)^{\dashv}) \\ & = & P-(S \cup T) \\ & = & (P-S) \cap (P-T) \\ & = & F(S^{\dashv}) \cap F(T^{\dashv}) \end{array}$$

Case $\vee$:

$$\begin{array}{rcl} F(S^{\dashv} \vee T^{\dashv}) & = & F((S \cap T)^{\dashv}) \\ & = & P-(S \cap T) \\ & = & (P-S) \cup (P-T) \\ & = & F(S^{\dashv}) \cup F(T^{\dashv}) \end{array}$$

■

Prototype complete tolerance spaces can be easily generated as follows. Given a set P, the *canonical tolerance space generated by* P is the pair $(\wp(P)\backslash\emptyset, \sim)$, where $A \sim B$ if and only if $A \cap B \neq \emptyset$. It is easy to see that the set $D_p = \{S \mid S \subseteq P \text{ and } p \in S\}$, for each $p \in P$, is an indistinguishability class and $\{p\} \subseteq S$ is a prototype. In fact it is straightforward to show that the set of all singletons is a complete set of prototypes. So we have the following

Proposition 4.10 Every canonical tolerance space generated by a set P is prototype complete with the set of singletons as a complete set of prototypes.

We will now show that prototype complete spaces are an appropriate candidate for measurement values. A measurement can be thought as an embedding of a set of objects in a prototype complete space. Given two tolerance spaces $R_1 = (X_1, \sim_1)$ and $R_2 = (X_2, \sim_2)$ then a map $f : X_1 \to X_2$ will be called an *embedding* if for all $x, y \in X_1$ we have $x \sim_1 y$ if and only if $f(x) \sim_2 f(y)$. If there is such an embedding

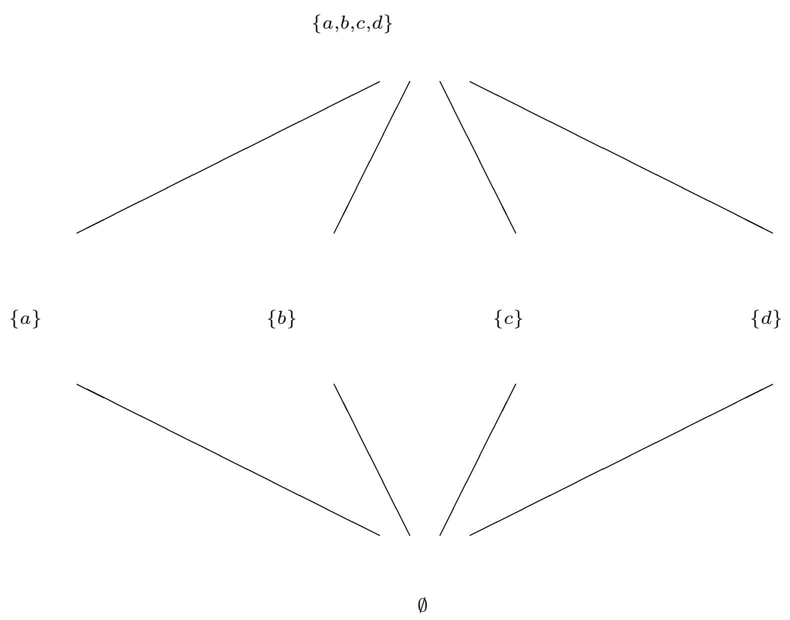

FIG. 4. The stable subset ortholattice of Figure 1.

then we will say that R_1 *embeds in* R_2. Therefore an embedding retains the original tolerance in the new space. Now we may use Theorem 3.5 of the previous section to show the following:

Corollary 4.11 Each tolerance space embeds in the canonical tolerance space generated by the set of its indistinguishability classes.

Therefore, a tolerance space might not have a complete set of prototypes, but embeds in one that has.

Using Corollary 4.11 and Theorem 4.6, we can embed the ortholattice of stable subsets of a tolerance space in a Boolean algebra. However, this embedding is not a homomorphism. This can be seen with the tolerance space of Example 2.2 whose ortholattice of stable subsets appears in Figure 4. This ortholattice is the well-known non-distributive lattice $\mathbf{M_{02}}$ that cannot be homomorphically embedded in a Boolean algebra which is distributive.

The above mismatch is illustrated with the prototype complete tolerance space generated by the indistinguishability classes of Example 2.5. We can define a measurement system as in Section 3, where S are the car states measured, P is the set of all pairs of consecutive integers, and the assignment function maps states to those pairs:

$$m(s) = \{(n\ n+1) | v(s) = n \text{ or } v(s) = n+1\}.$$

Now we have

$$s_1 \sim s_2 \text{ if and only if } m(s_1) \cap m(s_2) \neq \emptyset.$$

Each such pair of consecutive integers corresponds to those states which may be measured to be any of those two integers, i.e. the states whose actual velocity belongs to the interval $(n + 0.4, n + 0.6)$ $(= (n + 0.4, (n + 1) - 0.4))$. The choice of integer values as measurement values is not significant. What matters is the criterion we use to distinguish between those values. For example, if we assume that states are measured to any real number we can generate the same indistinguishability relation with

$$s_1 \sim s_2 \text{ if and only if } \min_{0 \leq \epsilon < 0.6} \{\lceil v(s_1) - \epsilon \rceil\} - \max_{0 \leq \epsilon < 0.6} \{\lfloor v(s_2) + \epsilon \rfloor\} \leq 1,$$

where $v(s_1) \leq v(s_2)$ is assumed. Observe that the underlying function of the criterion is no longer a metric on the larger set of real values. This example illustrates the difference between the stable subset ortholattice of the given tolerance space and the stable subset boolean algebra of the tolerance space generated by its indistinguishability classes. The reader may easily verify that the subset N of states that give a reading of some integer n, i.e. $N = \{s \mid v(s) = n\}$ is stable. Similarly, the subset $N' = \{s \mid v(s) = n + 1\}$ is stable. Their conjunction $N \wedge N'$ is the empty set. This is because we have no way to detect values on the interval $(n+0.4, n+0.6)$. For example, if $v(s_1) = n$ and $v(s_2) = n + 1$, then s_1 and s_2 are indistinguishable because their actual velocities could lie in the interval $(n + 0.4, n + 0.6)$. What makes them indistinguishable is the *possibility* of the velocities lying in this interval. In fact, s_1 and s_2 could be much farther apart, for example, their actual velocities could be $n - 0.5$ and $n + 1.5$, i.e. they could differ by 2. So measurement observations cannot guarantee that a certain value lies in the interval of any of the indistinguishability classes. The same sets are stable in the boolean algebra of the tolerance space generated by its indistinguishability classes. This time, however, the conjunction equals the singleton containing the indistinguishability class $(n\ n + 1)$, i.e. the interval $(n + 0.4, n + 0.6)$. This shows that the price for embedding the ortholattice into a boolean algebra is that propositions could be unobservable through measurement. Nevertheless, such unobservables could be observed in theory or in practice in some other tolerance space. For example, if we knew that $s_1 = s_2$ because we measured the same state with two identical devices simultaneously, then we can indeed detect values in the interval of an indistinguishability class. This, however, can only happen with a combination of two different devices and the tolerance space and its associated measurement system corresponding to this setting is significantly different from the one we assumed.

5 Conclusion

Prototypes form a necessary and simple part of the process of communicating and interpreting vague statements. This paper has shown that a mathematical formulation of prototypes is possible based on indistinguishability assertions as they arise in error-prone measurements that are not necessarily quantitative and may represent subjective judgements. In such cases, indistinguishability may be safely assumed to be reflexive, symmetric and exact, if interpreted as failure to distinguish. We showed that such indistinguishability assertions may be better understood as consistency assertions and this reading evades the pitfalls of relative indistinguishability. Moreover, each set of indistinguishability assertions can be grounded in measurements with a set of prototypes. Such a set can be canonically constructed although it might be unobservable in practice. Finally, whenever such prototypes can be identified or constructed, the logic of indistinguishability assertions may be embedded into classical logic.

Indistinguishability is just one of the causes of vagueness and does not account for all vagueness phenomena. In particular, indistinguishability does not account for vagueness observed in social contexts, which is of the utmost importance ([15]). Interaction among individuals depends on prototype formation, and the above study should contribute to a more comprehensive analysis of vagueness.

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The Soundness Paradox

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Abstract

An inference is standardly said to be sound just in case it is deductively valid and it has only true assumptions. The importance of a coherent concept of soundness to proof theory is obvious, in that it is only sound derivations, and not merely deductively valid arguments, that advance knowledge by providing proofs of theorems in logic and mathematics. The soundness paradox is informally albeit impredicatively formulated as argument (S): Argument (S) is unsound, therefore, argument (S) is unsound. This paper introduces and explains the importance of the soundness paradox, formally demonstrates how to avoid superficial impredication via Gödelization, and compares it with the similar but significantly different liar and validity or Pseudo-Scotus paradoxes. Although there are similarities in this family of semantic diagonalizations, the soundness paradox is not just a hybrid of the liar and validity paradoxes, but is more fundamental, belonging to a special category that resists the most powerful received solutions to the liar and validity paradoxes.

Keywords: Gödel arithmetization, impredicative definition, liar paradox, Ramsey disquotational analysis of truth, relevant reduction of paradox to another paradox, self-reference, soundness, soundness paradox, validity (Pseudo-Scotus) paradox.

1 Soundness

We are accustomed to regard any inference of the form $\varphi \vdash \varphi$ as deductively valid, and as sound just in case φ is true. There is, however, an apparent counterexample to these innocent-appearing assumptions, which I propose to call *the soundness paradox.* My purpose here is only to present the paradox, and not to entertain solutions or even a diagnosis.

Consider as an instantiation of the above schema the following argument:

(S) 1. Argument (S) is unsound.

———————————

2. Argument (S) is unsound.

If (S) is valid, as seems hard to deny, then a paradox arises when we ask whether or not the argument is also sound.

Assume as an application of excluded middle that, like any other argument, (S) is either sound or unsound. Suppose first that (S) is sound. Sound arguments by definition are deductively valid and have only true assumptions, and therefore only true conclusions. But the conclusion of argument (S) is that argument (S) is unsound. So, if argument (S) is sound, then argument (S) is unsound.

Alternatively, suppose that (S) is unsound. Then either (S) is deductively invalid, or it has at least one false assumption. An argument is deductively invalid if and only if it is logically possible for its assumptions to be true and its conclusions false. Argument (S) has only one assumption, which its conclusion restates. If the assumption and conclusion of (S) are the identical proposition, that argument (S) is unsound, then it is logically impossible for the assumption of (S) to be true and the conclusion of (S) to be false. If (S) is unsound, therefore, it cannot be because it is deductively invalid, but only because it has at least one false assumption. But the only assumption of (S) is that (S) is unsound. If the assumption is false, then it is false that argument (S) is unsound, which is to say that argument (S) is sound. So, if argument (S) is unsound, then it is sound.

Hence, argument (S), if deductively valid, is sound if and only if it is unsound, while if argument (S) is not deductively valid, then not all inferences of the logical form $\varphi \vdash \varphi$ are deductively valid.

2 Gödelizing Impredication Away

The paradox as presented is impredicative, where argument (S) is defined by an assumption and conclusion in which argument (S) is explicitly mentioned.[1] Impredicative reference, in this extreme case, the literal syntactical replication of the argument's external name or label '(S)' in its assumption and conclusion, can nevertheless be syntactically avoided by Gödel-arithmetizing the argument's syntax and inferential structure.

To symbolize the paradox requires a metalinguistic vocabulary to formally represent specific logical and semantic properties of propositions and inferences. We assume *Truth*, T, as a primitive bivalent relation of positive correspondence between a proposition and an existent state of affairs that the proposition describes or otherwise linguistically represents. If the state of affairs the proposition represents does not exist, then the proposition is false. A state of affairs is the possession of a property by or involvement in a relation of the objects in a well-defined semantic domain; a state of affairs Fa exists when an object a actually possesses a property or is involved in a relation F, and fails to exist when a does not actually possess or is not actually involved in relation F.[2] *Ramsey Reduction* states that for any proposition φ, φ is true if and only if φ. It allows us to move freely back and forth from true propositions to true metalinguistic propositions that state that the propositions are true. For convenience, we consider only inferences consisting of finitely many assumptions and finitely many conclusions, but the method is easly extendible to inferences of indefinite length. *Validity*, V, is defined as a relation among the truth conditions of the assumptions and conclusions of an inference, such that it is logically impossible for the assumptions to be true and the conclusions false. *Soundness*, S, is then standardly defined in terms of validity and truth. The metatheoretical relational property of being an *assumption*, A, (effectively) of an inference, and metatheoretical relational property of being a *conclusion*, C, (effectively) of an inference, reflect the presupposition that only inferences have assumptions or conclusions. The weakest alethic modal

[1] The concept of impredicative definition originates with Bertrand Russell, "Mathematical Logic as Based on the Theory of Types", *American Journal of Mathematics*, 30, 1908. See especially pp. 239–241, where Russell defines predicative functions, and speaks of his set theoretical paradox as arising through the definition of sets by 'non-predicative' functions. The discussion is replicated in connection with the 'vicious circle principle' in A.N. Whitehead and Russell, *Principia Mathematica*, second edition (Cambridge: Cambridge University Press, 1925–1927), Vol. I, Introduction, Chapter II. Charles S. Chihara, *Ontology and the Vicious-Circle Principle* (Ithaca: Cornell University Press, 1973), pp. 7–11, 138–144, offers an insightful account of Henri Poincaré's rejection of impredicative definitions in light of Richard's paradox.

[2] F.P. Ramsey, "Facts and Propositions", *Foundations: Essays in Philosophy, Logic, Mathematics and Economics*, edited by D.H. Mellor (Atlantic Highlands: Humanities Press, 1978), pp. 40–57.

logic, with appropriate standard set theoretical semantics, interprets the necessity symbol, $\Box$.

A Gödel substitution function, sub_g, substitutes for any whole number to which it is applied the unique syntax string, if any, that the Gödel number encodes. We define Gödel number, g, of the Gödelized soundness paradox inference, $\bar{S}[\mathrm{sub}_g(n)] \vdash \bar{S}[\mathrm{sub}_g(n)]$, as identical to n. Angle quotes, $\ulcorner, \urcorner$, are used conventionally to indicate that the Gödel-numbering context is intensional; a distinct Gödel number obtains for every distinct syntax combination, including logical equivalents, like $\varphi \vee \psi$, and $\neg\varphi \rightarrow \psi$, where $g\ulcorner\varphi \vee \psi\urcorner \neq g\ulcorner\neg\varphi\rightarrow\psi\urcorner$, even though $[\varphi \vee \psi] \leftrightarrow [\neg\varphi \rightarrow \psi]$. The paradox is complete when we conclude that the inference recovered by applying Gödel number substitution function sub_g to Gödel number n is sound or has metatheoretical property S if and only if it is unsound, or if and only if it has metatheoretical property $\bar{S}$. Impredication is avoided in the Gödelized formulation of the soundness paradox. It circumvents the need to designate the inference externally as in the original impredicative formulation by means of a term that is also required to designate the inference internally within the paradox assumption and conclusion. The self-reference of the Gödelized soundness paradox, unlike the original impredicative version, assumes no external labeling of the paradox inference, but achieves self-reference instead by means of its internal Gödel coding and executed by means of Gödel function sub_g, to recover an externally unlabeled form of the soundness paradox.[3]

Ramsey Reduction

$\forall\varphi[T\varphi \leftrightarrow \varphi]$

Validity

$\forall x[Vx \leftrightarrow \forall y_1 \ldots \forall y_n \forall z_1 \ldots \forall z_n[[Ay_1x \wedge \cdots \wedge Ay_nx \wedge$
$\neg\exists wAwx \wedge w \neq x_1 \wedge \cdots \wedge w \neq x_n \wedge Cz_1x \wedge \cdots \wedge Cz_nx \wedge \neg\exists wCwx \wedge w \neq y_1$
$\wedge \cdots \wedge w \neq y_n] \rightarrow \Box[Ty_1 \wedge \cdots \wedge Ty_n \rightarrow Tz_1 \wedge \cdots \wedge Tz_n]]]$

[3]The Gödel number of the argument is determined by assigning natural numbers to each syntax item in the expression to be arithmetized, each of which is then made the exponent of a corresponding prime number base taken in sequence in the same order of increasing magnitude as the syntax (standardly left-to-right) in the expression to be coded. The Gödel number of the expression is the product of these primes raised to the powers of the corresponding syntax item code numbers.

$\bar{S}$	[	sub_g	(	_	)	]	$\vdash$	$\bar{S}$	[	sub_g	(	_	)	]
|	|	|	|	|	|	|	|	|	|	|	|	|	|	|
1	2	3	4	5	6	7	8	1	2	3	4	5	6	7

The Gödel number of the soundness paradox argument on this assignment of Gödel numbers to syntax items in the formula is: $2^1 \times 3^2 \times 5^3 \times 7^4 \times 11^5 \times 13^6 \times 17^7 \times 19^8 \times 23^1 \times 29^2 \times 31^3 \times 37^4 \times 41^5 \times 43^6 \times 47^7 = n$. This enormous number is substituted for blank spaces (alternatively, free variables) to which the number 5 is here assigned in the open sentence above to complete the Gödel arithmetization in $g\ulcorner\bar{S}[\mathrm{sub}_g(n)] \vdash \bar{S}[\mathrm{sub}_g(n)]\urcorner = n$, where by stipulation, $\mathrm{sub}_g(n) = \ulcorner\bar{S}[\mathrm{sub}_g(n)] \vdash \bar{S}[\mathrm{sub}_g(n)]\urcorner$. The Fundamental Theorem of Arithmetic guarantees that every number can be decomposed into a unique factorization of prime number bases raised to particular natural number powers. When this is done to n and the factors put in ascending order, the expression mapped into Gödel-numbered space can be read directly from the exponents of each prime, and translated back into logical syntax by the glossary of natural number assignments.

Soundness

$\forall x[Sx \leftrightarrow [Vx \wedge \forall y[Ayx \rightarrow Ty]]]$

Inference

$\forall x[Ix \leftrightarrow \exists y \exists z[Ayz \wedge Czx]]$

Excluded Middle for Soundness of Inferences

$\forall x[Ix \leftrightarrow [Sx \vee \bar{S}x]]$

Gödelization of Soundness Paradox

(GS) $g\ulcorner \bar{S}[\mathrm{sub}_g(n)] \vdash \bar{S}[\mathrm{sub}_g(n)]\urcorner = n \wedge \mathrm{sub}_g(n) = \ulcorner \bar{S}[\mathrm{sub}_g(n)] \vdash \bar{S}[\mathrm{sub}_g(n)]\urcorner$
$\wedge [S[\mathrm{sub}_g(n)] \leftrightarrow \bar{S}[\mathrm{sub}_g(n)]]$

PROOF.

(1)	$g\ulcorner \bar{S}[\mathrm{sub}_g(n)] \vdash \bar{S}[\mathrm{sub}_g(n)]\urcorner = n \wedge$ $\mathrm{sub}_g(n) = \ulcorner \bar{S}[\mathrm{sub}_g(n)] \vdash \bar{S}[\mathrm{sub}_g(n)]\urcorner$	*Gödelization of Soundness Paradox*
(2)	$V[\mathrm{sub}_g(n)]$	*Validity*
(3)	$S[\mathrm{sub}_g(n)] \rightarrow T[\bar{S}[\mathrm{sub}_g(n)]]$	(1 *Soundness*)
(4)	$T[\bar{S}[\mathrm{sub}_g(n)]] \rightarrow \bar{S}[\mathrm{sub}_g(n)]$	(3 *Ramsey Reduction*)
(5)	$S[\mathrm{sub}_g(n)] \rightarrow \bar{S}[\mathrm{sub}_g(n)]$	(3–4)
(6)	$\bar{S}[\mathrm{sub}_g(n)] \rightarrow \bar{T}[\bar{S}[\mathrm{sub}_g(n)]]$	(1,2 *Soundness*)
(7)	$\bar{T}[\bar{S}[\mathrm{sub}_g(n)]] \rightarrow S[\mathrm{sub}_g(n)]$	(6 Bivalence of *Truth*)
(8)	$\bar{S}[\mathrm{sub}_g(n)] \rightarrow S[\mathrm{sub}_g(n)]$	(6–7)
(9)	$S[\mathrm{sub}_g(n)] \leftrightarrow \bar{S}[\mathrm{sub}_g(n)]$	(5,8)

■

The dilemma to prove the second conjunct formally in the Gödelization adheres closely to the informal reasoning. The dilemma horns are justified by *Excluded Middle for Soundness of Inferences*. We assume that the inference is sound, and then that it is unsound, in order to derive the two parts of the biconditional. The first conjunct in each formula assigns a Gödel number to the soundness paradox inference, and the derivation of the paradox is achieved directly by applying the definitions of *Ramsey Reduction and Soundness*.

3 Soundness, Validity, and the Liar

The soundness paradox is evidently related to the liar and to the so-called validity or Pseudo-Scotus paradox.[4] When the liar and the validity paradox are informally

[4] The validity paradox is also known as the Pseudo-Scotus paradox. For a detailed scholarly comparison of Pseudo-Scotus' theory of *consequentiae* with contemporary symbolic logic, see A. Charlene Senape McDermott, "Notes on the Assertoric and Modal Propositional Logic of the Pseudo-Scotus", *Journal of the History of Philosophy*, 10, 1972, pp. 273–306. Pseudo-Scotus, *In Librum Primum Priorum Analyticorum Aristotelis Quaestiones*, Question 10, Duns Scotus, *Ioannis Duns Scoti Opera Omnia*, edited by Luke Wadding [1639] (Paris: Vives, 1891–1895), Vol. II, p. 104. Translations of relevant passages are given in McDermott, pp. 288–291. See also G.B. Keene, "Self-Referent Inference and the Liar Paradox", *Mind*, 92, 1983, pp. 430–433. Benson Mates, "Pseudo-Scotus on the

considered, it is easy to see at least their superficial grammatical differences from the soundness paradox. The liar sentence says, in effect:

(L) Sentence (L) is false.

The validity paradox in its most streamlined impredicative formulation states:

(V) 1. Argument (V) is deductively valid.

---

2. Argument (V) is deductively invalid.

What is striking about these paradoxes in comparison with the soundness paradox is that the soundness paradox is not as obviously diagonal. Liar sentence (L) categorically declares its own falsehood. Soundness paradox (S), by contrast, says only conditionally that if it is assumed to be unsound, then it follows as a conclusion that it is unsound, which at first seems logically unproblematic. In validity paradox (V), the conclusion denies what the assumption asserts, whereas the conclusion of (S) merely repeats its assumption. The soundness paradox, despite its inferential form, is unlike the validity paradox, because the soundness paradox does not try to deduce the denial of a semantic predication from the predication itself. It is not possible to reformulate the liar sentence in inferential form modeled directly on the validity paradox. We can try to do so in only two ways, seeking to combine the liar and validity paradoxes so as to produce a version of the soundness paradox. An attempt to construct an inferential version of the liar paradox might take these forms:

(P) 1. Argument (P) is sound.

---

2. Argument (P) is unsound.

(Q) 1. Argument (Q) is unsound.

---

2. Argument (Q) is sound.

There is evidently no paradox in either case. Neither inference, for different reasons, is deductively valid, hence neither is sound. The soundness paradox gets its juice from the fact that it is only when we consider the soundness or unsoundness of argument (S) as an inference that is formally deductively valid that we run into contradiction. Then we find that an argument that declares its own unsoundness in assumption and conclusion proves to be sound if and only if it is unsound. Otherwise, as arguments (P) and (Q) indicate, there is nothing antithetical about an inference containing an

Soundness of *Consequentiae*", *Contributions to Logic and Methodology in Honor of J.M. Bocheński*, edited by A.T. Tymieniecka (Amsterdam: North-Holland Publishing Company, 1965), especially pp. 139–140. J.M. Bocheński, "De Consequentiis Scholasticorum Earumque Origine", *Angelicum*, 15, 1938, pp. 92–109, and "Notes Historiques sur les Propositiones Modales", *Revue des Sciences Philosophiques et Théologiques*, 26, 1937, pp. 673–699. Johannes Bendiek, "Die Lehre von den Konsequenzen bei Pseudo-Scotus", *Franziskanische Studien*, 34, 1952, pp. 205–234. Stephen Read, "Self-Reference and Validity", *Synthese*, 42, 1979, pp. 265–274. Roy A. Sorensen, *Blindspots* (Oxford: The Clarendon Press, 1988), pp. 301–303. Dale Jacquette, "The Validity Paradox in Modal S_5", *Synthese*, 109, 1996, pp. 47–62.

assumption or conclusion that declares the unsoundness of the inference to which it belongs.

The liar paradox, as we have seen, is reducible to the soundness paradox, but the soundness paradox is not reducible to the liar. A *reduction* of a paradox is the truth-preserving logical transformation of one paradox into another. The mode of paradox reduction is the deployment of a preferred logic to validly deduce the assumptions and conclusions of a target paradox P2 from the assumptions and conclusions of a given paradox P1. A *relevant reduction* of a paradox that is not merely based on the fact that in a classical validity semantics any proposition whatsoever is validly deducible from any contradiction, whereby any paradox is logically reducible to any other. A relevant reduction involves a connection of the concepts by which each of the paradoxes is defined, thereby revealing something interesting about the content of the ideas each of the paradoxes expresses. As such, paradox reduction is a special case of a general method for logically reducing any inference to another by validly deducing the assumptions and conclusions of one to the other. Consider the following equivalence principle:

(E) For any proposition φ, φ is true if and only if the inference $\varphi \vdash \varphi$ is sound.

The equivalence principle is unattractive for several reasons. It is blatantly circular if soundness is defined as above in terms of truth. In the present context, it may also be problematic to define truth in terms of soundness, because the ordinary concept of soundness is potentially jeopardized by the paradox. If these reservations are set aside, however, there is a relevant reduction of the liar to the soundness paradox effected by equivalence principle (E).

An inferential reformulation of the liar paradox can be given as a substitution instance of equivalence (E), which provides a relevant (and, incidentally, nonimpredicative) reduction of the liar to the soundness paradox, where (L*) is sound if and only if it is unsound:

(L*) 1. Sentence (L) is false.

---

2. Sentence (L) is false.

To reduce the soundness paradox to the liar via equivalence principle (E), we require that argument (S) is sound if and only if the assumption (or conclusion) of (S) is true.[5] The assumption of argument (S), call it (S*), states that:

(S*) Argument (S) is unsound.

[5] We can equally construe the soundness paradox as an inferential version of the liar without supposing that the soundness paradox is reducible to the liar by the following transformtion. Begin with the inference:

(S) 1. Argument (S) is unsound.

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2. Argument (S) is unsound.

According to the definition of soundness, this is equivalent to:

This sentence, like any other as we now classically suppose, is either true or false. We must ask whether (S*) is true if and only if it is false, in order to reduce the soundness paradox to the liar. If sentence (S*) is true, then argument (S) is unsound. But since (S) has a deductively valid structure, it can only be unsound if at least one of its assumptions, in this case only (S*), is false. So, if sentence (S*) is true, then (S*) is false. If (S*) is false, then argument (S) is sound. But a sound argument has only true assumptions, from which it follows that sentence (S*) as the only assumption of argument (S) is true. Thus, sentence (S*) is true if and only if (S*) is false — but not merely because in classical logic any proposition whatsoever is validly deducible from a contradiction. As a result, (S*) represents a single sentence equivalent of the liar paradox to which the soundness paradox can be relevantly reduced. The reduction may seem convincing, but it camouflages an essential distinction between paradoxes like the liar that can be written as individual sentences and paradoxes like the soundness paradox that are irreducibly inferential.

A relevant reduction of the soundness paradox to the liar must be formulated as a single sentence without internal inferential structure that is true if and only if it is false. There is no particular problem in using equivalence (E) to expand the liar sentence into an inferential equivalent that is sound if and only if it is unsound. The liar, as we have seen, by applying (E), can be relevantly reduced to the soundness paradox. To carry the reduction in the opposite direction, however, would require condensing the soundness paradox inference into a single paradoxical sentence, which turns out to be impossible. It may appear that sentence (S*) obtained by applying equivalence (E) to the soundness paradox (S) is exactly what is needed. But sentence (S*) is true if and only if it is false only by virtue of referring to the uncondensed inferential form of inference (S). The best we can do in that case is to reduce the soundness paradox to a single sentence of the form:

(S**) The argument whose assumption is (S*) and whose conclusion is (S*) is unsound.

But (S**) is not a reduction of the soundness paradox to the liar, because (S**) has an internal inferential structure that refers to an argument's assumption and conclusion.

There is a difference in the nature of the self-reference and self-non-attribution of semantic properties in the soundness paradox by contrast with the liar. The way

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(S') 1. Argument (S') is deductively invalid OR argument (S') has a false assumption.

---

2. Argument (S') is deductively invalid OR argument (S') has a false assumption.

Then, since (S') is evidently deductively valid, the inference in effect reduces to:

(S") 1. Argument (S") has a false assumption.

---

2. Argument (S") has a false assumption.

This version of the soundness paradox remains an impredicative variation of (L*), and for the same reasons is not further reducible from inferential form to the single sentence form required of the liar paradox.

in which diagonalization is achieved in the two paradoxes, despite superficial similarities, is inherently different, marking a significant disanalogy between the liar and the soundness paradox. We can appreciate the distinct modes of self-reference in the two paradoxes by considering their intuitive indexical formulations. When we expand the liar sentence (L) via equivalence (E) in the indexical inferential version (iL*), we obtain:

(iL*) 1. This sentence is false.

———————

2. This sentence is false

Here the question, '*Which* sentence is supposed to be false?', has a correct answer, intrinsically contained in both the assumption and conclusion of the inferentially expanded liar paradox. It is the very sentence (type or token in alternative interpretations of the inferential version of the liar) that indexically declares itself to be false — first in the assumption, and secondly in the conclusion. If we try to reduce the soundness paradox as above, by condensing it into a single liar type sentence in the indexical sentential version (iS*), we arrive only at something that does not express a self-contained paradox: '(iS*) This argument is unsound.' If we similarly ask, '*Which* argument is supposed to be unsound?', there is no correct answer intrinsically contained within (iS*). As a single sentence condensation of the soundness paradox, (iS*) by itself does not make sense and has no truth value. We can only recover the soundness paradox from (iS*) by referring back to the original inferential expression of (S), in order to know which argument is supposed to declare itself unsound.

The difference is reinforced when we go beyond eliminable indexical formulations to consider the proposed reduction of the soundness paradox to the liar in Gödel notation. The liar sentence is Gödelized as: $g\ulcorner\bar{T}[\mathrm{sub}_g(m)]\urcorner = m \wedge \mathrm{sub}_g(m) = \ulcorner\bar{T}[\mathrm{sub}_g(m)]\urcorner$. Then the Gödelized liar paradox states: $T[\mathrm{sub}_g(m)]\leftrightarrow\bar{T}[\mathrm{sub}_g(m)]$. The reduction of the liar to the soundness paradox is preserved when we reformulate it by expanding the liar sentence into an inference by equivalence (E). We distinguish between the liar sentence $\mathrm{sub}_g(m)$ and the Gödel number n of the inferential expansion of the liar paradox in $\mathrm{sub}_g(n)$, and we presuppose the following combined definition:

Propositionhood and Excluded Middle for Truth of Propositions

$\forall\varphi[T\varphi \vee \bar{T}\varphi]$

We first prove a Gödelized version of the liar paradox, and then show how to reduce the liar to the soundness paradox.

PROOF.

(1)	$g\ulcorner\bar{T}[\mathrm{sub}_g(n)] = m \wedge \mathrm{sub}_g(m) = \ulcorner\bar{T}[\mathrm{sub}_g(m)]\urcorner$	*Liar Gödelization*
(2)	$T[\mathrm{sub}_g(m)]\rightarrow T[\bar{T}[\mathrm{sub}_g(m)]]$	(1)
(3)	$T[\bar{T}[\mathrm{sub}_g(m)]]\rightarrow\bar{T}[\mathrm{sub}_g(m)]$	(2 *Ramsey Reduction*)
(4)	$T[\mathrm{sub}_g(m)]\rightarrow\bar{T}[\mathrm{sub}_g(m)]$	(2–3)

(5)	$\bar{T}[\text{sub}_g(m)] \rightarrow \bar{T}[T[\text{sub}_g(m)]]$	(1)
(6)	$\bar{T}[\bar{T}[\text{sub}_g(m)]] \rightarrow \bar{T}[\text{sub}_g(m)]$	(5 Bivalence of *Truth*)
(7)	$\bar{T}[\text{sub}_g(m)] \rightarrow T[\text{sub}_g(m)]$	(5–6)
(8)	$T[\text{sub}_g(m)] \leftrightarrow \bar{T}[\text{sub}_g(m)]$	(4,7)

∎

We then have the following relevant reduction of the sentential liar paradox to the inferential soundness paradox:

PROOF.

(1)	$g\ulcorner \bar{T}[\text{sub}_g(n)] = m \wedge \text{sub}_g(m) = \ulcorner \bar{T}[\text{sub}_g(m)]\urcorner$	*Liar Gödelization*
(2)	$g\ulcorner \bar{T}[\text{sub}_g(m)] \vdash \bar{T}[\text{sub}_g(m)]\urcorner = n \wedge \text{sub}_g(n) = \ulcorner \bar{T}[\text{sub}_g(m)] \vdash \bar{T}[\text{sub}_g(m)]\urcorner$	*Gödelization of Soundness Paradox*
(3)	$T[\text{sub}_g(m)] \leftrightarrow \bar{T}[\text{sub}_g(m)]$	*Liar*
(4)	$V[\text{sub}_g(n)]$	(2 *Validity*)
(5)	$S[\text{sub}_g(n)] \rightarrow T[T[\text{sub}_g(m)]$	(1,2 *Soundness*)
(6)	$S[\text{sub}_g(m)] \rightarrow \bar{T}[\text{sub}_g(m)]$	(3,5)
(7)	$\bar{T}[\text{sub}_g(m)] \rightarrow \bar{S}[\text{sub}_g(n)]$	(1,2 *Soundness*)
(8)	$S[\text{sub}_g(n)] \rightarrow \bar{S}[\text{sub}_g(n)]$	(3–7)
(9)	$\bar{S}[\text{sub}_g(n)] \rightarrow \bar{T}[\text{sub}_g(m)]$	(1,2 *Soundness*)
(10)	$\bar{S}[\text{sub}_g(n)] \rightarrow T[\text{sub}_g(m)]$	(3,9)
(11)	$T[\text{sub}_g(m)] \rightarrow T[\bar{T}[\text{sub}_g(m)]]$	(1,3 *Ramsey Reduction*)
(12)	$T[\bar{T}[\text{sub}_g(m)]] \rightarrow S[\text{sub}_g(n)]$	(1,2,4 *Soundness*)
(13)	$\bar{S}[\text{sub}_g(n)] \rightarrow S[\text{sub}_g(n)]$	(9–12)
(14)	$S[\text{sub}_g(n)] \leftrightarrow \bar{S}[\text{sub}_g(n)]$	(8,13)

∎

The paradox so derived remains intelligible as a diagonalization because a single noninferential sentence is true or false, and because there is a definite relation between the soundness of an inference containing a single noninferential sentence as assumption or conclusion and the sentence's truth value. But since the counterpart *Excluded Middle for Soundness of Inferences* does not apply to individual sentences lacking internal inferential structure (which may be true or false, but by themselves are neither sound nor unsound), there is no relevant reduction of the soundness paradox to an equivalent liar sentence formulation.

Although we can Gödelize sentence (S*) merely as a syntactical exercise in $g\ulcorner \bar{S}[\text{sub}_g(n)]\urcorner = n$, there is no valid inference from counterpart dilemma assumptions $\bar{S}[\text{sub}_g(n)]$ or $S[\text{sub}_g(n)]$. The reason is that, where $\text{sub}_g(n) = \ulcorner \bar{S}[\text{sub}_g(n)]\urcorner$, $\bar{S}[\text{sub}_g(n)]$ by *Excluded Middle for Soundness of Propositions* is undefined and altogether lacking in truth value. All this is a formally fancy way of saying that while we can get truth out of soundness, we cannot get soundness out of truth alone. We can discover the properties of individual sentences by themselves or as they occur in expanded inferences. But we cannot discover the properties of inferences in individual sentences, when by themselves they do not even implicitly colloquially express inferential relations between the assumptions and conclusion of an argument. Principle (E) implies that if an inference of the form $\varphi \vdash \varphi$ is sound, then proposition φ is true, and conversely. But it does not require that proposition φ be the liar sentence. If φ is stipulated to be the liar sentence, then the inference $\varphi \vdash \varphi$ is not the soundness paradox, but is instead argument (L*), which states that: 'Sentence (L) is false $\vdash$ Sentence (L) is false.' We cannot generally infer φ from the deductive validity of $\varphi \vdash \varphi$, unless φ is true — which is a sticking point, to say the least, when φ is the liar sentence. Despite the interdefinability of truth and soundness, we can relevantly reduce the liar paradox to the soundness paradox, but not the soundness paradox to the liar. The soundness paradox on the basis of this logical distinction is evidently a different and arguably more fundamental paradox than the liar.

There are further implications of the reduction disparity for how the soundness paradox might be solved by contrast with standard solutions to the liar. As an indication of what is at stake in the irreducibility of the soundness paradox to the liar, consider how the shift from sentential to inferential paradoxes complicates Tarski's approach to forestalling the liar paradox for the concept of truth in a hierarchy of formal object- and metalanguages.[6] Although it seems useful in avoiding the liar to hold that no sentence can assert its own truth or falsehood, the corresponding prohibition for inferences does not appear as unproblematic. Since validity is not at issue in the soundness paradox, we can focus on the question of whether we can invoke a Tarskian restriction on inferences affirming or denying the truth of their own assumptions and conclusions. The difficulty is that in a way every inference asserts (and does not merely imply) the truth of its own assumptions and conclusions, while the conclusions of an *argumentum reductio ad absurdum*, on the strength of a validly derived contradiction, asserts (and does not merely imply) the falsehood of one of its assumptions. As a consequence, it is by no means obvious that the Tarskian solution to the sentential liar paradox can be successfully applied to the inferential soundness paradox.[7]

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[6] See Alfred Tarski, "The Concept of Truth in Formalized Languages", *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*, translated by J.H. Woodger (Oxford: The Clarendon Press, 1956), pp. 152–278. Jon Barwise and John Etchemendy, *The Liar: An Essay on Truth and Circularity* (New York: Oxford University Press, 1987), pp. 692. Keith Simmons, "The Diagonal Argument and the Liar", *Journal of Philosophical Logic*, 19, 1990, especially pp. 277–278, 289–292. Simmons, *Universality and the Liar: An Essay on Truth and the Diagonal Argument* (Cambridge: Cambridge University Press, 1993), pp. 10–16. Tarski's method is criticized by Saul A. Kripke, "Outline of a Theory of Truth", *The Journal of Philosophy*, 72, 1975, pp. 690–716. Despite misgivings about Tarski's solution (especially p. 692), Kripke proposes a ramified Tarskian semantic hierarchy of object and metalanguages in order to avoid the liar paradox in a generalized theory of truth. Other objections to Tarski's theory of truth independent of its solution to the liar paradox are discussed by Jonathan Harrison, "The Trouble with Tarski", *The Philosophical Quarterly*, 48, 1998, pp. 1–22.

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QMML: Quantified Minimal Modal Logic and Its Applications

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Abstract

Although first-order Kripke semantics has become a well established branch of modal logic, very little - almost nothing - is written about logics with a weaker modal fragment. We try to help the situation by isolating principles determining the interaction between quantifiers and modalities in minimal semantics. First, we let the standard-model properties of monotonic and anti-monotonic domains clue us in on how to do this – i. e. we try to articulate, in terms of the inclusiveness of the domains of a certain set of worlds, a set of semantical restrictions that will validate the Barcan and converse Barcan formulae respectively. As it turns out, this can indeed by done, but only by adding assumptions strong enough to make the models virtually normal. Since the whole point of switching to a minimal framework would be to generalise the logic, we therefore abandon the worlds-objects thinking altogether, and switch to a much simpler and more direct validation strategy in which the propositions we are after are simply picked out as such.

Keywords: First-order modal logic, minimal semantics, permutation principles.

1 Introduction

It was a common feeling among our philosophical predecessors that there is precisely one correct system of logic, and that this system can be used to impose clarity and rigour on all significant contexts of thought. Kant, for instance says:

> Since Aristotle [logic] has not required to retrace a single step, unless, indeed, we care to count as improvements the removal of certain needless subtleties or the clearer exposition of its recognised teaching, features which concern the elegance rather than the certainty of science. It is remarkable also that to the present day logic has not been able to advance a single step, and is thus to all appearances a closed and completed body of doctrine.

Not many logicians today share Kant's view. There is presently a general consensus about the reasonableness of a multiplicity of logical systems. It is being recognised that which logic is the appropriate one for a certain task is to a large extent determined by the nature of that task. The discipline of logic has thus assumed a certain character of *design.* Logic is no longer seen first and foremost as the activity of digging up deeply veiled, but, once uncovered, self-evident truths, but rather as the enterprise of devising formal systems to meet specific needs or to capture essentially problematic views in a clear form. Logic construed as such – i. e. as *conceptual engineering* – has benefited enormously from the development of ever more rudimentary and general languages, languages from which the logic best cut to the relevant measure can be built, condition by condition.

For some reason, this process seems to have come to a temporary halt in the field of quantified modal logic. This is surprising, considering the virtually undisputed usefulness – for purposes of conceptual analysis and design – of first-order intensional languages. The sceptic reader may want to consider the following examples,

Sustenance: $A \wedge E_a \neg(\exists x) E_x \neg A \wedge ((\exists y) E_y \neg A \vee A)$

Distribution of responsibility: $\mathcal{O}\neg(\exists x) E_x A \rightarrow (\forall y)\mathcal{O} E_y \neg E_y A$

The E_x notation denotes indexed agency operators usually given the reading "agent x sees to it that ...", whereas $\mathcal{O}$ is an ideality operator of the kind familiar from deontic logic. Thus, the first of the concepts above consists of three conjuncts specifying a context, an action and a counter-factual condition respectively. The action is of type prevention, and the counter-factual condition says of it that if it did not occur then the context would have been different. Hence, the concept belongs to the *sustenance* variety. The second concept, assuming the form of a conditional, says that if a certain action A is in general forbidden then everyone is obliged to refrain from performing that action. Hence, the concept may be seen as a principle linking a set of individual duties to a general prohibition.

Note, that each of the concepts above involves an essential reference to a quantifier. The concept of sustenance is predicated on the *denial of the existence* of a certain kind of agency – expressible only as the external negation of an existential quantifier. The distribution of responsibility-principle, on the other hand, relies on the distinction between *de re* and *de dicto* necessity – it is due to this distinction that the formula is able to capture a relationship between a *general* prohibition and a set of *individual* duties.

According to current research in modal logic, both of the modalities above – obligatoriness and agency – are *non-normal* in the sense of not being closed under logical consequence. [1] Such modalities cannot be adequately described in standard semantics, but require a move to the more flexible framework of minimal models (familiar to some as neighbourhood semantics or Scott/Montague semantics). Alas, whereas the literature on first-order standard semantics is abundant, there is at present virtually no literature on the relation between worlds and objects in minimal models. [2]

Quantified modal logic, we know, is delimited and defined *syntactically* by a pair of principles expressing the possibilities of permutation between the quantifiers and the modal operators:

BF : $(\forall x)\Box\phi(x) \rightarrow \Box(\forall x)\phi(x)$

CBF : $\Box(\forall x)\phi(x) \rightarrow (\forall x)\Box\phi(x)$

These principles are usually referred to as, respectively, the Barcan and the converse Barcan formula (after Ruth Barcan Marcus), and have been extensively discussed in the philosophical literature. In a *standard model* setting, the permutation principles can be interpreted as making rather substantial philosophical claims (a good account

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[1] They are generally considered to be concepts of the EC variety. I. e. they are closed under classical equivalence and have a variant of the schema $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

[2] With a notable exception for [7] and [3]

may be found in [2]). This is due to the structuring they reflect of the standard model relation of accessibility along a scale of inclusiveness of domains: The Barcan formula thus corresponds to the following restriction on the relation R:

Monotonicity : if $\alpha R \beta$ then $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$,

whereas the converse Barcan formula corresponds to:

Anti-monotonicity : if $\alpha R \beta$ then $\mathcal{D}(\beta) \subseteq \mathcal{D}(\alpha)$,

To the extent we wish to utilise minimal models we should know whether the permutation principles mirror a similar – or, at least a vaguely analogous – ordering also minimally, and, if not, what other means we have of controlling the language. Hence, the central question of this paper becomes: What semantical restrictions must be imposed on a minimal model in order to validate the Barcan and converse Barcan formulae?

The paper is organised as follows: In section 2, we fix our notation, and introduce the basic semantical and syntactical notions. In section 3 we develop a validation strategy based, in analogy to the standard model case, on the contraction and expansion of certain domains. As we shall see, the resulting logics will turn out to be quite strong – so strong, in fact, as to verge on the brink of normality. This will lead us to consider, in section 4, the relationship between first-order minimal semantics and first-order standard semantics in a bit more detail. As a result we end up with a second validation strategy which is far more general than the first. In section 5 we prove the completeness of the systems discussed with regard to the latter classes of models.

2 Syntax and semantics essentials

2.1 Models, satisfaction and truth

The language takes as primitive a denumerably infinite set of n-place predicate letters $\mathbf{P}$, a denumerably infinite set of individual variables $\mathbf{Var}$, and the six symbols $\neg$, $\Box$, $\vee$ and $\forall$. Its syntax is defined from these as follows:

- if $P \in \mathbf{P}$, $x_1, x_2, \ldots, x_n \in \mathbf{Var}$, and P has arity n then $P(x_1, x_2, \ldots, x_n)$ is a wff.
- if ϕ is a wff then so are $\neg\phi$ and $\Box\phi$.
- if ϕ and ψ are wff then so is $(\phi \vee \psi)$.
- if $\phi(x)$ is a wff and $x \in \mathbf{Var}$, then $(\forall x)\phi(x)$ is a wff.

Note that there are no terms other than variables, and no identity sign in the language. Hence, the entire subject of rigid vs. non-rigid designation and necessary vs. contingent identity is cut away – the interaction between the quantifiers and the modal operators is this paper's only concern.

Our language is interpreted in models in which the domains are allowed to vary from world to world. Such a model $\mathcal{M} = \langle \mathcal{M}, \mathcal{N}, \mathcal{D}, \mathcal{V}, \mathbf{v} \rangle$ is specified as follows:

- $\mathcal{W}$ is a set of possible worlds.

- $\mathcal{N}$: $\mathcal{W} \to \mathcal{P}(\mathcal{P}(\mathcal{W}))$ is a function from worlds to propositions. Intuitively $\mathcal{N}$ picks out for each world α the set of necessary propositions at that world.
- $\mathcal{D}$ is a *domain function* assigning to each world $\alpha \in \mathcal{W}$ a set of objects $\mathcal{D}(\alpha)$.
- $\mathcal{V}$ is an interpretation function assigning to each n-place relation symbol ϕ, and to each possible world $\alpha \in \mathcal{W}$ a set of ordered n-tuples from $\bigcup\{\mathcal{D}(\alpha)\colon \alpha \in \mathcal{W}\}$ written $\mathcal{V}(\alpha, \phi)$.
- v is a set of valuation functions each of which assigns to each variable x an object from $\bigcup\{\mathcal{D}(\alpha)\colon \alpha \in \mathcal{W}\}$.

Generally we will assume familiarity with notions such as the scope of a quantifier and the notion of an x-variant [3] of a valuation of the free variables of a formula. The reader should keep in mind though, with regard to the *range* of the quantifiers, that varying domain QMML [4] – being varying domain – is a species of *free logic*. The characteristic feature of this particular brand of quantificational logic consists in the fact that terms need not refer to objects in the domain of quantification. The truth of a closed formula, therefore, requires, in addition to the truth of the relevant open sentences, the *existence* in the domain of reference of the objects designated by the terms of the open formulae:

Definition 2.1 Let w^i be any assignment from w^i be any assignment from $\bigcup\{\mathcal{D}(\alpha)\colon \alpha \in \mathcal{W}\}$ of values to the variables. Then:

A) $\mathcal{M} \vDash_{w^i} (\forall x)\phi(x)$ iff for any x-variant w^j of w^i s.t. $w^j(x) \in \mathcal{D}(\alpha) : \mathcal{M} \vDash_{w^j} \phi(x)$
B) $\mathcal{M} \vDash_{w^i} (\exists x)\phi(x)$ iff for some x-variant w^j of w^i s.t. $w^j(x) \in \mathcal{D}(\alpha) : \mathcal{M} \vDash_{w^j} \phi(x)$ [5]

In other words a closed sentence such as "some x is ϕ" is true at α precisely when the predicate ϕ is true of x and x exists at α.

Note, however, that since terms need not refer to objects in the domain of quantification, we must deal with a complication before we can define the truth in $\mathcal{M}$ of the atomic sentences on which the closure of a formula depends: A term can designate an existing as well as a non-existing object. Therefore, satisfaction and truth can be defined in several ways. The different possibilities are summarised below:

For satisfaction : We may decide that,

1. predicates are satisfied at a world α only by tuples from $\mathcal{D}(\alpha)$, or we may
2. allow predicates to be satisfied by any suitable tuple from $\bigcup\{\mathcal{D}(\alpha) : \alpha \in \mathcal{W}\}$.

Following the former course we would in effect deny that non-existent objects can have properties.[6] The latter course is less restrictive, and does not require the exis-

[3] Or, as it is often called, a bound alphabetic variant.

[4] which will henceforth be our official term for quantified *minimal* modal logic as opposed to QML which we use to refer to quantified modal logics of the normal variety.

[5] It is a well established result of first-order logic that if a closed sentence is true in a world with regard to a specific valuation, then it is true in that world with regard to *any* valuation. In what follows we will therefore drop the reference to a valuation when speaking about closed sentences

[6] Such a position has been fiercely defended by no lesser men than Leibniz and Kant. Thus Leibniz says "Nihile nullae prorietas sunt" – i.e. what is not has no properties – whereas Kant, speaking of non-existents says "Non-entis nulla sunt predicata"– i. e. all that is predicated of the object, whether affirmatively or negatively is erroneous (cf. [2]).

tence of the subject of predication. Intuitively speaking, therefore, it captures the view that, even though an object x might not exist in the domain associated with α it does exist under other circumstances we are willing to consider as possible, and so it is meaningful to speak about it [2]. Now,

For truth : We may either

1. choose the former of the above strategies making the concepts of truth and satisfaction coincide, or, we may, if we adopt the liberal view on satisfaction,
2. choose to regard satisfaction as the general case comprising every tuple that $\mathcal{V}$ returns from $\bigcup\{\mathcal{D}(\alpha)\colon \alpha \in \mathcal{W}\}$, while reserving the concept of truth for the special case in which the members of the satisfying tuple exist.

The choices made in the present paper is reflected by the fact that the interpretation function $\mathcal{V}$ is a totally defined function which ranges over the union of the individual domains. Hence, we allow existent as well as non-existent objects to have properties. As far as the concept of truth is concerned we shall make it synonymous with the concept of satisfaction, and we offer the following piece of reasoning (from [2]) as justification for this: Suppose that truth and satisfaction are distinct concepts and suppose that the formula $\Box(\phi(x) \vee \neg\phi(x))$ is true in some possible world α under some valuation v that assigns the object c to x. If $\Box$ is to be understood as a sentential operator from $\mathcal{W}$ to $\mathcal{P}(\mathcal{P}(\mathcal{W}))$ then for any world $\beta \in \|\phi(x) \vee \neg\phi(x)\|$ we should have the truth of $\phi(x) \vee \neg\phi(x)$. Now, $\mathcal{V}$ is not a partial function as we have defined it, so we know that ϕ is either satisfied or not satisfied by the object c at β. But we don't know that c exists at β, and since existence is assumed to be the difference between satisfaction and truth, we therefore don't know whether or not the $\phi(c) \vee \neg\phi(c)$ - *which is a tautology of PL* - is true at β. As we wish to exclude these cases, and to keep our proof procedures nice and simple, we shall treat satisfaction and truth as equivalent concepts. Thus, ϕ is either true or false of c at β, and in any event $\phi(x) \vee \neg\phi(x)$ comes out true. In a more compressed idiom:

Definition 2.2 $\mathcal{M}, \alpha \vDash_{w^i} \phi(x_1, ..., x_n)$ iff $\langle w^i(x_1), ..., w^i(x_n)\rangle \in \mathcal{V}(\alpha, \phi)$

2.2 Neologisms

Minimal semantics is not a relational semantics and hence the language in which it is formulated is not predicate logic but set-theory. In other words semantical restrictions in QMML will be formulated in terms of the membership of truth-sets in certain other sets, rather than in terms of accessibility relations between possible worlds. The problem is, however, that an atomic formula $\phi(x)$ of first-order logic is not really a proposition it is a pronominal construction for which no grammatical antecedent is expressed. Therefore it makes no sense to talk about the truth simpliciter, $\vDash \phi(x)$, of $\phi(x)$, nor does it make sense to talk about the truth-set, $\|\phi(x)\|$, of an open formula as such. Instead we must state the truth of $\phi(x)$ with reference to a specific assignment w^i of an object to the variable x, and, analogously, we must talk about the set of worlds $\|\phi(x)\|_{w^i}$ in which ϕ is true of the object assigned to x by this valuation. Hence,

Definition 2.3 $\mathcal{M}, \alpha \vDash_{w^i} \phi(x)$ iff $w^i(x) \in \mathcal{V}(\alpha, \phi)$

and

Definition 2.4 $\|\phi(x)\|_{w^i} =_{df} \{\alpha \in \text{W}: \mathcal{M}, \alpha \vDash_{w^i} \phi(\text{x})\}$

Now, among the truth-sets we will be particularly interested in – since this is what is required to have the universal closure of a formula – are those in which a predicate is satisfied by every object from some domain. To denote such a set we shall use the notation $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$. This set will be called the α-closure set for $\phi(x)$, and it is defined as follows:

Definition 2.5 $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i} =_{df} \|\phi(x)\|_{w^m} \cap, \ldots, \cap \|\phi(x)\|_{w^n}$ where $\{w^m(x), \ldots, w^n(x)\}$ $= \mathcal{D}(\alpha)$

Expressed in terms of truth at a world the set $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$ is equal to the set $\{\beta \in \mathcal{W}: \mathcal{M}, \beta \vDash_{\{w^i \in \mathbf{v}: w^i(x) \in \mathcal{D}(\alpha)\}} \phi(x)\}$, where $\mathcal{M}, \alpha \vDash_{\{w^i \in \mathbf{v}: w^i(x) \in \mathcal{D}(\alpha)\}} \phi(x)$ is defined as:

Definition 2.6 $\mathcal{M}, \alpha \vDash_{\{w^i \in \mathbf{v}: w^i(x) \in \mathcal{D}(\alpha)\}} \phi(x) =_{df} \mathcal{M}, \alpha \vDash_{w^m} \phi(x) \,\&, \ldots, \& \,\mathcal{M}, \alpha \vDash_{w^n}$ $\phi(x)$ where $\{w^m(x),$
$\ldots, w^n(x)\} = \mathcal{D}(\alpha)$

Intuitively speaking, therefore, the α-closure set for $\phi(x)$ is the set of worlds where a predicate ϕ is true of all objects from the domain of α.

Definition 2.6 is, quite obviously, a link between the notion of the truth of a closed formula and the notion of the truth of a relevant set of open sentences, in the sense that it allows us to derive the following lemma (the proof of which is trivial and therefore omitted):

Lemma 2.7 $\mathcal{M}, \alpha \vDash (\forall x)\phi(x)$ iff $\mathcal{M}, \alpha \vDash_{\{w^i \in \mathbf{v}: w^i(x) \in \mathcal{D}(\alpha)\}} \phi(x)$

The lemma expresses exactly the same as definition 2.1, but in our more compressed set-theoretical idiom. Together with definition 2.5 it thus enables us to formulate the truth of a sentence $(\forall x)\phi(x)$ in a world α set-theoretically by saying that $\alpha \in \bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$. Since this equivalence will be appealed to in proofs we single it out as a separate lemma:

Lemma 2.8 $\alpha \in \|(\forall x)\phi(x)\|$ iff $\alpha \in \bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$

Proof. : Left-to-right: By the standard definition of a truth set $\alpha \in \|(\forall x)\phi(x)\|$ iff $\mathcal{M}, \alpha \vDash (\forall x)\phi(x)$. By definition 2.1, then, we have that $\mathcal{M}, \alpha \vDash_{w^i} \phi(x)$ for any w^i such that $w^i(x) \in \mathcal{D}(\alpha)$. It follows, by definition 2.4, that $\alpha \in \|\phi(x)\|_{w^m}, \ldots \alpha \in \|\phi(x)\|_{w^n}$ where $\{w^m(x), \ldots w^n(x)\} = \mathcal{D}(\alpha)$. By the laws of elementary set-theory, then, $\alpha \in \|\phi(x)\|_{w^m} \cap, \ldots, \cap \|\phi(x)\|_{w^n}$, which means, by definition 2.5, that $\alpha \in \bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$.
Right-to-left: By definition 2.5 $\alpha \in \bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$ iff $\alpha \in \|\phi(x)\|_{w^m} \cap, \ldots, \cap \|\phi(x)\|_{w^n}$, where $\{w^m(x), \ldots, w^n(x)\} = \mathcal{D}(\alpha)$. By the standard definition of a truth set $\mathcal{M}, \alpha \vDash_{w^m}$ $\phi(x) \,\&, \ldots, \& \,\mathcal{M}, \alpha \vDash_{w^n} \phi(x)$. By definition 2.6 it follows that $\mathcal{M}, \alpha \vDash_{\{w^i \in \mathbf{v}: w^i(x) \in \mathcal{D}(\alpha)\}}$ $\phi(x)$, and hence we have, by lemma 2.7, that $\mathcal{M}, \alpha \vDash (\forall x)\phi(x)$ ∎

Taken together definition 2.3 - 2.6 gives us the following very useful chain of equivalences:

Corollary 2.9 $\mathcal{M}, \alpha \vDash (\forall x)\phi(x)$ iff $\alpha \in \|(\forall x)\phi(x)\|$ iff $\alpha \in \bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i}$ iff $\mathcal{M}, \alpha \vDash_{\{w^i \in : w^i(x) \in \mathcal{D}(\alpha)\}} \phi(x)$

Although they bring with them nothing novel, these definitions thus facilitate simplicity and compactness of expression – virtues the value of which will shortly become apparent.

Now, as QMML is about the interaction between quantifiers and modalities we need to relate the neologisms above to the notions of *de re* and *de dicto* necessity respectively. The following pair of lemmata take care of that.

Lemma 2.10 $\mathcal{M}, \alpha \vDash \Box(\forall x)\phi(x)$ iff $\{\beta : \beta \in \bigcap_{m \in \mathcal{D}(\beta)}^{n} \|\phi(x)\|_{w^i}\} \in \mathcal{N}(\alpha)$

The proof is immediate. It suffices to argue that $\{\beta : \beta \in \bigcap_{m \in \mathcal{D}(\beta)}^{n} \|\phi(x)\|_{w^i}\} = \|(\forall x)\phi(x)\|$ which is clearly implied by definition 2.5.

Lemma 2.11 $\mathcal{M}, \alpha \vDash (\forall x)\Box\phi(x)$ iff $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for any w^i such that $w^i(x) \in \mathcal{D}(\alpha)$

PROOF. We prove only the left-to-right direction: By definition 2.1 $\mathcal{M}, \alpha \vDash (\forall x)\Box\phi(x)$ iff $\mathcal{M}, \alpha \vDash_w \Box\phi(x)$ for any w^i s.t. $w^i(x) \in \mathcal{D}(\alpha)$. By definition 2.4, then, $\alpha \in \|\Box\phi(x)\|_{w^i}$. Now, from the definition of $\Box$ we have that $\alpha \in \|\Box\phi(x)\|_{w^i}$ if and only if $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ – which completes the argument. ∎

These last two results should come as no surprise to anyone. It is apparent, for instance, that what's stated by lemma 2.10 could equivalently be expressed by saying that $\mathcal{M}, \alpha \vDash \Box(\forall x)\phi(x)$ iff $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$. We shall use them interchangeably in what follows.

3 Validating the permutation principles

3.1 The general case: Varying domains

A standard model validates the Barcan formulae by contracting or expanding (as the case may be) the domains of a certain set of worlds, namely the set of worlds that is picked out by the necessity operator in the antecedent of the relevant Barcan formula. The domain of the world we start from either includes or is included in the domain of these worlds thus creating the effect of moving the necessity operator either into or out of a quantified formula. A minimal model does not have a relation predicate telling us which worlds are related to which others. Instead it has a function $\mathcal{N}$ picking out, for every world α, a set of necessary propositions. However, constraining $\mathcal{N}$ in an analogous way, imposing

[**Minimal Contraction**]: If $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$, for any w^i such that $w^i(x) \in \mathcal{D}(\alpha)$, then every β s.t. $\beta \in \|\phi(x)\|_{w^i}$ is such that $\mathcal{D}(\beta) \subseteq \mathcal{D}(\alpha)$

will not produce the same result in a minimal model. As it stands **Minimal Contraction** does not validate the Barcan formula. To see why, consider the following "dummy" model.

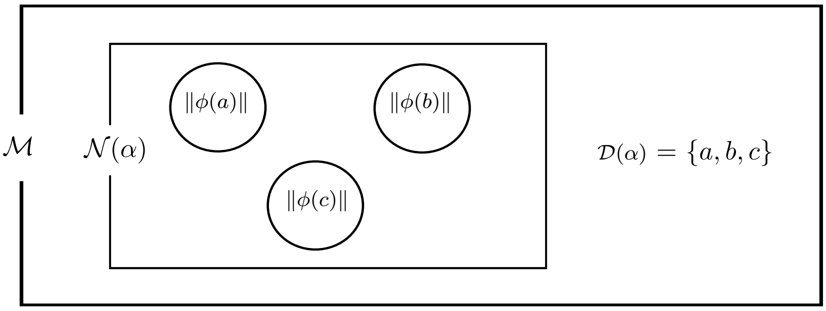

fig. 1

Since every atomic sentence formed from the predicate ϕ and an object from $\mathcal{D}(\alpha)$ is necessary at α itself, $(\forall x)\Box\phi(x)$ is true at α. However $\Box(\forall x)\phi(x)$ is not, since $(\forall x)\phi(x)$ is not in $\mathcal{N}(\alpha)$, so the Barcan formula fails.

This argument can obviously be made quite independently of whether or not each $\|\phi(x)\|_{w^i}$ [7] is minimally contracting. Minimal contraction simply does not come into play here, and may thus seem to be totally irrelevant to the truth of the Barcan formula.

However, this is not entirely true either. Minimal contraction can be brought into play by manipulating the following restrictions:

[m] if $X \cap Y$, then $X \in \mathcal{N}(\alpha)$ and $Y \in \mathcal{N}(\alpha)$
[c] if $X \in \mathcal{N}(\alpha)$ and $Y \in \mathcal{N}(\alpha)$, then $X \cap Y \in \mathcal{N}(\alpha)$
[n] $\mathcal{W} \in \mathcal{N}(\alpha)$

Following the terminology from [1] we shall say that a model satisfying **[m]** is *supplemented*, that a model satisfying **[c]** is *closed under intersections* and that a model satisfying **[n]** *contains the unit.* A model that is both supplemented and closed under intersections is a *quasi-filter*. If it also contains the unit it is a *filter*. Now, consider first the effect of intersection closure:

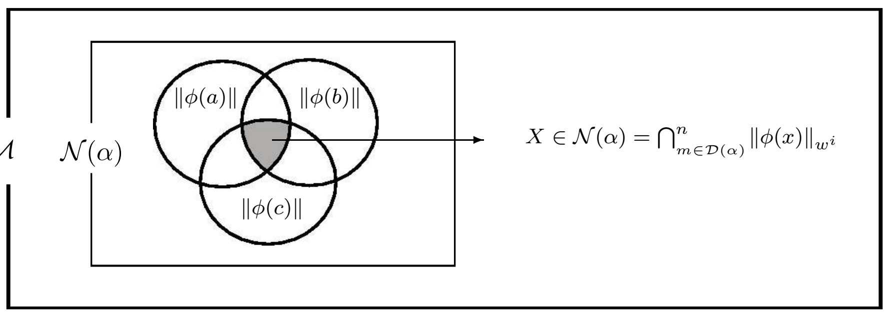

fig. 2

[7] As in this case, we shall omit the reference to the domain when this is clear from context

The effect is to bring an α-closure set into $\mathcal{N}(\alpha)$: The α-closure set for $\phi(x)$ is the intersection of all the atomic formulae formed from ϕ and the elements of $\mathcal{D}(\alpha)$. All the corresponding propositions are, by assumption, in $\mathcal{N}(\alpha)$. Condition **[c]** above says that the intersection of any finite collection of sets from $\mathcal{N}(\alpha)$ is itself in $\mathcal{N}(\alpha)$ – the α-closure set for $\phi(x)$ is obviously one of these.

Thus, minimal contraction does have an effect on intersection closures: If the domain of each member β of any $\|\phi(x)\|_{w^i}$ (such that $w^i(x) \in \mathcal{D}(\alpha)$) is included in the domain of α, and the α-closure set for $\phi(x)$ equals the intersection of all the $\|\phi(x)\|_{w^i}$, then the domain of each member γ of $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$ is obviously included in the domain of α too. The technical significance of this parallels, exactly, the significance of anti-monotonicity in the standard model case: If a predicate ϕ is true in a world γ of every object from the domain of α, and $\mathcal{D}(\gamma) \subseteq \mathcal{D}(\alpha)$ then it is also true of every object from the domain of γ itself – i. e. $\mathcal{M}, \gamma \vDash (\forall x)\phi(x)$. Now, since in this case γ is any member of $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$ it follows that $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i} \subseteq \|(\forall x)\phi(x)\|$. These latter sets do not necessarily *coincide*, though, as the following elaboration on figure 2 shows:

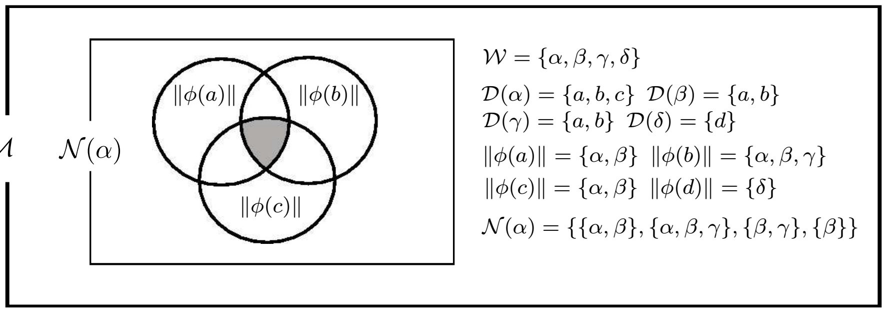

fig. 3

This model is closed under intersections and minimally contracting. Moreover, $(\forall x)\Box\phi(x)$ is true at α. However, $\Box(\forall x)\phi(x)$ is not since $\|(\forall x)\phi(x)\| = \{\alpha, \beta, \delta\}$, and this set is not in $\mathcal{N}(\alpha)$. The reason is this: Since $\mathcal{M}$ is varying domain there may be a world, e.g. δ, at which a predicate, e.g. ϕ, is true of every object, say d, existing at that world, without being true of any object existing at any other world – for instance α. Thus, δ will be in $\|(\forall x)\phi(x)\|$ but not in $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$.

This is easily changed though: The joint pull of intersection closure and minimal contraction makes $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$ a subset of $\|(\forall x)\phi(x)\|$. Hence, to have the truth of the Barcan formula in $\mathcal{M}$ it will suffice to supplement the model. Indeed, supplementation, intersection closure and minimal contraction will drive the Barcan formula through in *any* model:

Theorem 3.1 The Barcan formula is valid in the class of minimally contracting quasi-filters.

Proof. Let the model $\mathcal{M}$ be a minimally contracting quasi-filter and assume that $\mathcal{M}, \alpha \vDash (\forall x)\Box\phi(x)$. By the definition of $\forall$ and $\Box$, we have that $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for any valuation w^i such that $w^i(x) \in \mathcal{D}(\alpha)$. Since $\mathcal{M}$ is closed under intersection it follows that $\bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$. Now, every $\beta \in \bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i}$ is such that $\mathcal{M}, \beta \vDash_{\{w^i \in \mathbf{v} : w^i(x) \in \mathcal{D}(\alpha)\}} \phi(x)$ (cf. corollary 2.9) and minimal contraction guarantees that $\mathcal{D}(\beta) \subseteq \mathcal{D}(\alpha)$. Hence, every world β such that $\beta \in \bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i}$ is such that $\mathcal{M}, \beta \vDash_{\{w^i \in \mathbf{v} : w^i(x) \in \mathcal{D}(\beta)\}} \phi(x)$ In other words; the α-closure set for $\phi(x)$ is also a β-closure set for $\phi(x)$. It follows by corollary 2.9 that $\mathcal{M}, \beta \vDash (\forall x)\phi(x)$. Consequently we have that $\bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i}$ coincides with $\bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i} \cap \|(\forall x)\phi(x)\|$, and since the former set is in $\mathcal{N}(\alpha)$ the latter set is quite obviously in $\mathcal{N}(\alpha)$ too. Now, $\mathcal{M}$ is, by assumption, supplemented. Thus, since $\bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i} \cap \|(\forall x)\phi(x)\|$ is in $\mathcal{N}(\alpha)$, then so is $\|(\forall x)\phi(x)\|$. It follows by the definition of $\Box$ that $\mathcal{M}, \alpha \vDash \Box(\forall x)\phi(x)$. ∎

Now for the converse Barcan formula: If we stick to the procedure outlined above, giving the elements of the necessitated proposition in the antecedent of the relevant Barcan formula the adequate domain property – which in this case is the property of being a superset of the domain of the world we start from – we obtain the following minimal analogue to the standard model monotonicity property:

[**Minimal Expansion**]: If $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$ then every $\beta \in \|(\forall x)\phi(x)\|$ is such that $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$

Too see how this principle effects our semantics recall that $\|(\forall x)\phi(x)\| = \{\beta \in \mathcal{W} : \beta \in \bigcap_{m \in \mathcal{D}(\beta)}^{n} \|\phi(x)\|_{w^i}\}$ – i.e. the universal closure of $\phi(x)$ is the set of worlds β that are members of their own closure sets for $\phi(x)$. Now, minimal expansion tells us that β is such that $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$. Hence, the β-closure set for $\phi(x)$ is also an α-closure set for $\phi(x)$. It follows from this that $\|(\forall x)\phi(x)\| \subseteq \bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i}$ and since the former is in $\mathcal{N}(\alpha)$ supplementation sees to it that the latter is in $\mathcal{N}(\alpha)$ too. The situation is illustrated below:

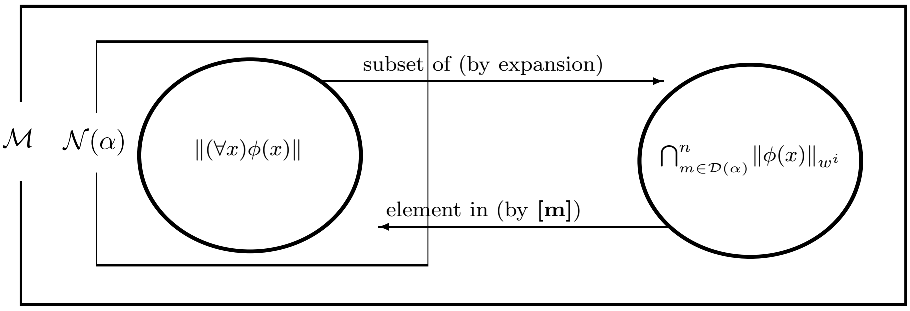

fig. 4

This does not bring us all the way, of course, since what we need to have (according to lemma 2.11 is the membership in $\mathcal{N}(\alpha)$, not of $\bigcap_{m \in \mathcal{D}(\alpha)}^{n} \|\phi(x)\|_{w^i}$, but of each of the atomic propositions $\|\phi(x)\|_{w^i}$ such that $w^i(x) \in \mathcal{D}(\alpha)$. We know, however, that the former set is formed by taking the intersection of all of the latter. If we wish to go in the other direction we may thus simply apply supplementation once more. Letting vision aid thought, what we do, then, is to move from figure 4 to figure 5 below.

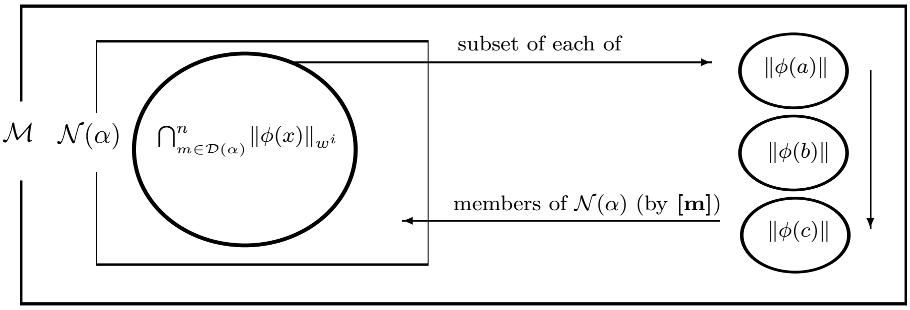

fig. 5

That this will indeed suffice is demonstrated in the proof of the following theorem:

Theorem 3.2 The converse Barcan formula is valid in the class of minimally expanding supplementations.

Proof. Let $\mathcal{M}$ be a minimally expanding supplementation and assume that $\mathcal{M}, \alpha \vDash \Box (\forall x)\phi(x)$. By the definition of $\Box$ it follows that $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$ which (by lemma 2.10) is equivalent to saying that $\{\beta \in \mathcal{W} : \beta \in \bigcap^n_{m\in\mathcal{D}(\beta)}\|\phi(x)\|_{w^i}\} \in \mathcal{N}(\alpha)$. Now, since every $\gamma \in \{\beta \in \mathcal{W} : \beta \in \bigcap^n_{m\in\mathcal{D}(\beta)}\|\phi(x)\|_{w^i}\}$ is such that $\mathcal{M}, \gamma \vDash_{\{w^i \in \mathbf{v} : w^i(x)\in\mathcal{D}(\gamma)\}} \phi(x)$, and since, by minimal expansion, $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\gamma)$, we have that $\mathcal{M}, \gamma \vDash_{\{w^i \in \mathbf{v}: w^i(x)\in\mathcal{D}(\alpha)\}} \phi(x)$. In other words; the γ-closure set for $\phi(x)$ is also an α-closure set for $\phi(x)$, which means that $\gamma \in \bigcap^n_{m\in\mathcal{D}(\alpha)}\|\phi(x)\|_{w^i}$.Consequently $\|(\forall x)\phi(x)\|$ coincides with $\bigcap^n_{m\in\mathcal{D}(\alpha)}\|\phi(x)\|_{w^i} \cap \|(\forall x)\phi(x)\|$, and since the former set is in $\mathcal{N}(\alpha)$ the latter set is in $\mathcal{N}(\alpha)$ too. It follows, by supplementation that, $\bigcap^n_{m\in\mathcal{D}(\alpha)}\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$. Now, $\bigcap^n_{m\in\mathcal{D}(\alpha)}\|\phi(x)\|_{w^i} =_{df} \|\phi(x)\|_{w^m} \cap, \ldots, \cap \|\phi(x)\|_{w^n}$ where $\{w^m(x), \ldots, w^n(x)\} = \mathcal{D}(\alpha)$ (definition 2.5). Applying supplementation once more gives $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for all w^i such that $w^i(x) \in \mathcal{D}(\alpha)$. By the definition of $\Box$ it follows that $\mathcal{M}, \alpha \vDash_{w^i} \Box\phi(x)$ and hence that $\mathcal{M}, \alpha \vDash (\forall x)\Box\phi(x)$. ■

The dynamics of this proof, although different in certain important respects [8] bears an interesting resemblance to the proof of the Barcan formula (as one would expect). In both cases a requirement on the size each of a certain set of sets of associated domains makes $\bigcap^n_{m\in\mathcal{D}(\alpha)}\|\phi(x)\|_{w^i}$ intersect with $\|(\forall x)\phi(x)\|$. More specifically, minimal contraction guarantees that $\bigcap^n_{m\in\mathcal{D}(\alpha)}\|\phi(x)\|_{w^i} \subseteq \|(\forall x)\phi(x)\|$, whereas minimal expansion gives us the converse. In each case the relevant fragments of this intersection is then "chopped" off and put into $\mathcal{N}(\alpha)$ by the condition $[\mathbf{m}]$. In the case of the Barcan formula the desired fragment is $\|(\forall x)\phi(x)\|$, and once we have this we're done. In the case of the converse Barcan formula we apply supplementation twice to get from $\bigcap^n_{m\in\mathcal{D}(\alpha)}\|\phi(x)\|_{w^i}$ to the individual $\|\phi(x)\|_{w^i}$.

[8]Notably, the proof does not require closure under intersections. Loosely speaking, this is due to the fact that the proposition we start with is already universally closed. It is not necessary, therefore, to "squeeze" a set of objects into the relevant set of worlds – they are already there.

3.2 The limiting case: Constant domains

As we have already mentioned, varying domain semantics is a species of free logic – a brand of quantificational logic in which a term need not refer to anything in the domain of quantification. Due to this, free logic differs from classical quantificational logic in certain important respects. Notably, it does not preserve the validity of the principle of universal instantiation:

[UI]: $(\forall x)\phi(x) \rightarrow \phi(y)$

Since domains are allowed to vary from world to world we cannot, from the fact that every object in the domain of α has the property ϕ in α, conclude that every object β from *any* world has the property ϕ in α, because β may well not be in the domain of quantification – in which case the formula $(\forall x)\phi(x)$ tells us nothing about it.

If one prefers to keep classical quantificational logic as a fragment of one's system, the domain of discourse must therefore be kept constant over the entire model. Since constant domain semantics is simply the limiting case of varying domain semantics, it is interesting, from a theoretical point of view, to see how constant domains affect our logic: [9]

Assume that the domain $\mathcal{D}$ of any world β in $\mathcal{M}$ is such that $\mathcal{D}(\beta) = \bigcup\{\mathcal{D}(\alpha) : \alpha \in \mathcal{W}\}$ (for convenience we shall refer to $\bigcup\{\mathcal{D}(\alpha) : \alpha \in \mathcal{W}\}$ as the domain $\mathcal{D}(\mathcal{M})$ of the *model*), i.e. that the domain is constant over the model. The following lemma shows in what way this assumption alters the semantical machinery:

Lemma 3.3 If every world $\alpha \in \mathcal{W}$ is such that $\mathcal{D}(\alpha) = \mathcal{D}(\mathcal{M})$ then $\{\beta \in \mathcal{W} : \beta \in \bigcap^n_{m\in\mathcal{D}(\beta)} \|\phi(x)\|_{w^i}\} = \bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i}$

Proof. Since every individual domain $\mathcal{D}(\beta)$ in $\mathcal{M}$ is equal to the domain $\mathcal{D}(\mathcal{M})$ of the model, $\beta \in \bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i}$ if and only if $\beta \in \bigcap^n_{m\in\mathcal{D}(\beta)} \|\phi(x)\|_{w^i}$. Thus $\{\beta : \beta \in \bigcap^n_{m\in\mathcal{D}(\beta)} \|\phi(x)\|_{w^i}\}$ is equal to $\{\beta : \beta \in \bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i}\}$. Now, from elementary set-theory we know that $\{\gamma : \gamma \in X\} = X$. Hence, we have that $\{\beta : \beta \in \bigcap^n_{m\in\mathcal{D}(\beta)} \|\phi(x)\|_{w^i}\}$ $= \{\beta : \beta \in \bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i}\} = \bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i}$ ∎

The significance of this lies in the fact that in constant domain semantics $\|(\forall x)\phi(x)\| = \bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$ for any $\alpha \in \mathcal{W}$. Unlike the varying domain case, therefore, we may always substitute the one for the other – a fact that simplifies the semantics and gives us the following theorems:

Theorem 3.4 The Barcan formula is valid in the class of constant domain intersection closures. [10]

---

[9] Much of the material in this section derives from Geir Waagbö [7] . Both of the theorems below were proved there (the second of these was independently proved in [6], in the varying domain framework employed here). The material in the present section contributes by providing a lemma bridging constant and varying domain semantics.

[10] As Waagböe has demonstrated this theorem holds only in the finite case. When $\mathcal{W}$ contains an infinite number of worlds it must be strengthened to closure under *arbitrary* intersections.

Proof. : Let $\mathcal{M}$ be closed under intersections and assume that $\mathcal{M}, \alpha \models (\forall x)\Box\phi(\mathrm{x})$. By the definition of $\forall$ and $\Box$ it follows that $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for any w^i such that $w^i(x) \in \mathcal{D}(\mathcal{M})$. Since $\mathcal{M}$ is closed under intersections, then, we have that $\bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$. By lemma 3.3 $\{\beta \in \mathcal{W} : \beta \in \bigcap^n_{m\in\mathcal{D}(\beta)} \|\phi(x)\|_{w^i}\} = \bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i}$. Since the latter set is in $\mathcal{N}(\alpha)$ the former is in $\mathcal{N}(\alpha)$, which means, by lemma 2.10, that $\mathcal{M}, \alpha \models \Box(\forall x)\phi(x)$ ∎

Theorem 3.5 The converse Barcan formula is valid in the class of constant domain supplementations.

Proof. : Let $\mathcal{M}$ be supplemented and assume that $\mathcal{M}, \alpha \models \Box(\forall x)\phi(x)$. It follows that $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$, by the definition of $\Box$, and that $\bigcap^n_{m\in\mathcal{D}(\mathcal{M})} \|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$, by lemma 3.3. Now, $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i} =_{df} \|\phi(x)\|_{w^m} \cap, \ldots, \cap \|\phi(x)\|_{w^n}$ where $\{w^m(x), \ldots, w^n(x)\} = \mathcal{D}(\alpha)$ (definition 2.5). Applying supplementation once more gives $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for all w^i such that $w^i(x) \in \mathcal{D}(\alpha)$. By the definition of $\Box$ it follows that $\mathcal{M}, \alpha \models_{w^i} \Box\phi(x)$ and hence that $\mathcal{M}, \alpha \models (\forall x)\Box\phi(x)$. ∎

As these proofs show constant domain semantics is simpler in the sense that the properties of minimal contraction and minimal expansion no longer figure explicitly in the proofs. More interestingly though, it is also simpler in the sense that supplementation – due to the coincidence of $\|(\forall x)\phi(x)\|$ with any $\bigcap^n_{m\in\mathcal{D}(\alpha)} \|\phi(x)\|_{w^i}$ – is not needed for the proof of the validity of the Barcan formula to go through. This feature lends constant domain semantics a grace and symmetry that is lacking from the varying domain case. Supplementation and intersection of sets thus forms an axis, so to speak, along which quantifiers and modalities permute.

4 The relationship between QML and QMML

The reader has probably noticed the rather conspicous divergence between our goals, as set forth in the introduction, and the amount of terrain that has been gained by our results: We wanted to have a quantified modal logic general enough to serve as a characterisation tool for first-order concepts involving non-normal modalities such as obligatoriness and agency. However, if we actually use either of the permutation principles as a part of such a characterisation, then we end up with a minimal system that is only marginally weaker than the normal systems we abandoned. This is true in particular of the class of minimally contracting quasi-filters. A quasi-filter is a filter whenever $\mathcal{N}(\alpha)$ is non-empty, and a finite filter is a normal model. Hence, a quasi-filters is, for all practical purposes, itself a normal model. Hence, validating the Barcan formula along the lines sketched above brings us very close to the class of standard models. So close, indeed, that it becomes natural to ask what the exact relationship between the two classes is. [11]

[11] In order to distinguih them clearly, we shall henceforth use 'monotonic'/'anti-monotonic' when speaking of standard models, and 'expanding'/'contracting' when speaking of minimal models.

A *propositional* standardmodel, we know ([1]), is essentially an *augmented* minimal model, where augmentation means:

[**Augmentation**]: $X \in \mathcal{N}(\alpha)$ iff $\bigcap \mathcal{N}(\alpha) \subseteq X$

In other words: An augmented model is a minimal model in which $\mathcal{N}(\alpha)$ contains the intersection of all of its members, and every superset thereof.

Moreover, one model being essentially another, means that it belongs to a class any member of which has a pointwise equivalent in the other class. A theorem to this effect can be found [1] where it is stated as follows (it is here numbered to fit the present essay):

Theorem 4.1 For every standard model $\mathcal{M}^{\mathcal{S}}$ there is a pointwise equivalent augmented minimal model $\mathcal{M}^{\mathcal{M}}$ and vice versa.

The theorem is proved by constructing a model from the one class out of a model from the other (and conversely), and by demonstrating, inductively, that these constructions preserve satisfaction. More specifically; a standardmodel $\mathcal{M}^{\mathcal{S}}$ is constructed from an arbitrary augmentation $\mathcal{M}^{\mathcal{M}}$ by defining R as $\alpha \mathrm{R} \beta$ iff $\beta \in \mathcal{N}(\alpha)$, and a minimal model is constructed from an arbitrary standardmodel by defining $\mathcal{N}$ as $X \in \mathcal{N}(\alpha)$ iff $\{\beta \in W : \alpha \mathrm{R} \beta\} \subseteq X$. Pointwise equivalence is proved by showing, on the assumption that the models are pointwise equivalent for all atomic sentences, that they are pointwise equivalent for any complex sentence formed from any atomic formulae and any combination of operators in the language. The only non-trivial case is the case where a sentence is a neccessitation (from [1]):

$\mathcal{M}^{\mathcal{M}}, \alpha \vDash \Box B$ iff $\|B\|^{\mathcal{M}^{\mathcal{M}}} \in \mathcal{N}(\alpha)$
- by the minimal definition of $\Box$

iff $\bigcap \mathcal{N}(\alpha) \subseteq \|B\|^{\mathcal{M}^{\mathcal{M}}}$
- by augmentation

iff for every $\beta \in \mathcal{W}$ such that $\alpha R \beta$; $\mathcal{M}^{\mathcal{S}}, \beta \vDash B$
- definition of R and the inductive hypothesis

iff $\mathcal{M}^{\mathcal{S}}, \alpha \vDash \Box B$
- by the standard defintion of $\Box$

Theorem 4.1 is easily extended to the first-order case. We just have to prove, on the basis of the hypothesis $\|\phi(x)\|_{w^i}^{\mathcal{M}^{\mathcal{M}}} = \|\phi(x)\|_{w^i}^{\mathcal{M}^{\mathcal{S}}}$ for any atomic formula $\phi(x)$ and any valuation w^i, that $\mathcal{M}^{\mathcal{M}}$ and $\mathcal{M}^{\mathcal{S}}$ are pointwise equivalent for universally closed sentences. The proof is trivial: If $\mathcal{M}^{\mathcal{M}}, \alpha \vDash (\forall x)\phi(x)$ then $\mathcal{M}^{\mathcal{M}}, \alpha \vDash_{w^i} \phi(x)$ for any object $w^i(x)$ from $\mathcal{D}(\alpha)$. By the inductive hypothesis it follows that $\mathcal{M}^{\mathcal{S}}, \alpha \vDash_{w^i} \phi(x)$ and hence that $\mathcal{M}^{\mathcal{S}}, \alpha \vDash (\forall x)\phi(x)$. Since this argument can clearly be repeated in the other direction we have that $\|(\forall x)\phi(x)\|^{\mathcal{M}^{\mathcal{M}}} = \|(\forall x)\phi(x)\|^{\mathcal{M}^{\mathcal{S}}}$ – i.e.;

Theorem 4.2 For every first-order standard model $\mathcal{M}^{\mathcal{S}}$ there is a pointwise equivalent first-order augmented minimal model $\mathcal{M}^{\mathcal{M}}$ and vice versa.

However, the pointwise equivalent *minimal* model that theorem 4.2 proves the existence of is not necessarily a (as the case may be) contracting or expanding one. Consider the following graphically represented quasi-filter:

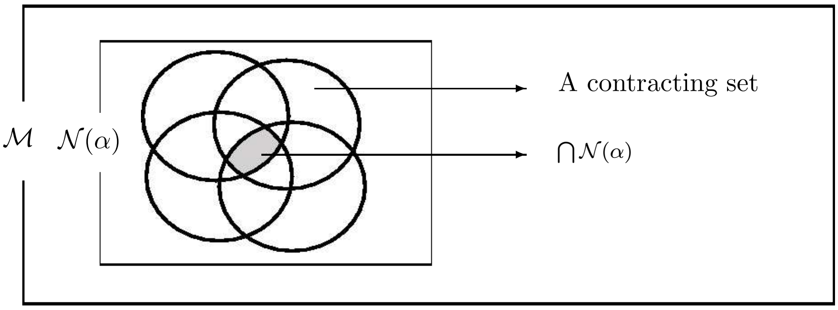

fig. 6

We clearly have a match in one direction: The set $\bigcap\mathcal{N}(\alpha)$ is itself contracting. This follows from the fact that $\bigcap\mathcal{N}(\alpha)$ is a subset of every other set in $\mathcal{N}(\alpha)$ and the assumption that at least one of these is contracting. Thus, since constructing the point-wise equivalent standard model is done by defining R in terms of $\bigcap\mathcal{N}(\alpha)$, minimal contraction translates nicely into anti-monotonicity :

PROOF. -sketch: Let $\mathcal{M}^M$ be a contracting minimal model and define the standard model $\mathcal{M}^S$ stipulating that $\alpha R\beta$ iff $\beta \in \bigcap\mathcal{N}(\alpha)$. Then;

$\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ is contracting
only if every $\beta \in \bigcap\mathcal{N}(\alpha)$ is such that $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$.
- by the inclusion of $\bigcap\mathcal{N}(\alpha)$ in $\|\phi(x)\|_{w^i}$
only if $\alpha R\beta$ then $\mathcal{D}(\beta) \subseteq \mathcal{D}(\alpha)$
- by the definition of R.

∎

However, things do not work out this nicely for the other direction. Even though $\bigcap\mathcal{N}(\alpha)$ is contracting it does not follow that any of its supersets are, which blocks the equivalence.

Hence, minimal contraction is a *sufficient* but not a *necessary* condition for the validity of the Barcan formula. We can generate an anti-monotonic standard-model from a contracting augmentation, but we cannot generate a contracting augmentation from an anti-monotonic standard model. This reflects the fact that minimal contraction is unnecessarily strong. To ensure the validity of the Barcan formula in a minimal model a much weaker principle which, tracking the movement of the necessity operator we have chosen (in want of a better name) to call closure under permutation inwards, will do:

[CUPI]: If $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for all w^i such that $w^i(x) \in \mathcal{D}(\alpha)$ then $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$

No proof, I am sure, is required to convince the reader of the adequacy of this restriction. Whereas minimal contraction validates the Barcan formula via a restriction on domains **[CUPI]** simply picks out the relevant sets and puts them in $\mathcal{N}(\alpha)$. Since **[CUPI]**, quite clearly, and unlike minimal contraction, *corresponds* to the Barcan formula (i.e. that the Barcan formula is a theorem *if and only if* **[CUPI]** is valid), demonstrating pointwise equivalence between anti-monotonic standard models and minimal CUPI models is trivial and almost empty of formal content. We shall do it anyway – for purposes of expository completeness:

PROOF. :

$\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for all w^i such that $w^i(x) \in \mathcal{D}(\alpha)$ then $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$
iff $\mathcal{M}^{\mathcal{M}}, \alpha \vDash (\forall x)\Box\phi(x) \to \Box(\forall x)\phi(x)$
- by correspondence
iff $\mathcal{M}^{\mathcal{S}}, \alpha \vDash (\forall x)\Box\phi(x) \to \Box(\forall x)\phi(x)$
- by pointwise equivalence
iff if $\alpha R \beta$ then $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$
- by correspondence ∎

The argument can clearly be repeated in the other direction, and so, every anti-monotonic Kripke model has a pointwise equivalent CUPI model. It is this that is captured by theorem 4.2. The argument for CUPO models and montonic Kripke models is similar.

5 Completeness

To our knowledge, there are as yet no completeness results for varying domain QMML. However, the situation is by no means one that calls for drastic measures or intense research activity because simply collecting together existing results almost gets the job done.

A complete, least system of free logic can be constructed by defining defining $\exists x$ as $\neg\forall x\neg$ and adding the following rules to propositional logic [4]:

[FUI] From $(\forall x)\phi(x)$ infer $E(y) \to \phi(y)$
[FUG] From $\psi \to (E(y) \to \phi(y))$ infer $\psi \to (\forall x)\phi(x)$

Here E is a primitive existence predicate used to adjust the familiar principles of universal instantiation and universal generalization to free logic. [12] Thus **[FUI]** says that what is true of everything is true of any arbitrary object on the condition that this arbitrary object exists. The rule of inference **[FUG]** is a similarly weakened principle saying that if a formula ϕ implies another formula ψ *whenever x exists*, then ϕ implies that every x is ψ.

The least modal logic, on the other hand, is constructed by defining $\Box A$ as $\neg\Diamond\neg A$, and adding the following rule to PL:

[RE] From $A \equiv B$ deduce $\Box A \equiv \Box B$

We shall argue a) that the first-order modal logic obtained by combining these sys-

[12] Note, that we could have taken advantage of the flexibility of varying domains and defined the notion of the existence in the following way: $E(x) =_{df} (\exists y)(y = x)$. However, since we have chosen to leave the identity sign out of our language we must work with the primitive

tems, we call it **QFE**, is complete with respect to the class of varying domain minimal models, and b) that the extensions obtained by adding the Barcan formulae are complete with regard to the class of models that are closed under **[CUPI]** and **[CUPO]** respectively.

A canonical model for *varying domain* first-order modal logic is built from sets of formulae Γ that are *saturated* (with respect to the language L) in the following sense:

1. Γ is consistent, i.e. there is some $A \subseteq L$ such that $A \notin \Gamma$
2. Γ is maximal, i.e. for all formulae $A \subseteq L$ either $A \in \Gamma$ or $\neg A \in \Gamma$, and
3. Γ is ω-complete, i.e. if $\phi[y/x] \in \Gamma$ for all terms y such that $E(y)$ then $(\forall x)\phi(x) \in \Gamma$

This definition differs from the corresponding definition for classical first-order logic in an instructive way: In classical first-order logic omega-completeness (3) means that

[ω] If $\Gamma \vdash \phi(y)$ for any term y of L then $\Gamma \vdash (\forall x)\phi(x)$ for any variable x

which is equivalent to,

[ω'] If $\Gamma \cup \{\neg(\forall x)\phi(x)\}$ is consistent, then for some term y of L, $\Gamma \cup \{\neg\phi(y)\}$ is consistent.

In free logic, on the other hand, we know that the meaning of the quantifiers is given, not by satisfaction alone, but by satisfaction + *existence* (cf. definition 2.1). The definition of saturation is adjusted to reflect this.

Now, as Henkin was the first to prove, Ω-completeness is guaranteed if we extend a first-order language L into a larger language L^+ which has all the terms of L and infinitely many new ones as well (cf. [5]). Thus,

Lemma 5.1 Any consistent set of free logic formulae of L can be extended to a free logic ω-complete set in L^+.

Moreover, it is a standard result of quantificational logic that once a set Γ is free logic ω-complete then any set (in the same language) Δ of which Γ is a subset is still ω-complete, which means that Γ can be extended into a maximally consistent set:

Lemma 5.2 If Γ is a omega-complete set of free logic formulae of L^+ then it can be extended to a saturated set Δ of formulae of L^+ which is such that $\Gamma \subseteq \Delta$.

Since lemma 5.2 guarantees the existence of a saturated extension of any consistent set of **QFE** we will have worlds enough to build a canonical model for the system. We call this model $\mathcal{M}^{Can}$ and define it as follows:[13]

1. $\mathcal{W}$ is the set of saturated extensions α of **QFE** each written in a language L^+_α such that L_α is an infinitely proper sublanguage of the language L^+_α
2. $\mathcal{N}$ satisfies $\Box A \in \alpha$ iff $|A| \in \mathcal{N}(\alpha)$ (where $|A|$ is the set of all maximally consistent sets containing A).
3. $\mathcal{D}(\alpha)$ is equal to the terms in L^+_α

[13] This model is a free logic version of one found in [7]

4. v maps terms to themselves and $\mathcal{V}(\alpha, \phi(x_1, \ldots, x_n)) = T$ iff $\phi(x_1, \ldots, x_n) \in \alpha$, for all $\phi(x_1, \ldots, x_n) \subseteq \bigcup L_\alpha$

As is well known completeness usually involves proving a lemma (the truth-lemma) which shows that membership in α and truth in α amounts to the same thing:

[TL] $\mathcal{V}(\alpha, A) = \mathrm{T}$ iff $A \in \alpha$

However, the members of $\mathcal{W}$ are sets written in different languages (corresponding to the different domains) so it is not obvious that the truth-lemma, as stated above, is provable in our model. For instance, if a term y does not appear in L_α we *could* have that $\mathcal{V}(\alpha, \neg\phi(y)) = T$, but $\neg\phi(y) \notin \alpha$. However, as a consequence of how we have defined our model, where truth is equivalent to satisfaction and predicates can be satisfied by objects from the union $\bigcup L_\alpha$ of the domains of all worlds α (cf. the definition of the canonical model, point 4), y would have to be a term from L_α^+. But, by the construction of the model α is written in L_α^+ which reduces the assumption that $\mathcal{V}(\alpha, \neg\phi(y)) = T$, and $\neg\phi(y) \notin \alpha$ to an absurdity. Hence **[TL]** holds for atomic sentences. That it also holds for all other formulae is proved by induction on their complexity. There are two non-trivial cases:

(A) $A = (\forall x)\phi(x)$. Proved in [4].

(B) $A = \Box B$. Assume that the truth lemma holds for the simpler formula B, i. e. that $\{\alpha \in \mathcal{W} : \mathcal{M}^{Can}, \alpha \vDash B\} = |B|$. Now, $\mathcal{M}^{Can}, \alpha \vDash \Box B$ iff $\{\alpha \in \mathcal{W} : \mathcal{M}^{Can}, \alpha \vDash B\} \in \mathcal{N}(\alpha)$ iff $|B| \in \mathcal{N}(\alpha)$. By the definition of $\mathcal{N}$ this is the case iff $\Box B \in \alpha$.

Completeness follows immediately. If A is a consistent formula then, by lemma 5.2, there must be a saturated extension Γ of **QFE** such that $A \in \Gamma$. Hence, a canonical model containing Γ exists, which means that A is satisfiable in a model in the relevant class [7]. That this model is written in different superlanguages L_α^+ doesn't matter since the valuation function v is in each case restricted to the set of variables in $\bigcup L_\alpha$. As a corollary we have the following theorem:

Theorem 5.3 The system **QFE** is complete with regard to the class of minimal free logic models.

As regards the permutation principles call the respective extensions of our basic system **QFE + BF** and **QFE + CBF**, and consider the two semantical requirements **[CUPI]** (closure under permutation inwards) and **[CUPO]** (closure under permutation outwards).

[CUPI]: If $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for all w^i such that $w^i(x) \in \mathcal{D}(\alpha)$ then $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$
[CUPO]: If $\|(\forall x)\phi(x)\| \in \mathcal{N}(\alpha)$ then $\|\phi(x)\|_{w^i} \in \mathcal{N}(\alpha)$ for all w^i such that $w^i(x) \in \mathcal{D}(\alpha)$.

To show that **QFE + BF** is complete with regard to the former and **QFE + CBF** with regard to the latter we must show that the respective canonical models belong to the class of CUPI and CUPO models, we will prove the former case only:

Theorem 5.4 The system **QFE + BF** is complete with regard to the class of minimal free logic CUPI models.

Proof. Let $\mathcal{M}$ be the smallest canonical minimal model for a classical first-order system containing the Barcan formula, and assume that an atomic sentence $\phi(x)$ is a member of the saturated set α for all assignments to x of values from $\mathcal{D}(\alpha)$ – in other words; assume that $|\phi(x)|_{w^i} \in \mathcal{N}(\alpha)$ for all w^i such that $w^i(x) \in \mathcal{D}(\alpha)$. By the truth lemma all these open sentences are true at α, and consequently so is $(\forall x)\Box\phi(x)$ (by the definition of a universal closure (2.1)). Since BF is a theorem of the system, it is true at α, which allows us to conclude, by modus ponens, that $\Box(\forall x)\phi(x)$ is true at α too. By the definition of the necessity operator in the canonical model, therefore, $|(\forall x)\phi(x)| \in \mathcal{N}(\alpha)$ and we are done. ∎

The proof of the completeness of **QFE + CBF** with respect to cupo models is similar in all relevant respects and has therefore been skipped.

The situation is rather more involved when it comes to minimal contraction and minimal expansion as we have to take the restrictions **[m]** and **[c]** into consideration as well. We leave this for future research and, for the moment, rest content with the fact that we have proved completeness for the most general minimal systems having the Barcan formulae. The language is thus under control, and the need for further developments is no longer pressing.

Acknowledgements

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Reviews

Review of *Relation algebras by games*, by Robin Hirsch & Ian Hodkinson, Elsevier; Amsterdam, 2002, *Studies in Logic and The Foundations of Mathematics*, V. 147, ISBN 0 444 50932 1, xviii+691pp.

There are twenty-one Chapters in six parts. Parts of eight chapters are modified versions of nine previously published papers (six by Hirsch and Hodkinson, one by Hirsch, one by Hodkinson, and one by Andréka and Hodkinson). There are more than 400 exercises ranging from drill to open problems. Chapter 21 has a list of twenty-one open problems, recalled from the body of the text, that the authors believe would "make a significant contribution to research in this field." The book is itself such a contribution, since it is built around the solutions to several open problems.

The first third of the book is occupied by the five chapters of Part I. It contains a historical introduction, preliminaries from model theory, universal algebra, and Boolean algebras, two chapters on relation algebras (with basic definitions, elementary facts, and many examples), and brief surveys of relativized representations, cylindric algebras, and other approaches to algebraic logic.

In Chapter 2, many basic definitions and facts from model theory are recalled, including the compactness theorem, the Löwenheim-Skolem-Tarski theorem, theorems on the existence of saturated models and ultraproducts, Los's theorem, and so on. The theory of Boolean algebras with operators is summarized up through the existence, uniqueness, and preservation theorems for completions and perfect extensions. This summary includes a survey of results on the interaction of atom structures, complex algebras, Sahlqvist equations, and modal logic.

In Chapters 3 and 4, representations of a relation algebra are viewed as models of an appropriate first-order theory constructed from the algebra. This allows the authors to prove Monk's theorem, that the class of representable relation algebras is closed under perfect extensions, by using the existence of saturated models of the associated first-order theories. Using this result, they then prove Tarski's theorem, that the class of representable relation algebras is closed under homomorphisms, by constructing a representation of a homomorphic image of a representable algebra from a representation of its perfect extension. Examples of relation algebras include proper relation algebras (algebras of binary relations), group relation algebras (whose elements are subsets of a group), Lyndon algebras (whose elements are sets of points in a projective geometry), and Monk algebras (which are designed to utilize some basic results of Ramsey theory).

Chapter 5 reviews the definitions, basic facts, and gives a brief survey of results about non-associative relation algebras, weakly associative relation algebras, semi-associative relation algebras, weakly representable relation algebras (wRRA), cylindric algebras, and the connections among these classes that arise through relativization and the formation of reducts. Chapter 6 is a short survey of other kinds of algebras of relations, including diagonal-free cylindric algebras, Halmos's polyadic algebras, and Pinter's substitution algebras. There is an informative discussion of the (as yet unresolved) finitization problem, and a list of representation results for relation algebras.

The second third of the book is comprised of Part II (Chapters 7–11, on games, networks, and axiomatization) and Part III (Chapters 12 and 13, on approximations to representations).

The general theme of Part II is that representability can be characterized by the existence of a winning strategy for a game played with networks. A network over some algebra is a (typically finite) graph whose edges are labeled with elements of the algebra. A network is essentially just a portion of a representation. Networks serve as positions in games. The object of a typical game on an algebra is, for the first player (male), to ferret out a difficulty that prevents the algebra from being representable (or having some other nice property), while the second player (female) attempts to build a representation (or some other appropriate structure) in response to challenges by the first player. These ideas are illustrated by a proof of Maddux's theorem that every weakly associative relation algebra has a relativized representation. An appropriate infinite game is defined and shown to be winnable by the second player, which implies that every countable weakly associative relation algebra has a relativized representation. This conclusion can then be extended to all weakly associative relation algebras by compactness or ultraproducts.

There are many variations on the games. They can be played on networks in general, or on networks restricted in some way, such as having atoms as labels or bounded size (no more than p nodes, for some fixed bound $p < \omega$). Depending on the application, moves may consist of choosing an edge and two elements, choosing two nearly identical networks, dropping a node and its edges from a network, properly labeling a set of edges of a network, or choosing one of two alternative networks. A given game can be last for countably many moves (an infinite game), or it can be terminated after a fixed finite number of rounds (an r-round game for some finite length $r < \omega$). A key lemma is that if the second player can win all the finite-length games, then she can win the infinite game. According to the central result of Chapter 7, if the second player can win a certain game on a relation algebra, then that algebra is representable. The proof relies on the fact that the representable relation algebras form an equational class.

The ability of the second player to win all finite-length games can be expressed as a sequence of first-order sentences, one for each finite-length game. The collection of such sentences characterizes representability. Each sentence in such collection has a quantifier prefix of the form $(\forall\exists)^r$, where r is the number of rounds in the corresponding game. The sentence says, roughly, that *for every* move by the first player, *there is* a response available to the second player such that *for every* move by the first player, *etc.* By restricting the second player to choosing only one of two alternative moves, the existential quantifier can be reduced to a disjunction and the resulting sentence is universal. For discriminator varieties (the typical case in this book) a universal sentence is equivalent in all simple algebras to an equation. These techniques are applied to obtained axiomatizations for various classes, including representable relation algebras and representable cylindric algebras in Chapter 8, and some pseudo-elementary classes and pseudo-universal classes in Chapter 9. Chapter 10 is a more formal presentation of some of this material, including precise definitions of games and strategies in terms of trees labeled with variables and formulas.

Chapter 11 is primarily about atomic relation algebras and Lyndon's conditions from his 1950 paper. Assume $\mathfrak{A}$ is an atomic relation algebra. Atomic networks are those networks whose labels are atoms of $\mathfrak{A}$. A new "atom-game" is introduced, played on atomic networks. The main results are that the atomic relation algebra $\mathfrak{A}$ satisfies the Lyndon conditions iff the second player wins all finite-length atom-games on $\mathfrak{A}$ iff $\mathfrak{A}$ is elementarily equivalent to a completely representable (hence also atomic) relation algebra. Furthermore, if $\mathfrak{A}$ is completely representable then the second player wins the infinite atom-game on $\mathfrak{A}$. The converse fails in general but holds if the number of atoms is countable. The exercises include examples of finite symmetric integral representable relation algebras with no finite representations, Monk's theorem that RRA, the class of representable relation algebra, is not finitely based, and Jónsson's proof that there is no equational axiomatization of RRA that uses only finitely many variables. The latter two results are both proved via Lyndon's algebras, obtained from projective lines, which were published in 1961. Jónsson's proof was found in 1988, but the result was stated by Tarski in a videotaped lecture in 1974. Perhaps this proof was at one time in the 1960's known to both Tarski and Jónsson.

For $n \geq 3$, RA_n is the class of relation algebras of dimension n, and $\mathbf{S}\mathsf{RaCA}_n$ is the class of subalgebras of relation-algebraic reducts of cylindric algebras of dimension n. It has long been known that these classes are canonical varieties. RA_n is defined via the concept of relational basis. Every atomic algebra in $\mathbf{S}\mathsf{RaCA}_n$ has a relational basis. This implies that $\mathbf{S}\mathsf{RaCA}_n \subseteq \mathsf{RA}_n$. In spite of the names, RA_3 and $\mathbf{S}\mathsf{RaCA}_3$ are strictly larger classes than RA, the class of all relation algebras. For $n = 4$ and $n = \omega$ the situation is quite nice, and provides characterizations of RA and RRA, since $\mathsf{RA} = \mathsf{RA}_4 = \mathbf{S}\mathsf{RaCA}_4$ and $\mathsf{RRA} = \mathsf{RA}_\omega = \mathbf{S}\mathsf{RaCA}_\omega$. When $4 \leq n < \omega$, these classes form descending chains of varieties of relation algebras that converge to RRA, that is, $\mathsf{RA}_{n+1} \subset \mathsf{RA}_n$ and $\mathbf{S}\mathsf{RaCA}_{n+1} \subset \mathbf{S}\mathsf{RaCA}_n$, where all inclusions were known to be strict, and $\mathsf{RRA} = \bigcap_{3\leq n<\omega} \mathsf{RA}_n = \bigcap_{3\leq n<\omega} \mathbf{S}\mathsf{RaCA}_n$. In Chapters 12 and 13 these definitions and results are reviewed and extended. A variation on the atom-game is introduced, in which there is a fixed finite bound p on the number of nodes in the networks that appear in the game. The second player wins the infinite p-bounded atom-game if and only if the algebra has a p-dimensional relational basis. An extension of the concept of relational basis, called "hyperbasis", is used to characterize $\mathbf{S}\mathsf{RaCA}_n$ in a manner similar to the definition of RA_n. The classes RA_n and $\mathbf{S}\mathsf{RaCA}_n$ are also characterized in terms of relativized representations that have certain nice properties.

The last third of the book consists of Part IV (Chapters 14–17), which contains various constructions of algebras, and Part V (Chapters 18 and 19) on decidability and the finite base property.

In Chapter 14 a relational structure $\mathfrak{S}(G)$ is associated with every graph G, with these crucial properties. If G has infinite chromatic number (and is itself necessarily infinite), then $\mathfrak{Cm}\mathfrak{S}(G)$, the complex algebra of $\mathfrak{S}(G)$, is a representable relation algebra. On the other hand, if G has a finite chromatic number, and G itself is either infinite or sufficiently large compared to its chromatic number, then $\mathfrak{Cm}\mathfrak{S}(G) \notin \mathsf{RRA}$. This allows two significant theorems. If, for example, G is a countable chain (undirected edges, all nodes of degree 2), then $\mathfrak{Cm}\mathfrak{S}(G) \notin \mathsf{RRA}$, but it turns out that if $\mathfrak{A}$ is the subalgebra of $\mathfrak{Cm}\mathfrak{S}(G)$ generated by the atoms of $\mathfrak{Cm}\mathfrak{S}(G)$, then $\mathfrak{A}$ is representable. It follows that RRA is not closed under completions, for $\mathfrak{Cm}\mathfrak{S}(G)$ is the completion of $\mathfrak{A}$. This result settles a 30-year old problem of Monk that was first solved by Hodkinson using a different example. Then, using a result of Erdös that there are finite graphs with arbitrarily large girth and chromatic number, the authors construct a sequence of infinite graphs G_i with infinite chromatic number and increasing girth, say G_i has girth i. The ultraproduct of the graphs G_i is 2-colorable because it has infinite girth, so the corresponding structure has a nonrepresentable complex algebra. But the complex algebras of the $\mathfrak{S}(G_i)$ are in RRA. Thus the class of structures whose complex algebras are in RRA does not form an elementary class. For $i > 15$, the algebra $\mathfrak{CmS}(G_i)$ satisfies only finitely many Lyndon conditions, so the second player will lose any sufficiently long finite-length atom-game played on $\mathfrak{CmS}(G_i)$. This algebra cannot be completely representable (for it would otherwise satisfy all the Lyndon conditions) but it is, nevertheless, representable. It is complete, atomic, and representable, but has no complete representations. The earliest example of such an algebra appears in Lyndon's 1950 paper. Additional examples, from Maddux's 1978 dissertation, conclude Chapter 14.

The main result of Chapter 15 is that $\mathbf{S}\mathsf{RaCA}_{n+1}$ is not finitely based relative to $\mathbf{S}\mathsf{RaCA}_n$ (and a similar statement for neat reducts of cylindric algebras). The proofs utilize variations on Monk algebras and techniques from Ramsey theory, and they produce other interesting results. For every finite $n \geq 5$, the inclusion $\mathbf{S}\mathsf{RaCA}_n \subseteq \mathsf{RA}_n$ is strict, and, in fact, $\mathbf{S}\mathsf{RaCA}_n$ is not finitely based relative to RA_n, nor is it finitely based relative to $\mathsf{RA}_n \cap \mathbf{S}\mathsf{RaCA}_{n-1}$. Furthermore, for every finite $n \geq 5$, RA_n is not included in $\mathbf{S}\mathsf{RaCA}_5$.

Chapters 16 and 17 are devoted to the construction and application of the rainbow algebra $\mathfrak{A}_{A,B}$, which is built from two relational structures A and B for a language having only binary and unary relation symbols. The rainbow algebra $\mathfrak{A}_{A,B}$ is always complete and atomic, and it is finite whenever A and B are finite. The number of atoms in $\mathfrak{A}_{A,B}$ is approximately the sum of the squares of the cardinalities of A and B. The idea is to translate Ehrenfeucht-Fraïssé games on relational structures to games on algebras. It is a result from model theory that the second player wins the standard p-pebble r-round Ehrenfeucht-Fraïssé "forth" game from A to B just in case every positive existential sentence with p variables and quantifier depth r true in A is also true in B. The main result is that if the second player can win a (slightly modified) p-pebble finite-length Ehrenfeucht-Fraïssé game from A to B, then the second player can win a p-bounded finite-length atom-game on $\mathfrak{A}_{A,B}$. This construction is quite powerful. It provides alternate proofs of earlier theorems, a proof of Haiman's theorem that wRRA is not finitely based, and some significant new results. For example, the class of completely representable relation algebras is not elementary. Perhaps most important is that RA_{n+1} is not finitely based relative to RA_n whenever $4 \leq n < \omega$, and that RA_n and $\mathbf{S}\mathsf{RaCA}_n$ are not closed under completions whenever $6 \leq n$ (leaving open the case $n = 5$.)

Chapter 18 is devoted to showing that if K is any class of algebras that contains RRA and is contained in either $\mathbf{S}\mathsf{RaCA}_5$ or wRRA, then there is no algorithm for determining whether a finite algebra belongs to K. In particular, the question "is a finite RA in RRA?" is undecidable. It follows readily from these results that if K is a variety, then the equational theory of K is also undecidable. The method is to encode each instance of the (undecidable) tiling problem as a relation algebra. Given a set of edge-labeled tiles τ, there is an algebra $\mathfrak{A}_\tau$ that is representable if and only if there is a tiling of plane using copies of tiles from τ so that adjacent tiles have matching labels on their common edge.

Every weakly associative relation algebra has a relative representation, but if the algebra is finite then, in fact, such a representation can be constructed on a finite set. In Chapter 19, this and similar results for finite algebras in RA_n and subalgebras of relation-algebraic reducts of finite n-dimensional cylindric algebras, involving appropriate notions of relative representation, are derived fairly directly from recent deep results in model theory that are stated with references but without proof.

Part VI, the Epilogue, has just two short chapters. One is a fifteen-page summary of the book, and the other is a list of twenty-one unsolved problems. There are 326 items in bibliography and ten pages of symbol indices, organized into ten tables according to subject.

This book is a significant advance in the theory of relation algebras. Many of its main results solve difficult and long-standing problems. Its methods, techniques, and constructions are powerful tools for exploring the intricate and varied world of relation algebras. Its many open problems indicate fruitful directions for further research.

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10 July 2003

Conferences

10th Workshop on Logic, Language, Information and Computation (*WoLLIC'2003*)

WoLLIC'2003 was held at the *Salão Nobre* of the *Escola de Minas* of Universidade Federal de Ouro Preto, Minas Gerais, Brazil. *WoLLIC* is a series of workshops which started in 1994 with **the aim of fostering interdisciplinary research in pure and applied logic**. The idea is to have a forum which is large enough in the number of possible interactions between logic and the sciences related to information and computation, and yet is small enough to allow for concrete and useful interaction among participants. Previous versions were held at: Recife (Pernambuco, Brazil) in 1994 and 1995; Salvador (Bahia, Brazil) in 1996; Fortaleza (Ceará, Brazil) in 1997; São Paulo (Brazil) in 1998; Itatiaia (Rio de Janeiro, Brazil) in 1999; Natal (Rio Grande do Norte) in 2000; Brasília (Distrito Federal, Brazil) in 2001; Rio de Janeiro (Brazil) in 2002.

Scientific sponsorship comes from the *Interest Group in Pure and Applied Logics* (*IGPL*), the *European Association for Logic, Language and Information* (*FoLLI*), the *Association for Symbolic Logic* (*ASL*), *European Association for Theoretical Computer Science* (*EATCS*), the *Sociedade Brasileira de Computação* (*SBC*), and the *Sociedade Brasileira de Lógica* (*SBL*).

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Contributions were received in the form of short papers in all areas related to logic, language, information and computation, including: pure logical systems, proof theory, model theory, algebraic logic, type theory, category theory, constructive mathematics, lambda and combinatorial calculi, program logic and program semantics, logics and models of concurrency, logic and complexity theory, proof complexity, foundations of cryptography (zero-knowledge proofs), descriptive complexity, nonclassical logics, nonmonotonic logic, logic and language, discourse representation, logic and artificial intelligence, automated deduction, foundations of logic programming, logic and computation, and logic engineering.

Apart from the contributed papers (15), and the invited talks (5), the programme includes 5 tutorial lectures:

• *Algorithmic Randomness and Derandomization*
by **Eric Allender** (Departament of Computer Science, Rutgers, the State University of NJ, USA)

• *Generalized Quantifiers*
by **Lauri Hella** (Department of Mathematics, Statistics and Philosophy, University of Tampere, Finland)

• *Implicit computational complexity*
by **Jean-Baptiste Joinet** (Équipe de Logique Matemátique, Université Paris 7, France)

• *Proof search foundations for logic programming*
by **Dale Miller** (Laboratoire d'Informatique, LIX, École Polytechnique, France)

• *Iterated theory change*
by **Hans Rott** (Institut für Philosophie, Universität Regensburg, Germany)

All papers in the volume were reviewed under the scientific responsibility of the programme committee consisting of

Argimiro Arratia (*Departamento de Matematicas, Universidad Simón Bolivar, Venezuela*)
Alessandra Carbone (*Institut des Hautes Études Scientifiques, and Université de Paris XII, France*)
Marcelo Coniglio (*Centro de Lógica e Epistemologia, Universidade Estadual de Campinas, Brazil*)
Gilles Dowek (*INRIA, France*)
Arnaud Fleury (*Facoltà di Scienze, Università di Verona, Italy*)
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The organising committee consisted of

Lucília Figueiredo (*Departamento de Computação, Universidade Federal de Ouro Preto*)
Fred Ulisses Maranhão (*Centro de Informática, Universidade Federal de Pernambuco*)
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Maria Ângela Weiss (*Departamento de Matemática, Universidade de São Paulo*)

The proceedings was published as volume 84 in the series *Electronic Notes in Theoretical Computer Science* (*ENTCS*). This series is published electronically through the facilities of Elsevier Science B.V. and its auspices. The volumes in the ENTCS series can be accessed at the URL

http://www.elsevier.nl/locate/entcs

Contributed papers: The programme committee received 21 (twenty-one) papers of high quality, from all over the world: Brazil: 7, France: 3, Brazil/New Zealand: 1, Germany: 1, Israel: 1, Italy: 1, Japan: 1, Mexico: 1, New Zealand: 1, New Zealand/Finland: 1, Norway: 1, Poland: 1, Spain: 1.

Each submitted paper was made anonymous and sent to 2 referees for evaluation. The referees gave 0-10 ratings to: (i) overall quality; (ii) soundness; (iii) originality; (iv) relevance to the workshop; (v) presentation, in this order of priority.

Full versions of the papers will go through another round of refereeing for possible publication in a special issue of the *Annals of Pure and Applied Logic* (to be confirmed).

Ruy J.G.B. de Queiroz
Elaine Pimentel
Lucília Figueiredo

Tutorials

***Tutorial*: Algorithmic Randomness and Derandomization**

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Kolmogorov complexity is a tool to measure the information content of a string. Strings with high Kolmogorov complexity are said to be "K-random". The study of this notion of randomness has a long history and it has close connections with the theory of computability. The set of K-random strings has long been known to be undecidable.

Derandomization is a fairly recent topic in complexity theory, providing techniques whereby probabilistic algorithms can be simulated efficiently by deterministic algorithms.

In this tutorial, we will survey a few of the important developments in these two fields (with particular emphasis on derandomization and pseudorandom generators), and we will learn what these two fields have to do with each other. In particular, we will see what derandomization techniques tell us about what can be "efficiently" reduced to the set of K-random strings.

***Tutorial*: Generalized Quantifiers**

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The idea of generalizing the usual existential and universal quantifiers is due to A. Mostowski (1957). He considered logical operators expressing cardinalities, like Q_0x and Q_1x which say that "there are infinitely many x" and "there are uncountably many x", respectively. Later Per Lindstrom (1966) generalized the notion of quantifier still further: according to his definition, any property of models of some fixed vocabulary can be taken as the interpretation of a quantifier. For example, if P is the property of graphs of being 3-colorable, and Q_P is the corresponding quantifier, then the formula $Q_Pxy\phi(x,y)$ says that the binary relation defined by $\phi(x,y)$ is a 3-colorable graph.

Extending a logic by generalized quantifiers is a minimal way of making undefinable properties definable. Indeed, if P is a property of models which is not definable in first order logic FO, then $FO(Q_P)$ is the least extension of FO, in which P is definable and which is closed under Boolean operations, first order quantifications and substituting relations by formulas. This fact has made generalized quantifiers a useful tool in studying th expressive power of various extensions of first order logic.

In this tutorial I will give a survey on the definability theory of generalized quantifiers. In particular, I will consider applications of quantifiers in the context of Finite Model Theory. I will also talk about some recent work on generalized second order quantifiers.

***Tutorial*: Implicit computational complexity**

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The proofs-as-programs paradigm, where computations are modelled by proofs conversion, offers a convenient theoretical framework for analysing properties of the computational dynamics of proofs/programs. In that frame, implicit computational complexity theory investigates computational complexity as a direct effect of the logical means involved (not as a by-product of ad hoc explicit external constraints). I will ouline the main recent approaches proposed in that field and will present how Linear Logic, having pull back to the logical level, the decomposition of dynamics into more fine grained operations, permits to design logically founded programming languages corresponding to several natural specific complexity classes.

***Tutorial*: Proof search foundations for logic programming**

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Sequent calculus is generally accepted as a foundation for logic programming, especially when the logic used is richer than first-order classical Horn clauses. I will outline a formal foundation of logic programming based on goal-directed sequent calculus provability, and will illustrate how higher-order quantification and linear logic all fit well into this foundation. Numerous examples of logic programming specifications using higher-order linear logic will be presented.

***Tutorial*: Iterated theory change**

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It is well known that early models of theory revision developed by Alchourrón, Makinson and Gärdenfors in the 1980s could account only for single changes of belief. Things changed only later when various authors addressed the problem of iterated theory change. The tutorial offers an overview over vari ous ideas and concepts for iterated theory change that were developed in the past ten years or so (with a certain emphasis on the speaker's own work). Topics include:
(1) revisions of "belief bases"
(2) revisions of "belief states"
(3) revisions by sequences
(4) revisions by comparisons.

Invited Papers

NL-printable sets and Nondeterministic Kolmogorov Complexity
Eric Allender
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P-printable sets were defined by Hartmanis and Yesha and have been investigated by several researchers. The analogous notion of L-printable sets was defined by Fortnow et al; both P-printability and L-printability were shown to be related to notions of resource-bounded Kolmogorov complexity. NL-printability was defined by Jenner and Kirsig, but some basic questions regarding this notion were left open. In this paper we answer a question of Jenner and Kirsig by providing a machine-based characterization of the NL-printable sets. The proof makes use of a simple hashing technique. We apply this same technique to investigate relationships among some resource-bounded notions of Kolmogorov complexity, based on nondeterministic Turing machines.

Quantifying over Quantifiers
Lauri Hella
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We study existential and universal quantification over quantifiers, i.e. quantification where the objects quantified over are Lindstrom quantifiers. First we consider the fragment where only existential quantification over quantifiers is allowed, denoted Σ_1^Q. We show that Σ_1^Q includes inflationary fixed-point logic extended with the ability to express that two defined structures are non-isomorphic, and that Σ_1^Q is included in existential second-order logic with the same extension.

The logic Σ_n^Q is defined as the fragment where we alternate existential and universal quantification for n levels. We show that Σ_{n+1}^Q is included in the n-th level of the complexity theoretical exponential hierarchy. We also show that there is a hierarchy on the arity of the quantifier variables, by showing that no fragment of Σ_n^Q with restricted arity of the quantifiers can express all Σ_1^Q properties.

Calculus of structures and proof-nets
Jean-Baptiste Joinet
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The Calculus of Structures is a new logical formalism developped by A.Gugliemi,

L.Strassburger et al. Using the central idea of CS, namely local rewritings of proofs at any depth, but directly in the formalism of Proof-Nets, I will give an alternative characterisation of the Multiplicative fragment of Multiplicative Linear Logic Proof Nets.

Encryption as an abstract data type

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At the Dolev-Yao level of abstraction, security protocols can be specified using multisets rewriting. Such rewriting can be modeled naturally using proof search in linear logic. The linear logic setting also provides a simple mechanism for generating nonces and session and encryption keys via eigenvariables. We illustrate several additional aspects of this direct encoding of protocols into logic. In particular, encrypted data can be seen naturally as an abstract data-type. Entailments between security protocols as linear logic theories can be surprisingly strong. We also illustrate how the well-known connection in linear logic between bipolar formulas and general formulas can be used to show that the asynchronous model of communication given by multiset rewriting rules can be understood, more naturally as asynchronous process calculus (also represented directly as linear logic formulas).

Economy and economics in the logic of theory change

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In this paper, I ask to what extent theory revision may be regarded as a branch of cognitive economics. Theory dynamics has long been said to be driven primarily by a concern for "informational economy". Asking about its descriptive as well as its normative adequacy, we discuss and criticize the idea of informational economy both with respect to theories and with respect to richer structures representing belief states (identified with theory-revision guiding structures). This view is contrasted with an alternative view of cognitve economics that takes theory change to be a problem of rational choice based on complete and transitive preferences. Under this interpretation, theory revision models are indeed amenable to an essentially economic interpretation, but they inherit the criticism that has been levelled against the classical theory of choice in wider contexts.

Contributed Papers

Intersection Types and Computational Rules

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The invariance of the meaning of a λ-term by reduction/expansion w.r.t. the con-

sidered computational rules is one of the minimal requirements for a λ-model. Being the intersection type systems a general framework for the study of semantic domains for the Lambda-calculus, the present paper provides a characterisation of "meaning invariance" in terms of characterisation results for intersection type systems enabling typing invariance of terms w.r.t. various notions of reduction/expansion, like β, η and a number of relevant restrictions of theirs.

k-Valued Non-Associative Lambek Grammars are Learnable from Function-Argument Structures

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This paper is concerned with learning categorial grammars in the model of Gold. We show that rigid and k-valued non-associative Lambek grammars are learnable from function-argument structured sentences. In fact, function-argument structures are natural syntactical decompositions of sentences in sub-components with the indication of the head of each sub-component.

This result is interesting and surprising because for every k, the class of k-valued NL grammars has infinite elasticity and one could think that it is not learnable, which is not true. Moreover, these classes are very close to unlearnable classes like k-valued associative Lambek grammars learned from function-argument sentences or k-valued non-associative Lambek calculus grammars learned from well-bracketed list of words or from strings. Thus, the k-valued non-associative Lambek grammars learned from function-argument sentences is at the frontier between learnable and unlearnable classes of languages.

Lowness Properties of Reals and Hyper-Immunity

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Ambos-Spies and Kučera [Problem 4.5, AK] asked if there is a non-computable set A which is low for the computably random reals. We show that no such A is of hyper-immune degree. Thus, each $g \leq_T A$ is dominated by a computable function. Ambos-Spies and Kučera [Problem 4.8, AK] also asked if every S-low set is S_0-low. We give a partial solution to this problem, showing that no S-low set is of hyper-immune degree.

Gap Embedding for Well-Quasi-Orderings

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Given a quasi-ordering of labels, a labelled ordered tree s is *embedded with gaps* in another tree t if there is an injection from the nodes of s into those of t that maps each edge in s to a unique disjoint path in t with greater-or-equivalent labels, and which

preserves the order of children. We show that finite trees are well-quasi-ordered with respect to gap embedding when labels are taken from an arbitrary well-quasi-ordering such that each tree path can be partitioned into a bounded number of subpaths of comparable nodes. This extends Kříž's result [**?**] and is also optimal in the sense that unbounded incomparability yields a counterexample.

The Universe of Approximations

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The idea of *approximate entailment* has been proposed in [Schaerf-Cadoli] as a way of modeling the reasoning of an agent with limited resources. They proposed a system in which a family of logics, parameterized by a set of propositional letters, approximates classical logic as the size of the set increases.

In this paper, we take the idea further, extending two of their systems to deal with full propositional logic, giving them semantics and sound and complete proof methods based on tableaux. We then present a more general system of which the two previous systems are particular cases and show how it can be used to formalize heuristics used in theorem proving.

Cut Elimination in a Class of Sequent Calculi for Pure Type Systems

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This paper present a new sequent calculus for Pure Type Systems (PTS). The calculus proposed is equiconsistent to the standard formulation (*natural deduction like*). The corresponding cut-free fragment makes it possible to introduce a notion of Cut Elimination. This property can be applied to develop proof-search strategies with dependent types.

We prove that Cut Elimination holds in two important families of normalizing systems, including, in particular, three systems in the Barendregt's λ-cube: $l\rightarrow$, $l2$, and $l\underline{\omega}$. In addition, a cut elimination result is obtained for the minimal implicational second-order sequent calculus.

On Functional Dependencies in Advanced Data Models

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One dilemma in the database community is the great variety of data models existing. We define an abstract data model that captures most of the relevant data models depending on the underlying type system. An algebraic foundation for the investigation of dependencies is presented similar to the one which is easily available for the relational data model (RDM). This may lead to a unifying dependency theory. A generalisation of Armstrong's Axioms for the implication of functional dependencies in the RDM to our abstract data model is given. The inference rules look similar to Armstrong's original axioms, thanks to the algebraic framework. The completeness result, however, requires a much finer analysis of the inference rules than in the RDM.

Expressibility of Higher Order Logics

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We study the expressive power of higher order logics on finite relational structures or databases. First, we give a characterization of the expressive power of the fragments Σ^i_j and Π^i_j, for each order $i \geq 2$ and each number of alternations of quantifier blocks j. Then we get as a corollary the expressive power of HO^i for each order $i \geq 2$. From our results, as well as from the results of D. Leivant and of R. Hull and J. Su, it turns out that no higher order logic can be complete. Even if we consider the union of higher order logics of all natural orders, i.e., $\bigcup_{i\geq 2} HO^i$, we still do not get a complete logic. So, we define a logic which we call *variable order logic* (VO) which permits the use of untyped relation variables, i.e., variables of variable order, by allowing quantification over orders. We show that this logic is *complete.*

Quantifier-free logic for multialgebraic theories

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We develop a new quantifier-free logic for deriving consequences of multialgebraic theories. Multilagebras are used as models for nondeterminism in the context of algebraic specifications. They are many sorted algebras with *set valued* operations. Formulae are sequents over atoms allowing one to state set-inclusion or identity of 1-element sets (determinacy). We introduce a sound and complete Rasiowa-Sikorski logic for proving multilagebraic tautologies. We then extend this system for proving consequences of specifications based on translation of finite theories into logical formulae. Finally, we show how such a translation may be avoided – introduction of the *specific cut* rules leads to a sound and complete Gentzen system for proving directly consequences of specifications. The improvements over earlier logics for multialgebras concern mainly the ability to handle empty carriers (as well as empty result-sets) without the use of quantifiers, and to derive consequences of (potentially infinite) theories without the use of general cut.

A Programming Language for the Interval Geometric Machine

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This paper presents an interval version of the Geometric Machine Model (GMM) and the programming language induced by its structure. The GMM is an abstract machine model, based on Girard's coherence space, capable of modelling sequential, alternative, parallel (synchronous) and non-deterministic computations on a (possibly infinite) shared memory. The processes of the GMM are inductively constructed in a Coherence Space of Processes. The memory of the GMM, supporting a coherence

space of states, is conceived as the set of points of a three dimensional euclidian space. The version of the GMM presented here operates with real intervals, and is defined to model the semantics of algorithms of Interval Mathematics. Using the programming language induced by such structure, simple interval algorithms are presented, and their domain-theoretic semantics in the machine model is given.

System NEL is Undecidable

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System NEL is a conservative extension of multiplicative exponential linear logic (extended by the rules mix and nullary mix) by a self-dual noncommutative connective called *seq* which has an intermediate position between the connectives *par* and *times*. In this paper, I will show that system NEL is undecidable by encoding two counter machines into NEL. Although the encoding is simple, the proof of the faithfulness is a little intricate because there is no sequent calculus and no phase semantics available for NEL.

Fusion of Pedigreed Preferential Relations

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Belief fusion, instead of AGM belief revision, was first proposed to solve the problem of inconsistency, that arise from repetitive application of the operation when agents' knowledge were amalgamated. However in the theory, all the sources must be totally ordered and thus applicable area is quite restrictive. In this paper, we realize the belief fusion of multiple agents for partially ordered sources. When we consider such a partial ranking over sources, there is no need to restrict that each agent has total preorders over possible worlds. The preferential model allows each agent to have strict partial orders over possible worlds. Especially, such an order is called a preferential relation, that prescribes a world is more plausible than the other. We introduce an operation which combines multiple preferential relations of agents. In addition, we show that our operation can properly include the ordinary belief fusion.

A Tableau Method for the Lambek Calculus based on a Matrix Characterization

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We propose a tableau method for the Lambek Calculus by adapting a method developed by Mantel and Otten for the multiplicative exponential fragment of the Linear Logic (MELL). We have incorporated new elements to the language of the tableau system and a new restriction to its closure conditions to deal with the noncommutative feature of the Lambek Calculus, considering a labelling technique developed for the matrix characterization method.

Statistics of implicational logic

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In this paper we investigate the size of the fraction of tautologies of the given length n against the number of all formulas of length n for implicational logic. We are specially interested in asymptotic behavior of this fraction. We demonstrate the relation between a number of premises of implicational formula and asymptotic probability of finding formula with this number of premises. Furthermore we investigate the distribution of this asymptotic probabilities. Distribution for all formulas is contrasted with the same distribution for tautologies only. We prove those distributions to be so different that enable us to estimate likelihood of truth for a given long formula. Despite of the fact that all discussed problems and methods in this paper are solved by mathematical means, the paper may have some philosophical impact on the understanding how much the phenomenon of truth is sporadic or frequent in random logical sentences.

Some properties of intercategorial entailment

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An equivalent definition of the intercategorial entailment (i.e. an entailment between expressions of different but functionally related categories) is given and some other formal properties are established. These show that the atomicity of denotational algebras plays an essential role in the phenomenon of intercategorial entailment. Various possible applications to the semantics of non-declaratives are indicated. They suggest that intercategorial entailment, although formally different from generalized entailment and from presupposition is a generalisation of both of these notions.

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[1] The list includes the referees for the papers in this issue, plus the referees of papers rejected meanwhile.

Interest Group in Pure and Applied Logics (IGPL)

The Interest Group in Pure and Applied Logics (IGPL) is sponsored by The European Association for Logic, Language and Information (FoLLI), and currently has a membership of over a thousand researchers in various aspects of logic (symbolic, mathematical, computational, philosophical, etc.) from all over the world (currently, more than 50 countries). Our main activity is that of a research and information clearing house.

Our activities include:

- Exchanging information about research problems, references and common interest among group members, and among different communities in pure and applied logic.
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- Supplying review copies of books through the journals on which some of us are editors.
- Helping to organise exchange visits and workshops among members.
- Advising on papers for publication.
- Editing and distributing a Newsletter and a Journal (the first scientific journal on logic which is FULLY electronic: submission, refereeing, revising, typesetting, publishing, distribution; first issue: July 1993): the Logic Journal of the Interest Group on Pure and Applied Logics. (For more information on the Logic Journal of the IGPL, see the Web homepage: http://www.oup.co.uk/igpl)
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