TECHNISCHE
UNIVERSITÄT
DRESDEN

Dresden University of Technology
Institute for Theoretical Computer Science
Chair for Automata Theory

LTCS–Report

A finite basis for the set of $\mathcal{EL}$-implications holding in a finite model

Franz Baader, Felix Distel

LTCS-Report 07-02

Lehrstuhl für Automatentheorie
Institut für Theoretische Informatik
TU Dresden
http://lat.inf.tu-dresden.de

Hans-Grundig-Str. 25
01062 Dresden
Germany

A finite basis for the set of $\mathcal{EL}$-implications holding in a finite model

Franz Baader, Felix Distel

Inst. für Theoretische Informatik
TU Dresden
Germany
{baader,felix}@tcs.inf.tu-dresden.de

Abstract

Formal Concept Analysis (FCA) can be used to analyze data given in the form of a formal context. In particular, FCA provides efficient algorithms for computing a minimal basis of the implications holding in the context. In this paper, we extend classical FCA by considering data that are represented by relational structures rather than formal contexts, and by replacing atomic attributes by complex formulae defined in some logic. After generalizing some of the FCA theory to this more general form of contexts, we instantiate the general framework with attributes defined in the Description Logic (DL) $\mathcal{EL}$, and with relational structures over a signature of unary and binary predicates, i.e., models for $\mathcal{EL}$. In this setting, an implication corresponds to a so-called general concept inclusion axiom (GCI) in $\mathcal{EL}$. The main technical result of this report is that, in $\mathcal{EL}$, for any finite model there is a *finite* set of implications (GCIs) holding in this model from which all implications (GCIs) holding in the model follow.

Contents

1 Introduction

Classical Formal Concept Analysis [10] assumes that data from an application are given by a formal context, i.e., by a set of objects G, a set of attributes M, and an incidence relation I that states whether or not an object satisfies a certain attribute. To analyze the data given by such a context, FCA provides tools for computing a minimal basis for the implications between sets of attributes holding in the context [9, 11]. An implication $A \to B$ between sets of attributes A, B holds in a given context if all objects satisfying every attribute in A also satisfy every attribute in B. A classical result by Duquenne and Guiges [12] says that such a unique minimal basis always exists. If the set of attributes is finite, which is usually assumed, this basis is trivially finite as well.

From a model-theoretic or (first-order predicate) logical point of view, a formal context is a very simple relational structure where all predicates (the attributes) are unary. In many applications, however, data are given by more complex relational structures where objects can be linked by relations of arities greater than 1. In order to take these more complex relationships between objects into account when analyzing the data, we consider concepts defined in a certain logic rather than simply sets of atomic attributes (i.e., conjunctions of unary predicates). Intuitively, a concept is a formula with one free variable, and thus determines a subset of the domain (the extension of the concept) for any model of the logic used to construct these formulae. We show that, under certain conditions on this logic, many of the basic results from FCA can be extended to this more general framework. Basically, this requirement is that a finite set of objects (i.e., elements of the domain of a given model) always has a most specific concept describing these objects. The operator that goes from a finite set of objects to its most specific concept corresponds to the prime operator in classical FCA, which goes from a set of objects A to the set of attributes A' that all objects from the set have in common. The classical prime operator in the other direction, which goes from a set of attributes B to the set of objects B' satisfying all these attributes, has as its corresponding operator the one that goes from a concept to its extension.

We instantiate this general framework with concepts defined in the Description Logic $\mathcal{EL}$ [2, 3], i.e., formal contexts are replaced by finite models of this DL and attributes are $\mathcal{EL}$-concepts. Though being quite inexpressive, $\mathcal{EL}$ has turned out to be very useful for representing biomedical ontologies such as SNOMED [21] and the Gene Ontology [22]. A major advantage of using an inexpressive DL like $\mathcal{EL}$ is that it allows for efficient reasoning procedures [3, 5]. Actually, it turns out that $\mathcal{EL}$ itself does not satisfy the requirements on the logic needed to transfer results from FCA since objects need not have a most specific concept. However, if we extend $\mathcal{EL}$ to $\mathcal{EL}_{\mathrm{gfp}}$ by allowing for cyclic concept definitions interpreted with greatest fixpoint semantics, then the resulting logic satisfies all the necessary requirements. Implications in this setting correspond to so-called general concept inclusion axioms (GCIs), which are available in modern ontology languages such as OWL [13] and are supported by most DL systems [14].

The main technical result of this paper is that, in $\mathcal{EL}$ and in $\mathcal{EL}_{\mathrm{gfp}}$, the set of GCIs holding in a finite model always has a finite basis, i.e., although there are in general infinitely many such GCIs, we can always find a finite subset from which the rest follows. We construct such a finite basis first for $\mathcal{EL}_{\mathrm{gfp}}$, and then show how this basis can be modified to yield one for $\mathcal{EL}$. Due to the space limitation, we cannot give complete proofs of these results. They can be found in [4].

Related work. There have been previous approaches for dealing with more complex contexts involving relations between objects. So-called power context families [23] allow for the representation of relational structures by using a separate (classical) context for each arity, where the objects of the context for arity n are n-tuples. As such, power context families are just an FCA-style way of representing relational structures. In order to make use of the more complex relational structure given by power context families, Prediger [15, 17, 16] and Priss [18] allow the knowledge engineer to define new attributes, and provide means for handling the dependencies between the newly defined attributes and existing attributes by means of formal concept analysis. However, rather than considering all complex attributes definable by the logical language, as our approach does, they restrict the attention to finitely many attributes explicitly defined by the knowledge engineer.

Similar to our general framework, Ferré [6] considers complex attributes definable by some logical language. The equivalent of a formal context, called logical context in [6], associates a formula (i.e., a complex attribute) with each object. Since it is assumed that formulae form a join-semilattice, the formula associated with a set of objects is obtained as the join of the formulae associated with the elements of the set. Our general framework can be seen as an instance of the one defined in [6], where the association of formulae to (sets of) objects is defined using the semantics of the logic in question. However, Ferré's work does not consider implications, which is the main focus of the present paper (see [4] for a more detailed comparison of our approach with the one in [6]).

The work whose objectives are closest to ours is the one by Rudolph [19, 20], who considers attributes defined in the DL $\mathcal{FLE}$, which is more expressive than $\mathcal{EL}$. However, instead of using one generalized context with infinitely many complex attributes, he considers an infinite family of contexts, each with finitely many attributes, obtained by restricting the so-called role depth of the concepts. He then applies attribute exploration [7] to the classical contexts obtained this way, in each step increasing the role depths until a certain termination condition applies. Rudolph shows that, for a finite DL model, this termination condition will always be satisfied eventually. However, the set of implications computed for the context considered at that point does not appear to be a basis for all the GCIs holding in the given finite model, though it might be possible to modify Rudolph's approach such that it produces a basis in our sense. The main problem with this approach is, however, that the number of attributes grows very fast when the role depth grows (this number increases at least by one exponential in each step).

2 The general framework

In classical FCA, a formal context (G, M, I) consists of a set of objects G, a set of attributes M, and an incidence relation $I \subseteq G \times M$. Such a formal context induces two operators (both usually denoted by $\cdot'$), one mapping each set of objects A to the set of attributes A' these objects have in common, and the other mapping each set of attributes B to the set of objects satisfying these attributes. A formal concept is a pair (A, B) such that $A = B'$ and $B = A'$. The set A is the extensional description of the concept whereas B is its intensional description. The two $\cdot'$ operators form a Galois connection, and if applied twice yield closure operators $\cdot''$ on the set of objects and the set of attributes, respectively.

Since data sometimes cannot be described only in terms of objects and attributes it is desirable to allow more expressive intensional descriptions than simple sets of attributes. In our general framework, we assume that intensional descriptions of sets of objects are given by concept descriptions. A *concept description language* is a pair $(\mathcal{L}, \mathcal{I})$, where $\mathcal{L}$ is a set, whose elements are called *concept descriptions*, and $\mathcal{I}$ is a set of tuples $i = (\Delta_i, \cdot^i)$, called *models*, consisting of a non-empty set Δ_i (of objects) and a mapping

$$\cdot^i : \mathcal{L} \to \mathfrak{P}(\Delta_i) : f \mapsto f^i$$

that assigns an *extension* $f^i \subseteq \Delta_i$ to each concept description $f \in \mathcal{L}$.

Since in FCA the closure operator $\cdot''$ is used extensively for constructing a minimal basis of the implications in a context, one may wish to define similar operators in our framework. Intuitively, models correspond to formal contexts, and the operator $\cdot^i$ corresponds to the $\cdot'$ operator that assigns an extension B' to each set of attributes B. In order to define an analogon to the $\cdot'$ operator in the other direction, we introduce the subsumption preorder on concept descriptions: $f_1 \in \mathcal{L}$ *is subsumed by* $f_2 \in \mathcal{L}$ (written $f_1 \sqsubseteq f_2$) if $f_1^i \subseteq f_2^i$ for all models $i \in \mathcal{I}$. If $f_1 \sqsubseteq f_2$ and $f_2 \sqsubseteq f_1$, then we say that f_1 and f_2 are *equivalent* ($f_1 \equiv f_2$). Given a set of objects A in a formal context, its intensional description A' is the largest set of attributes B such that $A \subseteq B'$. Since $B_1' \subseteq B_2'$ if $B_1 \supseteq B_2$, such a largest set should correspond to the least one w.r.t. subsumption. This motivates the following definition.

Definition 1 (Most specific concept). *Let $i \in \mathcal{I}$ be a model and X a set $X \subseteq \Delta_i$. Then $f \in \mathcal{L}$ is a* most specific concept *for X iff*

$$X \subseteq f^i \tag{1}$$

and f is a least concept description with this property, i. e. every other concept description g with $X \subseteq g^i$ also satisfies $f \sqsubseteq g$.

Observe that most specific concepts need not exist. There may for example be an infinite descending chain of concept descriptions whose models contain X. If $(\mathcal{L}/_{\equiv}, \sqsubseteq)$ seen as a partially ordered set does not satisfy the descending chain condition then there need not be a least description $f \in \mathcal{L}$ with $X \subseteq f^i$. There may also be two (or more) such descriptions f_1 and f_2 that are minimal with respect to $\sqsubseteq$ but satisfy neither $f_1 \sqsubseteq f_2$ nor $f_2 \sqsubseteq f_1$. Whether most specific concepts exist largely depends on $\mathcal{L}$ and its semantics. For example for the language that is presented in Section 3.1 most specific concepts always exists, and it can be shown that there is a 1-1-correspondence to the $\cdot'$ operator from FCA. Another example for a language for which most specific concepts always exists is $\mathcal{EL}_{\mathrm{gfp}}$ as we will see in Section 3.2.

If the most specific concept of a set $X \subseteq \Delta_i$ exists it is unique up to equivalence. We denote it (or, more precisely, an arbitrary element of its equivalence class) by X^i. The concept description X^i is called the *intensional description* of the set of objects X.

The following lemma shows that

$$\cdot^i : \mathfrak{P}(\Delta_i) \to \mathcal{L}$$

and

$$\cdot^i : \mathcal{L} \to \mathfrak{P}(\Delta_i)$$

do indeed form a Galois-connection with FCA-style properties. Because of these similarities to FCA we will sometimes use the term *description context* for a model $i \in \mathcal{I}$.

Lemma 2. *Let $(\mathcal{L}, \mathcal{I})$ be a concept description language such that X^i exists for every $i \in \mathcal{I}$ and every $X \subseteq \Delta_i$. Let $i \in \mathcal{I}$ be a model $X, X_1, X_2 \in \Delta_i$ sets of objects and $f, f_1, f_2 \in \mathcal{L}$ concept descriptions. Then the following statements hold*

(a) $X_1 \subseteq X_2 \Rightarrow X_1^i \sqsubseteq X_2^i$

(b) $f_1 \sqsubseteq f_2 \Rightarrow f_1^i \subseteq f_2^i$

(c) $X \subseteq X^{ii}$

(d) $f^{ii} \sqsubseteq f$

(e) $X^i \equiv X^{iii}$

(f) $f^i = f^{iii}$

(g) $X \subseteq f^i \Leftrightarrow X^i \sqsubseteq f$.

Proof. This follows directly from Lemma 3.6 in [6]. Despite this and the fact that it is purely technical to prove, the prove will be given here for matters of completeness.

(a) By definition it is $X_2 \subseteq X_2^{ii}$ so we get $X_1 \subseteq X_2^{ii}$. Hence the claim follows from Definition 1, since X_1^i is the least concept description with the property $X_1 \subseteq (X_1^i)^i$

(b) Follows immediately from the definition of $f_1 \sqsubseteq f_2$.

(c) cf Definition 1.

(d) $f^i \subseteq f^{iii}$ holds by Definition 1. Obviously it is $f^i \subseteq f^i$. Hence $f^{ii} \sqsubseteq f$, since by Definition 1 f^{ii} is the least description with this property.

(e) $X^i \sqsupseteq X^{iii}$ follows directly from (d). $X^i \sqsubseteq X^{iii}$ follows from (c) and (a).

(f) This can be proved in an analogous way to (e).

(g) Let $X \subseteq f^i$. Then we get $X^i \sqsubseteq f^{ii}$ from (a) and thus $X^i \sqsubseteq f$ follows from (d). Conversely let $X^i \sqsubseteq f$. Then $X^{ii} \subseteq f^i$ holds and hence $X \subseteq f^i$ follows from (c).

□

As in Formal Concept Analysis one may define the set of formal concepts for a given model $i \in \mathcal{I}$ as the set of pairs $\{(X^{ii}, X^i) \mid X \subseteq \Delta_i\}$. Ferré has shown that these formal concepts form a complete lattice (cf Section 2.1, [6]). Since there is a 1-1-correspondence between complete lattices and formal contexts, one may argue that Definition 1 is not really an extension to Formal Concept Analysis. Although this is true in a way, our definition's main advantage is that the intensional descriptions that are needed to describe the concepts are obtained in a natural way (i. e. as most specific concepts). In classical FCA it is totally unclear which concept descriptions are relevant to describe the data. So in the worst case one might have to start with an infinite context, containing all possible concept descriptions as attributes.

In the remainder of this section, we assume that $(\mathcal{L}, \mathcal{I})$ is an arbitrary, but fixed, concept description language. All definitions given below are implicitly parameterized with this language. Our goal is to characterize the subsumption relations that are valid in a given description context of this language by determining a minimal basis of implications comparable to the Duquenne-Guiges basis in classical FCA. We start by defining the notion of an implication and by showing some general results that hold for arbitrary concept description languages. Later on, we will look at the concept description language $\mathcal{EL}_{\mathrm{gfp}}$ in more detail.

Definition 3 (Implication). *An* implication *is a pair (f_1, f_2) of concept descriptions $(f_1, f_2) \in \mathcal{L} \times \mathcal{L}$, which we will usually denote as $f_1 \to f_2$. We say that the implication $f_1 \to f_2$* holds *in the description context $\iota = (\Delta_\iota, \iota)$ if $f_1^\iota \subseteq f_2^\iota$.*

Obviously, we have $f_1 \sqsubseteq f_2$ iff $f_1 \to f_2$ holds in every description context $\iota \in \mathcal{I}$. However, as said above, we are now interested in the implications that hold in a fixed description context rather than in all of them.

In order to define the notion of a basis of the implications holding in a description context, we must first define a consequence operator on implications. Let $\mathcal{B} \subseteq \mathcal{L} \times \mathcal{L}$ be a set of implications and $f_1 \to f_2$ an implication. If $f_1 \to f_2$ holds in all description contexts $i \in \mathcal{I}$ in which all implications from $\mathcal{B}$ hold, then we say that $f_1 \to f_2$ *follows* from $\mathcal{B}$. It is not hard to see that the relation *follows* is

- *reflexive*, i. e. every implication $f_1 \to f_2 \in \mathcal{B}$ follows from $\mathcal{B}$, and
- *transitive*, i. e. if $f_1 \to f_2$ follows from $\mathcal{B}_2$, and every implication in $\mathcal{B}_2$ follows from $\mathcal{B}_1$, then $f_1 \to f_2$ follows from $\mathcal{B}_1$.

Definition 4 (Basis). *For a given description context ι we say that $\mathcal{B} \subseteq \mathcal{L} \times \mathcal{L}$ is a* basis *for the implications holding in ι if $\mathcal{B}$ is*

- sound *for ι, i.e., it contains only implications holding in ι;*
- complete *for ι, i.e., any implication that holds in ι follows from $\mathcal{B}$; and*
- minimal *for ι, i.e., no strict subset of $\mathcal{B}$ is complete for ι.*

Since the above definitions use only the $\cdot^\iota$ operator that assigns an extension to every concept description, but not the one in the other direction, they also make sense for concept description languages where the most specific concept of a set of objects need not always exist. An example of such a language is $\mathcal{EL}$, i.e., the sublanguage of $\mathcal{EL}_{\mathrm{gfp}}$ that does not allow for cyclic concept definitions (see Section 3.2 below).

The description language $(\mathcal{L}',\mathcal{I}')$ is a *sublanguage* of the description language $(\mathcal{L},\mathcal{I})$ if $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathcal{I}' = \left\{ i|_{\mathcal{L}'} \;\middle|\; i \in \mathcal{I} \right\}$, where $i|_{\mathcal{L}'}$ is the restriction of i to $\mathcal{L}'$, i.e., $\Delta_i = \Delta_{i|_{\mathcal{L}'}}$ and $\cdot^{i|_{\mathcal{L}'}}$ is the restriction of the mapping $\cdot^i$ to $\mathcal{L}'$.

Proposition 5. *Assume that $(\mathcal{L}',\mathcal{I}')$ is a sublanguage of $(\mathcal{L},\mathcal{I})$, that $f_1 \to f_2 \in \mathcal{L}' \times \mathcal{L}'$, and that $\mathcal{B} \subseteq \mathcal{L}' \times \mathcal{L}'$. Then $f_1 \to f_2$ follows from $\mathcal{B}$ in $(\mathcal{L},\mathcal{I})$ iff $f_1 \to f_2$ follows from $\mathcal{B}$ in $(\mathcal{L}',\mathcal{I}')$.*

Proof. $f^i = f^{i|_{\mathcal{L}'}}$ holds for all $f \in \mathcal{L}'$ and all $i \in \mathcal{I}$. Therefore an implication $g_1 \to g_2$ holds in the $\mathcal{L}$-description context i if and only if it holds in the $\mathcal{L}'$-description context $i|_{\mathcal{L}'}$. The claim follows directly from this fact. □

In the remainder of this section, we will characterize complete subsets of the set of all implications holding in a given description context ι. Whenever we use the $\cdot^\iota$ operator from sets of objects to concept descriptions, we implicitly assume that it is defined. By definition X^{I} is the most precise concept description such that X is contained in its extension. One can even say that it captures all the information about X that can be expressed in $\mathcal{L}$. This is the reason why we can restrict ourselves to implications that only contain implications whose right hand sides are of the form f^{II} for some $f \in \mathcal{L}$.

Lemma 6. *If the implication $f_1 \to f_2$ holds in ι, then it follows from $\{f_1 \to f_1^{\iota\iota}\}$, and the set $\{f_1 \to f_1^{\iota\iota}\}$ is sound for ι.*

Proof. By Lemma 2(f), all implications of the form $f \to f^{\iota\iota}$ hold in ι, which yields soundness of $\{f_1 \to f_1^{\iota\iota}\}$.

Let $f_1 \to f_2$ be any implication that holds in ι. Then by definition $f_1^\iota \subseteq f_2^\iota$ holds. By Lemma 2 (g) this is equivalent to

$$f_1^{\iota\iota} \sqsubseteq f_2. \tag{2}$$

Let j be some model in which $f_1 \to f_1^{\iota\iota}$ holds. By definition this implies that $f_1^j \subseteq (f_1^{\iota\iota})^j$ is true. Using Lemma 2 (g) again we get

$$f_1^{jj} \sqsubseteq f_1^{\iota\iota}. \tag{3}$$

From (2) we get

$$f_1^{jj} \sqsubseteq f_2. \tag{4}$$

and hence $f_1^j \subseteq f_2^j$. So $f_1 \to f_2$ holds in j. □

Corollary 7. *The set of implications*

$$\{f \to f^{\iota\iota} \mid f \in \mathcal{L}\}$$

is sound and complete in ι.

Having reduced the number of right hand sides that are needed to construct a complete set of implications, one may wonder whether something similar can be done for the left hand sides as well. This is possible if we can find a so-called dominating set of concept descriptions.

Definition 8 (dominating sets of concept descriptions). *Let $\mathcal{D} \subseteq \mathcal{L}$ be a set of concept descriptions. We say that $\mathcal{D}$* dominates *the description context ι iff for every $f \in \mathcal{L}$ there is some $\bar{f} \in \mathcal{D}$ such that*

$$f \sqsubseteq \bar{f}$$

and

$$f^\iota = \bar{f}^\iota.$$

It is sufficient to consider implications whose left-hand sides belong to a dominating set.

Lemma 9. *Let $\mathcal{D} \subseteq \mathcal{L}$ be a set that dominates ι. Then*

$$\mathcal{B} = \{f \to f^{\iota\iota} \mid f \in \mathcal{D}\}$$

is sound and complete for ι.

Proof. Soundness has already been shown. To show completeness, let $f_1 \to f_2$ be an implication that holds in ι. Lemma 6 states that $f_1 \to f_2$ follows from $f_1 \to f_1^{\iota\iota}$. Hence it is sufficient to show that $f_1 \to f_1^{\iota\iota}$ follows from $\mathcal{B}$. Since $\mathcal{D}$ dominates ι there exists $g \in \mathcal{D}$, such that $g^\iota = f_1^\iota$ and $f_1 \sqsubseteq g$.

Let j be a model in which all implications of $\mathcal{B}$ hold. From $f_1 \sqsubseteq g$ and Lemma 2 it follows that

$$f_1^j \subseteq g^j. \tag{5}$$

As $g \to g^{\iota\iota} \in \mathcal{B}$ holds in j, we have $g^j \subseteq (g^{\iota\iota})^j$. Thus

$$f_1^j \subseteq (g^{\iota\iota})^j. \tag{6}$$

On the other hand $g^\iota = f_1^\iota$ implies that $g^{\iota\iota} = f_1^{\iota\iota}$, and so

$$f_1^j \subseteq (f_1^{\iota\iota})^j. \tag{7}$$

Hence $f_1 \to f_1^{\iota\iota}$ holds in j. □

2.1 Related Work

A similar definition to Definition 1 can be found in Ferré [6] and shall briefly be explained here. Like us, Ferré starts with some logic $\mathcal{L}$ and a preorder $\sqsubseteq$. Then $\sqsubseteq$ induces a partial order on the set of equivalence classes of $\equiv$, i. e. $(\mathcal{L}/\equiv, \sqsubseteq)$ is a partially ordered set. If the least upper bound of two such equivalence classes $[f_1]$ and $[f_2]$ exists in $(\mathcal{L}/\equiv, \sqsubseteq)$, we call this bound the *least common subsumer* of $[f_1]$ and $[f_2]$. For matters of simplicity we may sometimes write f_1 when we actually mean the equivalence class $[f_1]$. Similarly we will denote the least common subsumer of $[f_1]$ and $[f_2]$ simply by $\mathrm{lcs}(f_1, f_2)$. We define least common subsumers for arbitray sets of concept descriptions analogously and denote these by $\mathrm{lcs}_{k \in K} f_k$.

Definition 10 (from [6]). *A* logical context *is a triple* $K = (O, \mathcal{L}, d)$ *where*

- O *is a finite set of objects,*
- $\mathcal{L}$ *is a logic, such that* $(\mathcal{L}/\equiv, \sqsubseteq)$ *forms a join-semilattice,*
- d *is a mapping*
$$d : O \to \mathcal{L}$$
that associates to every object $o \in O$ *a concept description* $d(o) \in \mathcal{L}$.

In a logical context Ferré defines the mappings

$$\sigma_K : \mathfrak{P}(O) \to \mathcal{L},\ \sigma_K(A) = \mathrm{lcs}_{o \in A}\, d(o)$$
$$\tau_K : \mathcal{L} \to \mathfrak{P}(O),\ \tau_K(f) = \{o \in O \mid d(o) \sqsubseteq f\}.$$

The most striking difference between Ferré's definition and ours is that in Ferré's work the concept descriptions that are associated to singleton sets $\{x\}$ can be chosen arbitrarily. We can show that Definition 1 is a special case of Definition 10, if we choose an appropriate function d. Let $\mathcal{L}$ and i be such that $\{x\}^i$ exists for all $x \in \Delta_i$. If we define $O = \Delta_i$, and $d(x) = \{x\}^i$ for all $x \in \Delta_i$ then the two definitions 10 and 1 match for all singletons $\{x\}$, i. e. $\sigma_K(\{x\}) = \{x\}^i$ for all $x \in \Delta_i$. The following results show that they also match for arbitrary sets instead of singletons and that there is a similar correspondence for τ_K. The correspondence for τ_K is not hard to see:

Corollary 11. *Let* $\mathcal{L}$ *and* i *be such that* $\{x\}^i$ *exists for all* $x \in \Delta_i$. *Let* $d(x)$ *be defined as above. Then*
$$f^i = \tau_K(f).$$

Proof.

$$x \in f^i \Leftrightarrow \{x\} \subseteq f^i \overset{\text{Lemma 2 (g)}}{\Leftrightarrow} \{x\}^i \sqsubseteq f \Leftrightarrow d(x) \sqsubseteq f \overset{\text{Def. 10}}{\Leftrightarrow} x \in \tau_K(f)$$

□

The following proposition shows that the two definitions match for sets of arbitrary cardinality.

Proposition 12. *Let* $\mathcal{L}$ *be a language and* i *a model. Let* $\{X_m\}_{m \in M}$ *be a family of sets* $A_m \subseteq \Delta_i$ *for which* A_m^i *exists for all* $m \in M$. *Then* $\mathrm{lcs}_{m \in M} A_m^i$ *exists iff* $(\bigcup_{m \in M} A_m)^i$ *exists. In this case*
$$\mathrm{lcs}_{m \in M} A_m^i = \Big(\bigcup_{m \in M} A_m\Big)^i.$$

Proof. First assume that $f = \mathrm{lcs}_{m \in M} A_m^i$ exists. Then f by definition subsumes all concept descriptions A_m^i. Therefore

$$f \sqsupseteq A_m^i \quad \forall m \in M.$$

So by Lemma 2 (g)
$$f^i \supseteq A_m \quad \forall m \in M$$

and thus

$$f^i \supseteq \bigcup_{m \in M} A_m.$$

Now let $g \in \mathcal{L}$ be another concept description such that

$$g^i \supseteq \bigcup_{m \in M} A_m.$$

Using the same arguments as above, but in the other direction, we get that

$$g \sqsupseteq A_m^i \quad \forall m \in M,$$

i. e. g is an upper bound for the A_m^i. Since f by definition is the least upper bound for the A_m^i, we get $f \sqsubset g$. So we have shown that $f^i \supseteq \bigcup_{m \in M} A_m$ and that for every other concept description g with $g^i \supseteq \bigcup_{m \in M} A_m$ we have $f \sqsubset g$. By Defintion 1 it follows that $f = (\bigcup_{m \in M} A_m)^i$. The other direction can be shown analogously. □

Corollary 11 and Proposition 12 show that if we define $d(x) = \{x\}^i$, Definitions 1 and 10 match, in the sense that $\sigma_K(A) = A^i$ for all $A \subseteq \Delta_i$ and $\tau_K(f) = f^i$ for all $f \in \mathcal{L}$. So Definition 1 is in fact a specialisation of Definition 10. The main reason why we restrict ourselves to Definition 1 is that it uses the semantics of $\mathcal{L}$ in a natural way, whereas Definition 10 does not use it at all. In fact since semantics are not used in Ferré's definition it would even suffices to use any join-semilattice $(P, \leq)$ instead of $(\mathcal{L}/\equiv, \sqsubseteq)$. This has been done by Ganter and Kuznetsov in [8].

Proposition 12 also provides a criterion for the existence of the $\cdot^i$ operator:

Corollary 13. *Let $\mathcal{L}$ be a language and $i \in \mathcal{I}$ a model. Then A^i exists for all sets $A \subseteq \Delta_i$ iff*

- *$\{x\}^i$ exists for every $x \in \Delta_i$, and*
- *$\mathrm{lcs}_{a \in A}\{a\}^i$ exists for all $A \subseteq \Delta_i$.*

3 Instances of the general framework

3.1 Classical FCA

In this section we show how classical FCA can be obtained as a special case of the above definitions. We define a language $\mathcal{L}_{\mathrm{FCA}}$ and an appropriate semantics such that the operators $\cdot^I$ behave like the operators $\cdot'$. In classical FCA concepts are described intentionally by listing all properties that are common to a group of objects. Therefore we define the language $\mathcal{L}_{\mathrm{FCA}}$ to be

$$\mathcal{L}_{\mathrm{FCA}} = \mathfrak{P}(M)$$

for a fixed set of attributes M. We define a *primitive interpretation* to be a mapping

$$i : M \to \mathfrak{P}(\Delta_i).$$

For every such primitive interpretation we can define an extension (denoted by $\cdot^i$) as follows

$$\begin{aligned} \cdot^i : \mathcal{L}_{\mathrm{FCA}} &\to \mathfrak{P}(\Delta_i) \\ B &\mapsto \bigcap_{m \in B} i(m). \end{aligned}$$

As the set of models $\mathcal{I}_{\mathrm{FCA}}$ we use the set of all mappings that can be obtained as such an extension of some primitive interpretation. However, observe that larger sets of attributes yield narrower extensions. Hence the direction of the inclusion is reversed, when we view the attribute sets as concept descriptions.

Proposition 14. *Let $A, B \in \mathcal{L}_{\mathrm{FCA}}$. Then $A \subseteq B$ as sets iff $A \sqsupseteq B$ as concept descriptions.*

Proof. Suppose $A \subseteq B$. Then for every $i \in \mathcal{I}_{\mathrm{FCA}}$

$$\begin{aligned} B^i &= \bigcap_{m \in B} \{m\}^i \\ &= \bigcap_{m \in A} \{m\}^i \cap \bigcap_{m \in B \setminus A} \{m\}^i \\ &\subseteq \bigcap_{m \in A} \{m\}^i \\ &= A^i. \end{aligned}$$

Hence $B \sqsubseteq A$.

Now suppose $B \sqsubseteq A$. Let $i_\star \in \mathcal{I}_{\mathrm{FCA}}$ be the extension of the primitive interpretation $i_\star$ with the domain $\Delta_{i_\star} = M$ and $i_\star(m) = M \setminus \{m\}$ for every $m \in M$. $B \sqsubseteq A$ implies $B^{i_\star} \subseteq A^{i_\star}$. Thus

$$\begin{aligned} B^{i_\star} &\subseteq A^{i_\star} \\ \bigcap_{m \in B} i_\star(m) &\subseteq \bigcap_{m \in A} i_\star(m) \\ \bigcap_{m \in B} M \setminus \{m\} &\subseteq \bigcap_{m \in A} M \setminus \{m\} \\ M \setminus B &\subseteq M \setminus A \\ B &\supseteq A. \end{aligned}$$

□

Using the language and semantics defined above we obtain classical FCA from Definition 1. Before we can prove this, we need to show that the operators $\cdot^i$ are well-defined. With the above semantics A^i exists for every $i \in \mathcal{I}$ and every $A \subseteq \Delta_i$. More precisely we get

$$A^i = \{m \in M \mid A \subseteq \{m\}^i\},$$

because then

$$A \subseteq \bigcap_{m \in M : A \subseteq \{m\}^i} \{m\}^i = A^{ii}$$

and for every set $B \subseteq M$

$$\begin{aligned} A \subseteq B^i \Leftrightarrow & A \subseteq \bigcap_{m \in B} \{m\}^i \\ \Leftrightarrow & \forall m \in B : A \subseteq \{m\}^i \\ \Leftrightarrow & \forall m \in B : m \in \{\mu \in M \mid A \subseteq \{\mu\}^i\} \\ \Leftrightarrow & B \subseteq A^i \\ \Leftrightarrow & B \sqsupseteq A^i. \end{aligned}$$

Now every model $i \in \mathcal{I}_{\mathrm{FCA}}$ corresponds to some classical FCA-context (G_i, M, I_i) where $G_i = \Delta_i$ and $I_i = \{(x, m) \mid x \in \{m\}^i\}$. For then for all $A \subseteq \Delta_i$ we get

$$\begin{aligned} A^i = \{m \in M \mid A \subseteq \{m\}^i\} &= \{m \in M \mid \forall x \in A : x \in \{m\}^i\} \\ &= \{m \in M \mid \forall x \in A : xI_im\} = A' \end{aligned}$$

and for all $C \subseteq M$, we get

$$C^i = \bigcap_{m \in C} \{m\}^i = \bigcap_{m \in C} \{x \in G \mid xI_im\} = \{x \in G \mid \forall m \in C : xI_im\} = C'.$$

Conversely every FCA-context (G, M, I) corresponds to a model $i_I \in \mathcal{I}_{\mathcal{FCA}}$ where we define $\Delta_{i_I} = G$ and $\{m\}^{i_I} = \{g \in G \mid gIm\}$. For all $A \subseteq \Delta_i$ we get

$$\begin{aligned} A' = \{m \in M \mid \forall x \in A : xIm\} &= \{m \in M \mid \forall x \in A : x \in \{m\}^{i_I}\} \\ &= \{m \in M \mid A \subseteq \{m\}^{i_I}\} = A^{i_I} \end{aligned}$$

and for all $C \subseteq M$, we get

$$C' = \{x \in G \mid \forall m \in C : xIm\} = \bigcap_{m \in C} \{x \in G \mid xIm\} = \bigcap_{m \in C} \{m\}^{i_I} = C^{i_I}.$$

This shows that classical FCA can be expressed in terms of description contexts.

It is well-known that for implications in classical FCA, we can always find a set of implications which is not just complete and irredundant, but also minimal with respect to the number of implications in the basis. This set is called the Duquenne-Guiges-basis [12]. It is constructed using so-called pseudo-intents.

Definition 15. *$P \subseteq M$ is called a pseudo-intent of $i \in \mathcal{I}_{\mathrm{FCA}}$ iff $P \neq P^{ii}$ and $P \sqsubseteq Q^{ii}$ holds for every pseudo-intent $Q \sqsupseteq P$, $Q \neq P$.*

Theorem 16. *The set of implications*

$$\mathcal{L} = \{P \to P^{ii} \mid P \text{ pseudo-intent}\}$$

is irredundant and complete.

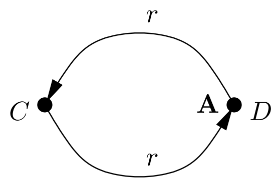

Figure 1: Example of a simple $\mathcal{EL}$-description graph

This is proved in [12] and [10]. There are two major problems, why the concept of a Duquenne-Guiges-basis cannot be extended to most languages other than classical FCA. First, for most languages the lattice $(\mathcal{L}/\equiv, \sqsubseteq)$ does not satisfy the ascending chain condition. Therefore pseudo-intents cannot be defined recursively as in Definition 15. Another major issue arises from the fact that '*follows*' in FCA can be characterised like this:

Proposition 17. *$A \to B$ follows from a set of $\mathcal{L}_{\mathrm{FCA}}$-implications $\mathcal{B}$ iff for every $E \in \mathcal{L}_{\mathrm{FCA}}$ with*

$$\forall C \to D \in \mathcal{B} : E \sqsubseteq C \Rightarrow E \sqsubseteq D$$

we also have

$$E \sqsubseteq A \Rightarrow E \sqsubseteq B.$$

This proposition does not necessarily hold for other description languages than $\mathcal{L}_{\mathrm{FCA}}$. However, since it is crucial in proving the non-redundance of the Duquenne-Guigues-basis, we need to find other ways to determine non-redundant implication bases.

3.2 $\mathcal{EL}$ with terminological cycles and greatest fixpoint semantics

We start by defining $\mathcal{EL}$, and then show how it can be extended to $\mathcal{EL}_{\mathrm{gfp}}$. Concept descriptions of $\mathcal{EL}$ are built from a set $\mathcal{N}_c$ of concept names and a set $\mathcal{N}_r$ of role names, using the constructors top concept, conjunction, and existential restriction:

- concept names and the top concept $\top$ are $\mathcal{EL}$-concept descriptions;
- if C, D are $\mathcal{EL}$-concept descriptions and r is a role name, then $C \sqcap D$ and $\exists r.C$ are $\mathcal{EL}$-concept descriptions.

In the following, we assume that the sets $\mathcal{N}_c$ and $\mathcal{N}_r$ of concept and role names are finite. This assumption is reasonable since in practice data are usually represented over a finite signature.

Models of this language are pairs $(\Delta_I, \cdot^I)$ where Δ_I is a finite,[1] non-empty set, and $\cdot^I$ maps role names r to binary relations $r^I \subseteq \Delta_I \times \Delta_I$ and $\mathcal{EL}$-concept descriptions to

[1]Usually, the semantics given for description logics allows for models of arbitrary cardinality. However, in the case of $\mathcal{EL}$ the restriction to finite models is without loss of generality since it has the finite model property, i.e., a subsumption relationship holds w.r.t. all models iff it holds w.r.t. all finite models.

subsets of Δ_I such that

$$\top^I = \Delta_I, \qquad (C \sqcap D)^I = C^I \cap D^I, \quad \text{and}$$
$$(\exists r.C)^I = \{d \in \Delta_i \mid \exists e \in C^I \text{ such that } (d,e) \in r^I\}.$$

Subsumption and equivalence between $\mathcal{EL}$-concept descriptions is defined as in our general framework, i.e., $C \sqsubseteq D$ iff $C^I \sqsubseteq D^I$ for all models I, and $C \equiv D$ iff $C \sqsubseteq D$ and $D \sqsubseteq C$.

Unfortunately, $\mathcal{EL}$ itself cannot be used to instantiate our framework since in general a set of objects need not have a most specific concept in $\mathcal{EL}$. This is illustrated by the following simple example. Assume that $\mathcal{N}_c = \{P\}$, $\mathcal{N}_r = \{r\}$, and consider the model I with $\Delta_I = \{a,b\}$, $r^I = \{(a,b),(b,a)\}$, and $P^I = \{b\}$ (cf Fig. 1 for a graphical representation of this model). To see that the set $\{a\}$ does not have a most specific concept, consider the $\mathcal{EL}$-concept descriptions

$$C_k := \underbrace{\exists r.\exists r \dots \exists r.}_{k \text{ times}} \top.$$

We have $\{a\} \subseteq C_k^I = \{a,b\}$ for all k, and thus a most specific concept C for $\{a\}$ would need to satisfy $C \sqsubseteq C_k$ for all $k \geq 0$. However, it is easy to see that $C \sqsubseteq C_k$ can only be true if the role depth of C, i.e., the maximal nesting of existential restrictions, is at least k. Since any $\mathcal{EL}$-concept description has a finite role depth, this shows that such a most specific concept C cannot exist.

However, most specific concepts always exist in $\mathcal{EL}_{\text{gfp}}$, the extension of $\mathcal{EL}$ by cyclic concept definitions interpreted with greatest fixpoint (gfp) semantics.[2] In $\mathcal{EL}_{\text{gfp}}$, we assume that the set of concept names is partitioned into the set $\mathcal{N}_{\text{prim}}$ of primitive concepts and the set $\mathcal{N}_{\text{def}}$ of defined concept. A *concept definition* is of the form

$$B_0 \equiv P_1 \sqcap \dots \sqcap P_m \sqcap \exists r_1.B_1 \sqcap \dots \sqcap \exists r_n.B_n$$

where $B_0, B_1, \dots, B_n \in \mathcal{N}_{\text{def}}$, $P_1, \dots, P_m \in \mathcal{N}_{\text{prim}}$, and $r_1, \dots, r_n \in \mathcal{N}_r$. The empty conjunction (i.e., $m = 0 = n$) stands for $\top$. A *TBox* is a finite set of concept definitions such that every defined concept occurs at most once as a left-hand side of a concept definition.

Definition 18 ($\mathcal{EL}_{\text{gfp}}$-concept description). *An* $\mathcal{EL}_{\text{gfp}}$-concept description *is a tuple* $(A, \mathcal{T})$ *where* $\mathcal{T}$ *is a TBox and* A *is a defined concept occurring on the left-hand side of a definition in* $\mathcal{T}$.

For example, $(A, \mathcal{T})$ with $\mathcal{T} := \{A \equiv \exists r.B, B \equiv P \sqcap \exists r.A\}$ is an $\mathcal{EL}_{\text{gfp}}$-concept description. Any $\mathcal{EL}_{\text{gfp}}$-concept description $(A, \mathcal{T})$ can be represented by a directed, rooted, edge- and node-labeled graph: the nodes of this graph are the defined concepts in $\mathcal{T}$, with A being the root; the edge label of node B_0 is the set of primitive concepts occurring in the definition of B_0; and every conjunct $\exists r_i.B_i$ in the definition of B_0 gives rise to an edge from B_0 to B_i with label r_i. In the following, we call such graphs

[2] Because of the space restriction, we can only give a very compact introduction of this DL. See [1, 4] for more details.

description graphs. The description graph associated with the $\mathcal{EL}_{\text{gfp}}$-concept description from our example is shown in Fig. 1, where A is the root.

Models of $\mathcal{EL}_{\text{gfp}}$ are of the form $I = (\Delta_I, \cdot^I)$ where Δ_I is a finite, non-empty set, and $\cdot^I$ maps role names r to binary relations $r^I \subseteq \Delta_I \times \Delta_I$ and primitive concepts to subsets of Δ_I. The mapping $\cdot^I$ is extended to $\mathcal{EL}_{\text{gfp}}$-concept descriptions $(A, \mathcal{T})$ by interpreting the TBox $\mathcal{T}$ with gfp-semantics: consider all extensions of I to the defined concepts that satisfy the concept definitions in $\mathcal{T}$, i.e., assign the same extension to the left-hand side and the right-hand side of each definition. Among these extensions of I, the *gfp-model of* $\mathcal{T}$ *based on* I is the one that assigns the largest sets to the defined concepts (see [1] for a more detailed definition of gfp-semantics). The *extension* $(A, \mathcal{T})^I$ *of* $(A, \mathcal{T})$ in I is the set assigned to A by the gfp-model of $\mathcal{T}$ based on I.

Again, subsumption and equivalence of $\mathcal{EL}_{\text{gfp}}$-concept descriptions is defined as in the general framework.

Let $\mathcal{U} = (R_\mathcal{U}, \mathcal{T}_\mathcal{U}) \in \mathcal{EL}_{\text{gfp}}$ and $\mathcal{V} = (R_\mathcal{V}, \mathcal{T}_\mathcal{V}) \in \mathcal{EL}_{\text{gfp}}$ be two concept descriptions. Then we write $\exists r.\mathcal{U}$ as an abbreviation for the pair $(R_{\exists r.\mathcal{U}}, \mathcal{T}_{\exists r.\mathcal{U}})$, where without loss of generality $R_{\exists r.\mathcal{U}}$ is a concept name that does not occur in $\mathcal{T}_\mathcal{U}$ and

$$\mathcal{T}_{\exists r.\mathcal{U}} = \mathcal{T}_\mathcal{U} \cup \{R_{\exists r.\mathcal{U}} \equiv \exists r.R_\mathcal{U}\}.$$

The concept description $\mathcal{U} \sqcap \mathcal{V} = (R_{\mathcal{U}\sqcap\mathcal{V}}, \mathcal{T}_{\mathcal{U}\sqcap\mathcal{V}})$ is defined similarly. First assume without loss of generality that the sets of defined concept names in $\mathcal{U}$ and $\mathcal{V}$ are disjoint. We define a new TBox $\mathcal{T}_{\mathcal{U}\sqcap\mathcal{V}}$ as follows

$$\mathcal{T}_{\mathcal{U}\sqcap\mathcal{V}} = \mathcal{T}_\mathcal{U} \cup \mathcal{T}_\mathcal{V} \cup \{R_{\mathcal{U}\sqcap\mathcal{V}} \equiv \bigsqcap_{i=1}^{k} A_i \sqcap \bigsqcap_{i=1}^{l} C_i \sqcap \bigsqcap_{i=1}^{m} B_i \sqcap \bigsqcap_{i=1}^{n} D_i\}\,,$$

where

$$R_\mathcal{U} = \bigsqcap_{i=1}^{k} A_i \sqcap \bigsqcap_{i=1}^{l} C_i$$

and

$$R_\mathcal{V} = \bigsqcap_{i=1}^{m} B_i \sqcap \bigsqcap_{i=1}^{n} D_i$$

with primitive concept names A_i, B_i and defined concept names C_i, D_i. Then the semantics behave like we know it from $\mathcal{EL}$, i. e. for all $I \in \mathcal{I}$

$$(\exists r.\mathcal{U})^I = \{x \in \Delta_I \mid \exists y \in \mathcal{U}^I : (x, y) \in r^I\} \tag{8}$$

and

$$(\mathcal{U} \sqcap \mathcal{V})^I = \mathcal{U}^I \cap \mathcal{V}^I\,. \tag{9}$$

Using $\mathcal{EL}_{\text{gfp}}$ the most specific concept $\{a\}^I$ exists for the simple example in the beginning of the chapter. However it is still unclear whether most specific concepts exist for all sets $X \subseteq \Delta_I$ and all models $I \in \mathcal{I}$. To show this, we need some definitions and results from Baader [2]. Baader shows how instance and subsumption relations in $\mathcal{EL}_{\text{gfp}}$ can be characterised using so called $\mathcal{EL}$-description graphs and simulations of such graphs.

Definition 19 ($\mathcal{EL}$-description graphs). *An $\mathcal{EL}$-description graph is a graph $\mathcal{G} = (V, E, L)$ where*

- *V is a set of nodes*
- *$E \subseteq V \times \mathcal{N}_{\text{role}} \times V$ is a set of directed edges labeled by role names*
- *$L : V \to \mathfrak{P}(\mathcal{N}_{\text{prim}})$ is a labeling function*

For a normalized $\mathcal{EL}$-TBox $\mathcal{T}$ the corresponding $\mathcal{EL}$-description graph $\mathcal{G}_{\mathcal{T}}$ is the graph $\mathcal{G} = (V_{\mathcal{T}}, E_{\mathcal{T}}, L_{\mathcal{T}})$ where

- *the vertices of $\mathcal{G}_{\mathcal{T}}$ are the defined concepts of $\mathcal{T}$*
- *if A is a defined concept and*

$$A \equiv P_1 \sqcap \ldots \sqcap P_m \sqcap \exists r_1.B_1 \sqcap \exists r_l.B_l$$

  its definition in $\mathcal{T}$, then

  - *$L_{\mathcal{T}}(A) = \{P_1, \ldots, P_m\}$, and*
  - *A is the source of the edges (A, r_1, B_1), $\ldots$, $(A, r_2, B_l) \in E_{\mathcal{T}}$.*

Conversely, every $\mathcal{EL}$-description graph can be transformed into an $\mathcal{EL}$-TBox.

A model I can also be transformed into an $\mathcal{EL}$-description graph.

- *The vertices of $\mathcal{G}_I$ are the elements of Δ_I.*
- *$E_I = \{(x, r, y) \mid (x, y) \in r^I\}$*
- *$L_I(x) = \{P \in \mathcal{N}_{\text{prim}} \mid x \in P^I\}$ for all $x \in \Delta_I$.*

Definition 20 (Simulation). *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two $\mathcal{EL}$-description graphs. The binary relation $Z \subseteq V_1 \times V_2$ is a* simulation *from $\mathcal{G}_1$ to $\mathcal{G}_2$ iff*

(a) $(v_1, v_2) \in Z$ implies $L_1(v_1) \subseteq L_2(v_2)$, and

(b) if $(v_1, v_2) \in Z$ and $(v_1, r, v_1') \in E_1$, then there exists a node $v_2' \in V_2$ such that $(v_1', v_2') \in Z$ and $(v_2, r, v_2') \in E_2$.

We write $Z : \mathcal{G}_1 \overrightarrow{\sim} \mathcal{G}_2$ to express that Z is a simulation from $\mathcal{G}_1$ to $\mathcal{G}_2$.

Then instance relations in a given model can be characterised as follows.

Proposition 21. *Let $I \in \mathcal{I}$ be a gfp-model. Then the following are equivalent for any $\mathcal{U} = (A, \mathcal{T}) \in \mathcal{EL}_{\text{gfp}}$ and $x \in \Delta_I$.*

- *$x \in \mathcal{U}^I$*
- *There is a simulation $Z : \mathcal{G}_{\mathcal{T}} \overrightarrow{\sim} \mathcal{G}_I$ such that $(A, x) \in Z$.*

This result eventually leads to the following theorem which characterises subsumption.

Theorem 22. *Let $\mathcal{U}_1 = (A_1, \mathcal{T}_1), \mathcal{U}_2 = (A_2, \mathcal{T}_2) \in \mathcal{EL}_{\text{gfp}}$. Then the following two statements are equivalent.*

- $\mathcal{U}_1 \sqsubseteq \mathcal{U}_2$
- *There is a simulation $Z : \mathcal{G}_{\mathcal{T}_2} \overrightarrow{\sim} \mathcal{G}_{\mathcal{T}_1}$ such that $(A_2, A_1) \in Z$.*

Both results have been proved by Baader in [2]. We are now able to prove the existence of most specific concepts in $\mathcal{EL}_{\text{gfp}}$.

Corollary 23. *Let $I \in \mathcal{I}$ be a model and $x \in \Delta_I$. Then $(x, \mathcal{T}_x) \in \mathcal{EL}_{\text{gfp}}$ where $\mathcal{T}_x$ is the TBox defined by $\mathcal{G}_I$ is the most specific concept of x.*

Proof. As $\mathcal{G}_I = \mathcal{G}_{\mathcal{T}_x}$ it is obvious that the identity relation $\text{id}_{\mathcal{G}_I}$ satisfies the conditions of Proposition 21. Hence $x \in \mathcal{T}_x^I$. Now assume that there is another $\mathcal{EL}_{\text{gfp}}$-concept description $(A, \bar{\mathcal{T}})$ such that $x \in (A, \bar{\mathcal{T}})^I$. Then by Proposition 21 there is a simulation $Z : \mathcal{G}_{\bar{\mathcal{T}}} \overrightarrow{\sim} \mathcal{G}_I$ such that $(A, x) \in Z$. Then Z is also a bisimulation $Z : \mathcal{G}_{\bar{\mathcal{T}}} \overrightarrow{\sim} \mathcal{G}_{\mathcal{T}_x}$ with $(A, x) \in Z$. By Theorem 22 this proves $\mathcal{T}_x \sqsubseteq \bar{\mathcal{T}}$. Therefore $\mathcal{T}_x$ is the least concept description with the desired properties. □

Theorem 24. *In $\mathcal{EL}_{\text{gfp}}$ the most specific concept X^I exists for every $X \subseteq \Delta_I$.*

Proof. First assume that $X \neq \emptyset$. In [1] it is shown that least common subsumers exist and are unique up to equivalence for any finite set of $\mathcal{EL}_{\text{gfp}}$-concept descriptions. From Corollary 23 and Corollary 13 it follows that X^I exists. To be precise X^I is the lcs of all $\mathcal{T}_x, x \in X$.

In the case that $X = \emptyset$ we define $\mathcal{T}_{\text{all}}$ to be the TBox that contains only one defined concept, namely the root concept $R_{\mathcal{T}_{\text{all}}}$ defined as

$$R_{\text{all}} \equiv \bigsqcap_{B \in \mathcal{N}_{\text{prim}}} B \quad \sqcap \quad \bigsqcap_{r \in \mathcal{N}_{\text{role}}} \exists r.R_{\mathcal{T}_{\text{all}}}.$$

Then every concept description $\mathcal{T} \in \mathcal{EL}_{\text{gfp}}$ has $(R_{\text{all}}, \mathcal{T}_{\text{all}}) \sqsubseteq \mathcal{T}$. Obviously also $\emptyset \subseteq (R_{\text{all}}, \mathcal{T}_{\text{all}})^I$. Therefore $\emptyset^I = (R_{\text{all}}, \mathcal{T}_{\text{all}})$. □

Because of this result $\mathcal{EL}_{\text{gfp}}$ is a lot easier to handle with our methods than $\mathcal{EL}$ since we do not have to worry about the existence of X^I when using $\mathcal{EL}_{\text{gfp}}$. However Proposition 5 can be used to show that any set of implications that is complete for $\mathcal{EL}_{\text{gfp}}$ must also be complete for $\mathcal{EL}$—as long as both the left-hand-sides and the right-hand-sides of the implications do not contain terminological cycles. So from now on we shall work with $\mathcal{EL}_{\text{gfp}}$ which is more convenient and then try to transfer the result to $\mathcal{EL}$.

4 A finite basis for $\mathcal{EL}_{\text{gfp}}$-implications

We show that the set of implications holding in a given model always has a finite basis in $\mathcal{EL}_{\text{gfp}}$. A first step in this direction is to show that it is enough to restrict the attention to implications with acyclic $\mathcal{EL}_{\text{gfp}}$-concept descriptions as left-hand sides. The $\mathcal{EL}_{\text{gfp}}$-concept description $(A, \mathcal{T})$ is *acyclic* if the graph associated with it is acyclic. It is easy to see that there is a 1–1-relationship between $\mathcal{EL}$-concept descriptions and acyclic $\mathcal{EL}_{\text{gfp}}$-concept descriptions. For example, $(A, \{A \equiv B \sqcap \exists r.B, B \equiv P\})$ corresponds to $P \sqcap \exists r.P$, and $\exists r.P$ corresponds to $(A, \{A \equiv \exists r.B, B \equiv P\})$. This shows that $\mathcal{EL}$ can indeed be seen as a sublanguage of $\mathcal{EL}_{\text{gfp}}$. In the following, we will not distinguish an acyclic $\mathcal{EL}_{\text{gfp}}$-concept description from its equivalent $\mathcal{EL}$-concept description.

Given an $\mathcal{EL}_{\text{gfp}}$-concept description, its *node size* is the number of nodes in the description graph corresponding to it.

Theorem 25. *In $\mathcal{EL}_{\text{gfp}}$ the set*

$$\{\mathcal{U} \in \mathcal{EL}_{\text{gfp}} \mid \mathcal{U} \text{ is acyclic}\}$$

dominates every description context I with finite Δ_I.

The proof requires some technical work that will be provided after this short corollary.

Corollary 26. *The set of implications $\{\mathcal{U} \to \mathcal{U}^{II} \mid \mathcal{U} \in \mathcal{EL}_{\text{gfp}},\ \mathcal{U} \text{ is acyclic}\}$ is sound and complete for I.*

Proof. Follows immediately from Lemma 9 and Theorem 25. □

In order to prove Theorem 25 we define a family $((A, \mathcal{T})_d)_{d \in \mathbb{N}}$ of acyclic approximations of a concept description $(A, \mathcal{T}) \in \mathcal{EL}_{\text{gfp}}$. To obtain $(A, \mathcal{T})_d$, the description graph associated with $(A, \mathcal{T})$ is unraveled into a (possibly infinite) tree, and then all branches are cut at depth d. More formally, we first define $\mathcal{T}_0$ to be the TBox defined by the graph G_0, where

- $V_0 = \big\{(A)\big\}$
- $E_0 = \emptyset$
- $L_0\big((A)\big) = L_{\mathcal{T}}(A)$.

The $\mathcal{EL}_{\text{gfp}}$-concept graphs $\mathcal{G}_d$ corresponding to the TBoxes $\mathcal{T}_d$, $d > 0$, are defined recursively:

- $V_d = V_{d-1} \cup \Big\{(C_1, r_1, C_2, \ldots, C_{d-1}, r_{d-1}, C_d) \;\Big|\; (C_1, r_1, C_2, \ldots, C_{d-1}) \in V_{d-1}, (C_{d-1}, r_{d-1}, C_d) \in E_{\mathcal{T}}\Big\}$
- $E_d = E_{d-1} \cup \Big\{\big((C_1, \ldots, C_{d-1}), r_{d-1}, (C_1 \ldots, C_{d-1}, r_{d-1}, C_d)\big) \;\Big|\; (C_1 \ldots, C_{d-1}, r_{d-1}, C_d) \in V_d\}$
- $L_d\big((C_1, r_1, C_2, \ldots, C_k)\big) = L(C_k)$ for all $(C_1, r_1, C_2, \ldots, C_k) \in V_d$.

Then define $(A,\mathcal{T})_d = ((A),\mathcal{T}_d)$.

$V_{\mathcal{G}_d}$ can be seen as the set of all directed paths in $\mathcal{G}_\mathcal{T}$ of length at most d. Two such paths are connected by an r-edge in $\mathcal{G}_d$ if one path can be obtained from the other by adding an r-edge in $\mathcal{G}_\mathcal{T}$. The graph $\mathcal{G}_d$ is a directed tree, i. e. there is exactly one directed path from C_0 to each vertex.

For all $d \in \mathbb{N}$ we furthermore define the mappings

$$\begin{aligned} \zeta_{d,\mathcal{T}} : \qquad & V_d \to V_\mathcal{T} \\ (C_1, r_1 C_2, \ldots, C_k) & \mapsto C_k. \end{aligned}$$

It is purely technical to check that $\zeta_{d,\mathcal{T}}$ induces the simulation

$$\bar{\zeta}_{d,\mathcal{T}} = \{(\mathbf{p}, \zeta_{d,\mathcal{T}}(\mathbf{p})) \,|\, \mathbf{p} \in V_d\} : \mathcal{G}_d \overrightarrow{\sim} \mathcal{G}_\mathcal{T}.$$

Also note that $\zeta_{d,\mathcal{T}}$ leaves labels unchanged.

Lemma 27. *Let $\mathcal{U} = (A,\mathcal{T})$ be an $\mathcal{EL}_{\mathrm{gfp}}$-concept description of node size m, I a model of cardinality n, and $d = m \cdot n + 1$. Then $x \in (\mathcal{U}_d)^I$ implies $x \in \mathcal{U}^I$.*

Proof. Let $\mathcal{G}_d$ be the description graph corresponding to $\mathcal{T}_d$ whose vertices are denoted as in the above construction. Since $x \in (\mathcal{U}_d)^I$ we know from Proposition 21 that there is a simulation

$$Z_d : \mathcal{G}_d \overrightarrow{\sim} \mathcal{G}_I$$

such that $((A), x) \in Z_d$. Using this simulation we construct a mapping

$$z : \mathcal{G}_d \to \mathcal{G}_I$$

such that $z((A)) = x$ and for all $(C_1, r_1, C_2 \ldots, C_k) \in V_d$ we have

$$\Big((C_1, r_1, C_2 \ldots, C_k), z\big((C_1, r_1, C_2 \ldots, C_k)\big)\Big) \in Z_d \tag{10}$$

and

$$\begin{aligned} &\Big((C_1, r_1, C_2 \ldots, C_{k-1}), r, (C_1, r_1, C_2 \ldots, C_k)\Big) \in E_d \\ &\qquad\qquad \Rightarrow \Big(z\big((C_1, r_1, C_2 \ldots, C_{k-1})\big), r, z\big((C_1, r_1, C_2 \ldots, C_k)\big)\Big) \in E_I. \end{aligned} \tag{11}$$

This can be done recursively by first defining $z((A)) = x$. Now assume that we have already assigned a value to $z((C_1, r_1, C_2 \ldots, C_k))$. Then for every $(C_1, r_1, C_2 \ldots, C_k, r_k, C_{k+1}) \in V_d$ we know from the construction of V_d that

$$(C_k, r_k, C_{k+1}) \in E_\mathcal{T} \tag{12}$$

and

$$\big((C_1, r_1, C_2 \ldots, C_k), r_k, (C_1, r_1, C_2 \ldots, C_k, r_k, C_{k+1})\big) \in E_d \tag{13}$$

from the construction of E_d. Since $\big((C_1, r_1, C_2 \ldots, C_k), z((C_1, r_1, C_2 \ldots, C_k))\big) \in Z_d$ there must be some $y \in \Delta_I$ such that $\big((C_1, r_1, C_2 \ldots, C_k, r_k, C_{k+1}), y\big) \in Z_d$ and $(x, r_k, y) \in E_{\mathcal{G}_I}$. Defining $z\big((C_1, r_1, C_2 \ldots, C_k, r_k, C_{k+1})\big) = y$ suffices (10) and (11).

Since $\mathcal{G}_d$ is a directed tree, there is exactly one path from (A) to every other vertex in V_d. We define $\bar{V}$ to be the set of vertices $\mathbf{p} \in V_d$ such that on the path from (A) to $\mathbf{p}$ there are no two distinct vertices $\mathbf{q}$ and $\mathbf{r}$ with

$$\big(\zeta_{d,\mathcal{T}}(\mathbf{q}), z(\mathbf{q})\big) = \big(\zeta_{d,\mathcal{T}}(\mathbf{r}), z(\mathbf{r})\big).$$

Since there are only $m \cdot n = d - 1$ possible values for $(\zeta_{d,\mathcal{T}}(\mathbf{q}), z(\mathbf{q}))$, such a path can have at most length $d-1$. In other words, $\bar{V}$ contains only vertices with $\text{depth}(\mathbf{p}) < d$.

Define

$$Z = \big\{ \big(\zeta_{d,\mathcal{T}}(\mathbf{p}), z(\mathbf{p})\big) \bigm| \mathbf{p} \in \bar{V} \big\}.$$

We show that Z is a simulation $Z : \mathcal{G}_\mathcal{T} \to \mathcal{G}_I$ with $(A, x) \in Z$. For every pair $\big(\zeta_{d,\mathcal{T}}(\mathbf{p}), z(\mathbf{p})\big) \in Z$ we know that

$$L_\mathcal{T}\big(\zeta_{d,\mathcal{T}}(\mathbf{p})\big) = L_d(\mathbf{p})$$

because $\zeta_{d,\mathcal{T}}$ preserves labels. Since $(\mathbf{p}, z(\mathbf{p})) \in Z_d$ and Z_d is a simulation we have

$$L_d(\mathbf{p}) \subseteq L_I\big(z(\mathbf{p})\big).$$

Hence

$$L_\mathcal{T}\big(\zeta_{d,\mathcal{T}}(\mathbf{p})\big) \subseteq L_I\big(z(\mathbf{p})\big).$$

Now let $\big(\zeta_{d,\mathcal{T}}(\mathbf{p}), r, v\big) \in E_\mathcal{T}$ be an edge in $\mathcal{G}_\mathcal{T}$. Since $\mathbf{p} \in \bar{V}$ and thus $\text{depth}(\mathbf{p}) < d$ we know from the construction of $\mathcal{G}_d$ that there is some vertex $\mathbf{p}' \in V_d$ such that $(\mathbf{p}, r, \mathbf{p}') \in E_d$. By (11) this implies that $\big(z(\mathbf{p}), r, z(\mathbf{p}')\big) \in E_I$.

To prove that $(\zeta_{d,\mathcal{T}}(\mathbf{p}'), z(\mathbf{p}')) \in Z$ we look at two cases. Either $\mathbf{p}' \in \bar{V}$. Then $(\zeta_{d,\mathcal{T}}(\mathbf{p}'), z(\mathbf{p}')) \in Z$ by definition. In the other case that $\mathbf{p}' \notin \bar{V}$ there must be two distinct vertices $\mathbf{q}$ and $\mathbf{r}$ on the path that connects $(R_\mathcal{T})$ and $\mathbf{p}'$ with

$$\big(\zeta_{d,\mathcal{T}}(\mathbf{q}), z(\mathbf{q})\big) = \big(\zeta_{d,\mathcal{T}}(\mathbf{r}), z(\mathbf{r})\big).$$

However, since $\mathbf{p} \in \bar{V}$, $\mathbf{r}$ (the later node among $\mathbf{q}$ and $\mathbf{r}$) must be equal to p'. Thus

$$\big(\zeta_{d,\mathcal{T}}(\mathbf{p}'), z(\mathbf{p}')\big) = \big(\zeta_{d,\mathcal{T}}(\mathbf{r}), z(\mathbf{r})\big) = \big(\zeta_{d,\mathcal{T}}(\mathbf{q}), z(\mathbf{q})\big) \in Z.$$

This proves that Z is a simulation from $\mathcal{G}_\mathcal{T}$ to $\mathcal{G}_I$ such that $(A, x) \in Z$. Hence $x \in \mathcal{U}^I$ follows from Proposition 21. □

Proof of Theorem 25. Let $\mathcal{U}$ be an $\mathcal{EL}_{\text{gfp}}$-concept description and I a description context. We must find an acyclic $\mathcal{EL}_{\text{gfp}}$-concept description $\mathcal{V}$ such that $\mathcal{U} \sqsubseteq \mathcal{V}$ and $\mathcal{U}^I = \mathcal{V}^I$.

Let m be the node size of $\mathcal{U}$, n the cardinality of I, and $d = m \cdot n + 1$. We know that $\mathcal{U} \sqsubseteq \mathcal{U}_d$, and thus also $\mathcal{U}^I \subseteq (\mathcal{U}_d)^I$. Lemma 27 shows that the inclusion in the other direction holds as well. Thus, $\mathcal{V} := \mathcal{U}_d$ does the job. □

The complete set of implications given in the corollary is, of course, infinite. Also note that, though the left-hand sides $\mathcal{U}$ of implications in this set are acyclic, the right-hand sides $\mathcal{U}^{II}$ need not be acyclic. We show next that there is also a *finite* sound and complete set of implications. As mentioned before, a finite basis can then be obtained by removing redundant elements.

Theorem 28. *In $\mathcal{EL}_{\text{gfp}}$, for any description context I, there exists a* finite *set $\mathcal{B}$ of implications that is sound and complete for I.*

Proof. By Corollary 26 it suffices to find a finite and sound set of implications from which all implications of the form $\mathcal{U} \to \mathcal{U}^{II}$, where $\mathcal{U}$ is an acyclic $\mathcal{EL}_{\text{gfp}}$-concept description, follow. To this purpose, consider the set $\mathcal{E} := \{\mathcal{U}^I \mid \mathcal{U} \text{ is an } \mathcal{EL}_{\text{gfp}}\text{-concept description}\}$, and let $\mathcal{C}$ be a set of $\mathcal{EL}_{\text{gfp}}$-concept descriptions that contains, for each set $X \in \mathcal{E}$, exactly one element $\mathcal{V}$ with $\mathcal{V}^I = X$. Because of Theorem 25, we can assume without loss of generality that $\mathcal{C}$ contains only acyclic descriptions. Since Δ_I is finite, the sets $\mathcal{E}$ and $\mathcal{C}$ are also finite.

Consider the following finite set of implications, which is obviously sound:

$$\begin{aligned}\mathcal{B} := {} & \{P \to P^{II} \mid P \in \mathcal{N}_{\text{prim}} \cup \{\top\}\} \\ & \cup \{\exists r.C \to (\exists r.C)^{II} \mid r \in \mathcal{N}_r, C \in \mathcal{C}\} \\ & \cup \{C_1 \sqcap C_2 \to (C_1 \sqcap C_2)^{II} \mid C_1, C_2 \in \mathcal{C}\}.\end{aligned}$$

We show that, for any acyclic $\mathcal{EL}_{\text{gfp}}$-concept description $\mathcal{U}$, the implication $\mathcal{U} \to \mathcal{U}^{II}$ follows from $\mathcal{B}$. Since $\mathcal{U}$ is acyclic, we can view it as an $\mathcal{EL}$-concept description. The proof is by induction on the structure of this description.

Base case: $\mathcal{U} = P \in \mathcal{N}_{\text{prim}} \cup \{\top\}$. Then $P \to P^{II}$ is in $\mathcal{B}$ by definition. Thus, it also follows from $\mathcal{B}$.

Step case 1: $\mathcal{U} = \exists r.\mathcal{V}$ for some $r \in \mathcal{N}_r$ and some $\mathcal{EL}$-concept description $\mathcal{V}$. Let J be a description context in which all implications from $\mathcal{B}$ hold. The semantics of existential restrictions yields

$$\mathcal{U}^J = (\exists r.\mathcal{V})^J = \{x \in \Delta_J \mid \exists y \in \mathcal{V}^J : (x, y) \in r^J\}.$$

By the induction hypothesis, $\mathcal{V} \to \mathcal{V}^{II}$ follows from $\mathcal{B}$, and thus holds in J. Therefore $\mathcal{V}^J \subseteq (\mathcal{V}^{II})^J$, which yields

$$\mathcal{U}^J \subseteq \{x \in \Delta_J \mid \exists y \in (\mathcal{V}^{II})^J : (x, y) \in r^J\}.$$

Now, choose $C \in \mathcal{C}$ such that $C^I = \mathcal{V}^I$. Lemma 2(g) yields $\mathcal{V}^{II} \sqsubseteq C$, and thus

$$\begin{aligned}\mathcal{U}^J & \subseteq \{x \in \Delta_J \mid \exists y \in C^J : (x, y) \in r^J\} \\ & = (\exists r.C)^J.\end{aligned}$$

Since $\exists r.C \to (\exists r.C)^{II} \in \mathcal{B}$ holds in J by assumption, we get

$$\begin{aligned}\mathcal{U}^J & \subseteq ((\exists r.C)^{II})^J \\ & = (\{x \in \Delta_I \mid \exists y \in C^I : (x, y) \in r^I\}^I)^J \\ & = (\{x \in \Delta_I \mid \exists y \in \mathcal{V}^I : (x, y) \in r^I\}^I)^J \\ & = ((\exists r.\mathcal{V})^{II})^J = (\mathcal{U}^{II})^J.\end{aligned}$$

Thus, we have shown that $\mathcal{U} \to \mathcal{U}^{II}$ holds in every description context J in which all implications from $\mathcal{B}$ hold.

Step case 2: $\mathcal{U} = \mathcal{U}_1 \sqcap \mathcal{U}_2$ for $\mathcal{EL}$-concept descriptions $\mathcal{U}_1, \mathcal{U}_2$. Let J be a description context in which all implications from $\mathcal{B}$ hold. By the induction hypothesis, $\mathcal{U}_1^J \subseteq (\mathcal{U}_1^{II})^J$ and $\mathcal{U}_2^J \subseteq (\mathcal{U}_2^{II})^J$. Therefore

$$\mathcal{U}^J = (\mathcal{U}_1 \sqcap \mathcal{U}_2)^J = \mathcal{U}_1^J \cap \mathcal{U}_2^J \subseteq (\mathcal{U}_1^{II})^J \cap (\mathcal{U}_2^{II})^J.$$

We choose $C_1, C_2 \in \mathcal{C}$ such that $C_1^I = \mathcal{U}_1^I$ and $C_2^I = \mathcal{U}_2^I$. Then

$$\mathcal{U}^J \subseteq (C_1^{II})^J \cap (C_2^{II})^J \subseteq C_1^J \cap C_2^J = (C_1 \sqcap C_2)^J,$$

where the second inclusion holds due to Lemma 2(d). Since the implication $C_1 \sqcap C_2 \to (C_1 \sqcap C_2)^{II} \in \mathcal{B}$ holds in J, we get

$$\begin{aligned} \mathcal{U}^J &\subseteq ((C_1 \sqcap C_2)^{II})^J \\ &= ((C_1^I \cap C_2^I)^I)^J \\ &= ((\mathcal{U}_1^I \cap \mathcal{U}_2^I)^I)^J \\ &= ((\mathcal{U}_1 \sqcap \mathcal{U}_2)^{II})^J = (\mathcal{U}^{II})^J. \end{aligned}$$

This shows that $\mathcal{U} \to \mathcal{U}^{II}$ follows from $\mathcal{B}$. ☐

Corollary 29. *In $\mathcal{EL}_{\mathrm{gfp}}$, for any description context I there exists a* finite *basis for the implications holding in I.*

Proof. Starting with $\mathcal{B}^* := \mathcal{B}$, where in the beginning all implications are unmarked, take an unmarked implication $\mathcal{U} \to \mathcal{V} \in \mathcal{B}^*$. If this implication follows from $\mathcal{B}^*$, then remove it, i.e., $\mathcal{B}^* := \mathcal{B}^* \setminus \{\mathcal{U} \to \mathcal{V}\}$; otherwise, mark $\mathcal{U} \to \mathcal{V}$. Continue with this until all implications in $\mathcal{B}^*$ are marked. The final set $\mathcal{B}^*$ is the desired basis. ☐

5 A finite basis for the implications in standard $\mathcal{EL}$

Although the sublanguage $\mathcal{EL}$ of $\mathcal{EL}_{\mathrm{gfp}}$ is not an instance of our general framework, we can nevertheless show the above corollary also for this language. Because of Proposition 5, it is sufficient to show that in $\mathcal{EL}_{\mathrm{gfp}}$ any description context I has a finite basis consisting of implications where both the left-hand *and the right-hand sides* are acyclic.

The following proposition will allow us to construct a finite set of implications *with acyclic right-hand sides* from which a given implication $\mathcal{U} \to \mathcal{U}^{II}$ (with potentially cyclic right-hand side) follows. Recall that, for any $\mathcal{EL}_{\mathrm{gfp}}$-concept description $\mathcal{U}$, we obtain the acyclic description $\mathcal{U}_d$ by unraveling the description graph and then cutting all branches at depth d.

Lemma 30. *Let k_0 be any natural number. Then for every concept description $\mathcal{U} \in \mathcal{EL}_{\mathrm{gfp}}$ the implication $\mathcal{U} \to \mathcal{U}^{II}$ follows from*

$$\begin{aligned} \mathcal{B} = &\{(X^I)_{k_0} \to (X^I)_{k_0+1} \mid X \subseteq \Delta_I\} \\ &\cup \{\mathcal{U} \to (U^{II})_{k_0}\} \end{aligned}$$

where $(X^I)_k$ denotes the unraveling of X^I up to depth k and $(U^{II})_k$ denotes the unraveling of $\mathcal{U}^{II}$ up to depth k.

Once again, the proof requires some technical work and will be provided after the following results.

Lemma 31. *Let $\mathcal{U} = (A, \mathcal{T}) \in \mathcal{EL}_{\text{gfp}}$. Then we can find a set $\mathcal{P} \subseteq \mathcal{N}_{\text{prim}}$ and a set $\mathcal{X} \subseteq \mathcal{N}_{\text{role}} \times \mathcal{EL}_{\text{gfp}}$ such that*

$$\mathcal{U} \equiv \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S}) \in \mathcal{X}} \exists r.\mathcal{S}$$

Proof.

$$\mathcal{U} \equiv \bigsqcap_{B \in L_{\mathcal{T}}(A)} B \sqcap \bigsqcap_{(A,r,C) \in E_{\mathcal{T}}} \exists r.(C, \mathcal{T}).$$

□

Lemma 32. *Let $X \subseteq \Delta_I$. Then we can find a set $\mathcal{P} \subseteq \mathcal{N}_{\text{prim}}$ and a set $\mathcal{Y} \subseteq \mathcal{N}_{\text{role}} \times \mathfrak{P}(\Delta_I)$ such that*

$$X^I \equiv \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \exists r.Y^I$$

Proof. By Lemma 31 we know that we can write X^I as

$$X^I \equiv \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S}) \in \mathcal{X}} \exists r.\mathcal{S}$$

for some $\mathcal{P} \subseteq \mathcal{N}_{\text{prim}}$ and some $\mathcal{X} \subseteq \mathcal{N}_{\text{role}} \times \mathcal{EL}_{\text{gfp}}$. From Lemma 2 we know that $\mathcal{S}^{II} \sqsubseteq \mathcal{S}$ for any $\mathcal{S} \in \mathcal{EL}_{\text{gfp}}$ and hence

$$\exists r.\mathcal{S}^{II} \sqsubseteq \exists r.\mathcal{S}.$$

This implies that

$$\begin{aligned} X^I &\equiv \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S}) \in \mathcal{X}} \exists r.\mathcal{S} \\ &\sqsupseteq \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S}) \in \mathcal{X}} \exists r.\mathcal{S}^{II}. \end{aligned} \tag{14}$$

Looking at the extension of $\bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S}) \in \mathcal{X}} \exists r.\mathcal{S}^{II}$ we get

$$\begin{aligned} \Big(\bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S}) \in \mathcal{X}} \exists r.\mathcal{S}^{II}\Big)^I &= \bigcap_{P \in \mathcal{P}} P^I \cap \bigcap_{(r,\mathcal{S}) \in \mathcal{X}} \big(\exists r.\mathcal{S}^{II}\big)^I \\ &= \bigcap_{P \in \mathcal{P}} P^I \cap \bigcap_{(r,\mathcal{S}) \in \mathcal{X}} \{x \in \Delta_I \mid \exists y \in \mathcal{S}^{III} : (x, y) \in r^I\} \end{aligned}$$

and by Lemma 2 (f)

$$
\begin{aligned}
\Big(\bigsqcap_{P\in\mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S})\in\mathcal{X}} \exists r.\mathcal{S}^{II}\Big)^I &= \bigcap_{P\in\mathcal{P}} P^I \cap \bigcap_{(r,\mathcal{S})\in\mathcal{X}} \{x \in \Delta_I \mid \exists y \in \mathcal{S}^{III} : (x,y) \in r^I\} \\
&= \bigcap_{P\in\mathcal{P}} P^I \cap \bigcap_{(r,\mathcal{S})\in\mathcal{X}} \{x \in \Delta_I \mid \exists y \in \mathcal{S}^{I} : (x,y) \in r^I\} \\
&= \Big(\bigsqcap_{P\in\mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S})\in\mathcal{X}} \exists r.\mathcal{S}\Big)^I \\
&= X^{II}.
\end{aligned}
$$

So X^I and $\bigsqcap_{P\in\mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S})\in\mathcal{X}} \exists r.\mathcal{S}^{II}$ have the same extension in I. By definition we know that X^I is the least concept description with the extension X^{II} and therefore

$$
X^I \sqsubseteq \bigsqcap_{P\in\mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S})\in\mathcal{X}} \exists r.\mathcal{S}^{II}. \tag{15}
$$

From (14) and (15) we get

$$
X^I \equiv \bigsqcap_{P\in\mathcal{P}} P \sqcap \bigsqcap_{(r,\mathcal{S})\in\mathcal{X}} \exists r.\mathcal{S}^{II}. \tag{16}
$$

With $\mathcal{Y} := \{(r, \mathcal{S}^I) \mid (r, \mathcal{S}) \in \mathcal{X}\}$ we get the desired result. □

Proposition 33. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{EL}_{\text{gfp}}$ be concept descriptions. Let $r \in \mathcal{N}_{\text{role}}$ be a role name. Then*

$$
\begin{aligned}
(\exists r.\mathcal{S})_k &\equiv \exists r.(\mathcal{S}_{k-1}) \\
(\mathcal{S} \sqcap \mathcal{T})_k &\equiv \mathcal{S}_k \sqcap \mathcal{T}_k,
\end{aligned}
$$

where $(\exists r.\mathcal{S})_k$ denotes the unraveling of $\exists r.\mathcal{S}$ up to depth k, etc.

Proof. From the definition of $\exists r.\mathcal{S}$ we know that except for the root vertex $\big(R_{\exists r.\mathcal{S}}\big)$ every vertex in $\mathcal{G}_{(\exists r.\mathcal{S})_k}$ is of the form

$$
\big(R_{\exists r.\mathcal{S}}, r, R_{\mathcal{S}}, r_1, C_1, \ldots, C_l\big),
$$

where $\big(R_{\mathcal{S}}, r_1, C_1, \ldots, C_l\big)$ is a path of length l in $\mathcal{S}$ and $l \leq k-1$. Then it is purely technical and not very hard to check that Z defined as

$$
\begin{aligned}
Z = &\big\{\big((R_{\exists r.\mathcal{S}}), R_{\exists r.\mathcal{S}_{k-1}}\big)\big\} \cup \\
&\big\{\big((R_{\exists r.\mathcal{S}}, r, R_{\mathcal{S}}, r_1, C_1, \ldots, C_l), (R_{\mathcal{S}}, r_1, C_1, \ldots, C_l)\big) \mid (R_{\mathcal{S}}, r_1, C_1, \ldots, C_l) \in \mathcal{S}_{k-1}\big\}
\end{aligned}
$$

is a simulation from $\mathcal{G}_{(\exists r.\mathcal{S})_k}$ to $\mathcal{G}_{\exists r.\mathcal{S}_{k-1}}$ and that Z^{-1} is a simulation from $\mathcal{G}_{\exists r.\mathcal{S}_{k-1}}$ to $\mathcal{G}_{(\exists r.\mathcal{S})_k}$. This shows that $(\exists r.\mathcal{S})_k \equiv \exists r.(\mathcal{S}_{k-1})$. □

Proof of Lemma 30. To prove the proposition, we first show, by induction on ℓ, that the implications $(X^I)_\ell \to (X^I)_{\ell+1}$ follow from $\mathcal{B}$ for all $\ell \geq k_0$. For $\ell = k_0$ this is trivial because $(X^I)_{k_0} \to (X^I)_{k_0+1} \in \mathcal{B}$.

Now, assume that $(Y^I)_k \to (Y^I)_{k+1}$ follows from $\mathcal{B}$ for every $Y \subseteq \Delta_I$ and every $k, k_0 \leq k < \ell$. Let J be a model in which all implications from $\mathcal{B}$ hold. Then, by the induction hypothesis, we get

$$((Y^I)_k)^J \subseteq ((Y^I)_{k+1})^J \tag{17}$$

for all $k_0 \leq k < \ell$ and all $Y \subseteq \Delta_I$. By $(*)$, for any set $X \subseteq \Delta_I$, there exist sets $\mathcal{P} \subseteq \mathcal{N}_{\text{prim}}$ and $\mathcal{Y} \subseteq \mathcal{N}_r \times \mathfrak{P}(\Delta_I)$ such that

$$X^I \equiv \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \exists r.Y^I.$$

It is easy to see that this implies

$$(X^I)_\ell \equiv \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \exists r.(Y^I)_{\ell-1} \tag{18}$$

and

$$(X^I)_{\ell+1} \equiv \bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \exists r.(Y^I)_\ell. \tag{19}$$

Thus, we have

$$\begin{aligned}
\Big((X^I)_\ell\Big)^J &\overset{(18)}{=} \Big(\bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \exists r.(Y^I)_{\ell-1}\Big)^J \\
&= \bigsqcap_{P \in \mathcal{P}} P^J \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \{x \in \Delta_J \mid \exists y \in ((Y^I)_{\ell-1})^J : (x,y) \in r^J\}.
\end{aligned}$$

From (17) we obtain $((Y^I)_{\ell-1})^J \subseteq ((Y^I)_\ell)^J$, and thus

$$\begin{aligned}
\Big((X^I)_\ell\Big)^J &\subseteq \bigsqcap_{P \in \mathcal{P}} P^J \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \{x \in \Delta_J \mid \exists y \in ((Y^I)_\ell)^J : (x,y) \in r^J\} \\
&= \Big(\bigsqcap_{P \in \mathcal{P}} P \sqcap \bigsqcap_{(r,Y) \in \mathcal{Y}} \exists r.(Y^I)_\ell\Big)^J \\
&\overset{(19)}{=} \Big((X^I)_{\ell+1}\Big)^J.
\end{aligned}$$

Hence we have shown that $(X^I)_\ell \to (X^I)_{\ell+1}$ follows from $\mathcal{B}$, which concludes the induction proof.

Now, let J again be a model in which all implications from $\mathcal{B}$ hold, and let $x \in \mathcal{U}^J$. We must show that this implies $x \in (\mathcal{U}^{II})^J$. We have $x \in ((\mathcal{U}^{II})_{k_0})^J$ because $\mathcal{U} \to (\mathcal{U}^{II})_{k_0} \in \mathcal{B}$. Hence $x \in ((\mathcal{U}^{II})_k)^J$ for all $k \leq k_0$ since $(\mathcal{U}^{II})_{k_0} \sqsubseteq (U^{II})_k$ for all $k \leq k_0$. From what we have shown above, we know that

$$(U^{II})_k \to (\mathcal{U}^{II})_{k+1}$$

follows from $\mathcal{B}$ for all $k \geq k_0$. Thus $((\mathcal{U}^{II})_k)^J \subseteq ((\mathcal{U}^{II})_{k+1})^J$ holds in J for all $k \geq k_0$, which yields $x \in ((\mathcal{U}^{II})_k)^J$ also in this case.

Therefore $x \in ((\mathcal{U}^{II})_k)^J$ for $k = |\mathcal{G}_\mathcal{U}|\cdot|\Delta_J|+1$, independently of whether this number is smaller or larger than k_0. It follows directly from Lemma 27 that $x \in (\mathcal{U}^{II})^J$. Thus, we have shown that

$$\mathcal{U}^J \subseteq (\mathcal{U}^{II})^J$$

if all implications from $\mathcal{B}$ hold in J. This means that $\mathcal{U} \to \mathcal{U}^{II}$ follows from $\mathcal{B}$. □

Having proved Proposition 30, we are almost finished with constructing a finite, sound and complete set of acyclic implications for the implications holding in a description context I. The idea is to replace any implication $\mathcal{U} \to \mathcal{U}^{II}$ in the finite, sound and complete set of implications constructed in the proof of Theorem 28 by the corresponding implications from Proposition 30.

The remaining problems is, however, that the set of implications obtained this way need not be sound for I. Indeed, if k_0 is too small, then the implications in $\{(X^I)_{k_0} \to (X^I)_{k_0+1} \mid X \subseteq \Delta_I\}$ need not hold in I. Therefore, we define for every $X \subseteq \Delta_I$

$$d_X := m_X \cdot n + 1,$$

where m_X is the node size of X^I and n is the cardinality of the model I. The number k_0 is the maximum of these numbers, i.e.,

$$k_0 := \max_{X \subseteq \Delta_I} d_X. \tag{20}$$

Then, because $d_X \leq k_0$ for every $X \subseteq \Delta_I$, we have

$$X^I \sqsubseteq (X^I)_{k_0+1} \sqsubseteq (X^I)_{k_0} \sqsubseteq (X^I)_{d_X}.$$

By Lemma 2(b), this implies

$$X^{II} \subseteq ((X^I)_{k_0+1})^I \subseteq ((X^I)_{k_0})^I \subseteq ((X^I)_{d_X})^I.$$

From Lemma 27 we obtain $X^{II} \supseteq ((X^I)_{d_X})^I$, and thus

$$X^{II} = ((X^I)_{k_0+1})^I = ((X^I)_{k_0})^I = ((X^I)_{d_X})^I.$$

In particular, this shows

$$((X^I)_{k_0})^I \subseteq ((X^I)_{k_0+1})^I.$$

Hence, all implications in $\{(X^I)_{k_0} \to (X^I)_{k_0+1} \mid X \subseteq \Delta_I\}$ hold in I.

Theorem 34. *In $\mathcal{EL}_{\mathrm{gfp}}$ for any description context I there exists a finite set $\mathcal{B}$ of implications that is complete, such that for any implication*

$$(A \to B) \in \mathcal{B}$$

both A and B are acyclic.

Proof. Let $\mathcal{C}$ be the set of acyclic $\mathcal{EL}_{\mathrm{gfp}}$-concept descriptions defined in the proof of Theorem 28. We have shown in that proof that the set

$$\begin{aligned}\mathcal{B}_\star := \{P \to P^{II} \mid P \in \mathcal{N}_{\mathrm{prim}} \cup \{\top\}\}\\ \cup \{\exists r.C \to (\exists r.C)^{II} \mid r \in \mathcal{N}_r, C \in \mathcal{C}\}\\ \cup \{C_1 \sqcap C_2 \to (C_1 \sqcap C_2)^{II} \mid C_1, C_2 \in \mathcal{C}\}\end{aligned}$$

is complete for I.

Let k_0 be defined as in (20). Then, by Proposition 30, the fact that $\mathcal{B}_\star$ is complete also implies that the following set of implications is complete for I:

$$\begin{aligned}\mathcal{B} := \{(X^I)_{k_0} \to (X^I)_{k_0+1} \mid X \subseteq \Delta_I\}\\ \cup \{P \to (P^{II})_{k_0} \mid P \in \mathcal{N}_{\mathrm{prim}} \cup \{\top\}\}\\ \cup \{\exists r.C \to ((\exists r.C)^{II})_{k_0} \mid r \in \mathcal{N}_r, C \in \mathcal{C}\}\\ \cup \{C_1 \sqcap C_2 \to ((C_1 \sqcap C_2)^{II})_{k_0} \mid C_1, C_2 \in \mathcal{C}\}.\end{aligned}$$

Regarding soundness, we have shown above that, due to the fact that k_0 was chosen large enough, all implications of the form $(X^I)_{k_0} \to (X^I)_{k_0+1}$ hold I. The implications $P \to (P^{II})_{k_0}$ hold because $P \to P^{II}$ holds in I, and $P^{II} \sqsubseteq (P^{II})_{k_0}$. The same arguments can be used to show that the implications of the forms $\exists r.C \to ((\exists r.C)^{II})_{k_0}$ and $C_1 \sqcap C_2 \to ((C_1 \sqcap C_2)^{II})_{k_0}$ hold in I.

The left-hand sides of implications in $\mathcal{B}$ are acyclic since the elements of $\mathcal{C}$ are acyclic, primitive concepts and $\top$ are acyclic, and any concept description of the form $\mathcal{U}_k$ is acyclic. This last argument also shows that the right-hand sides of implications in $\mathcal{B}$ are acyclic. □

Corollary 35. *In $\mathcal{EL}$, for any description context I, there exists a* finite *basis for the implications holding in I.*

6 Conclusion

We have shown that any description context I (i.e., any finite relational structure over a finite signature of unary and binary predicate symbols) has a finite basis for the $\mathcal{EL}$- and $\mathcal{EL}_{\mathrm{gfp}}$-implications holding in I. Such a basis provides the knowledge engineer with interesting information on the application domain described by the context. The knowledge engineer can, for example, use these implications as starting point for building an ontology describing this domain.

In this paper, we have concentrated on showing the existence of a finite basis. Of course, if this approach is to be used in practice, we also need to find efficient algorithms for computing the basis. After that, the next step will be to generalize attribute exploration [7] to our more general setting. This would allow us to consider also relational structures that are not explicitly given, but rather "known" by a domain expert.

Finally, we will also try to show similar results for other DLs. For the DL $\mathcal{FL}_0$, which differs from $\mathcal{EL}$ in that existential restrictions are replaced by value restrictions, we are quite confident that this is possible. For more expressive DLs, like $\mathcal{ALC}$, this is less clear.

References

[1] Franz Baader. Least common subsumers and most specific concepts in a description logic with existential restrictions and terminological cycles. In Georg Gottlob and Toby Walsh, editors, *Proceedings of the 18th International Joint Conference on Artificial Intelligence*, pages 319–324. Morgan Kaufmann, 2003.

[2] Franz Baader. Terminological cycles in a description logic with existential restrictions. In Georg Gottlob and Toby Walsh, editors, *Proc. of the 18th Int. Joint Conf. on Artificial Intelligence (IJCAI 2003)*, pages 325–330, Acapulco, Mexico, 2003. Morgan Kaufmann, Los Altos.

[3] Franz Baader, Sebastian Brandt, and Carsten Lutz. Pushing the $\mathcal{EL}$ envelope. In Leslie Pack Kaelbling and Alessandro Saffiotti, editors, *Proc. of the 19th Int. Joint Conf. on Artificial Intelligence (IJCAI 2005)*, pages 364–369, Edinburgh (UK), 2005. Morgan Kaufmann, Los Altos.

[4] Franz Baader and Felix Distel. A finite basis for the set of EL-implications holding in a finite model. Technical Report 07-02, Inst. für Theoretische Informatik, TU Dresden, Dresden, Germany, 2007.

[5] Franz Baader, Carsten Lutz, and Bontawee Suntisrivaraporn. CEL—a polynomial-time reasoner for life science ontologies. In Ulrich Furbach and Natarajan Shankar, editors, *Proc. of the Int. Joint Conf. on Automated Reasoning (IJCAR 2006)*, volume 4130 of *Lecture Notes in Artificial Intelligence*, pages 287–291. Springer-Verlag, 2006.

[6] Sébastien Ferré. *Systèmes d'information logiques: un paradigme logico-contextuel pour interroger, naviguer et apprendre.* PhD thesis, IRISA, France, 2002.

[7] Bernhard Ganter. Attribute exploration with background knowledge. *Theoretical Computer Science*, 217(2):215–233, 1999.

[8] Bernhard Ganter and Sergei O. Kuznetsov. Pattern structures and their projections. *Lecture Notes in Computer Science*, 2120:129–144, 2001.

[9] Bernhard Ganter and Rudolf Wille. Implikationen und abhängigkeiten zwischen merkmalen. In P. O. Degens, H.-J. Hermes, and O. Opitz, editors, *Die Klassifikation und ihr Umfeld*, Frankfurt, 1986. Indeks-Verlag.

[10] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis: Mathematical Foundations.* Springer, New York, 1997.

[11] Bernhard Ganter, Rudolf Wille, and Karl Erich Wolff, editors. Mannheim, 1987. B. I.-Wissenschaftsverlag.

[12] J.-L. Guigues and V. Duquenne. Familles minimales d'implications informatives résultant d'un tableau de données binaires. *Math. Sci. Humaines*, 95:5–18, 1986.

[13] Ian Horrocks, Peter F. Patel-Schneider, and Frank van Harmelen. From SHIQ and RDF to OWL: The making of a web ontology language. *Journal of Web Semantics*, 1(1):7–26, 2003.

[14] Ralf Möller and Volker Haarslev. Description logic systems. In Franz Baader, Diego Calvanese, Deborah McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, editors, *The Description Logic Handbook: Theory, Implementation, and Applications*, pages 282–305. Cambridge University Press, 2003.

[15] Susanne Prediger. Logical scaling in formal concept analysis. In *International Conference on Conceptual Structures*, pages 332–341, 1997.

[16] Susanne Prediger. Terminologische merkmalslogik in der formalen begriffsanalyse. In Gerd Stumme and Rudolf Wille, editors, *Begriffliche Wissensverarbeitung: Methoden und Anwendungen*, pages 99–124, Berlin, Heidelberg, New York, 1999. Springer.

[17] Susanne Prediger and Rudolf Wille. The lattice of concept graphs of a relationally scaled context. In *International Conference on Conceptual Structures*, pages 401–414, 1999.

[18] U. Priss. The formalization of wordnet by methods of relational concept analysis. In C. Fellbaum, editor, *WordNet: An Electronic Lexical Database and some of its applications*, Cambridge, Massachusetts, 1998. MIT Press.

[19] Sebastian Rudolph. Exploring relational structures via fle. In Karl Erich Wolff, Heather D. Pfeiffer, and Harry S. Delugach, editors, *Proc. of the 12th Int. Conf. on Conceptual Structures (ICCS'04)*, volume 3127 of *Lecture Notes in Computer Science*, pages 196–212, 2004.

[20] Sebastian Rudolph. *Relational Exploration: Combining Description Logics and Formal Concept Analysis for Knowledge Specification.* PhD thesis, Technische Universität Dresden, 2006.

[21] K.A. Spackman, K.E. Campbell, and R.A. Cote. SNOMED RT: A reference terminology for health care. *J. of the American Medical Informatics Association*, pages 640–644, 1997. Fall Symposium Supplement.

[22] The Gene Ontology Consortium. Gene Ontology: Tool for the unification of biology. *Nature Genetics*, 25:25–29, 2000.

[23] Rudolf Wille. Conceptual graphs and formal concept analysis. In Dickson Lukose, Harry S. Delugach, Mary Keeler, Leroy Searle, and John F. Sowa, editors, *Proc. of the 5th Int. Conf. on Conceptual Structures (ICCS'97)*, volume 1257 of *Lecture Notes in Computer Science*, pages 290–303, 1997.