Bounded Degree

$\mathcal{D}_k$—the class of structures $\mathbb{A}$ in which every element has at most k neighbours in $G\mathbb{A}$.

$O(f(k) \cdot n^1)$ FPT

Theorem (Seese)

For every sentence φ of FO and every k there is a linear time algorithm which, given a structure $\mathbb{A} \in \mathcal{D}_k$ determines whether $\mathbb{A} \models \varphi$.

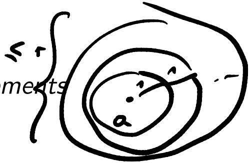
Note: this is not true for MSO unless P = NP.

The proof is based on *locality* of first-order logic. Specifically, *Hanf's theorem*.

Hanf Types

For an element a in a structure $\mathbb{A}$, define

> $N_r^{\mathbb{A}}(a)$—*the substructure of* $\mathbb{A}$ *generated by the elements whose distance from* a *(in* $G\mathbb{A}$*) is at most* r*.*

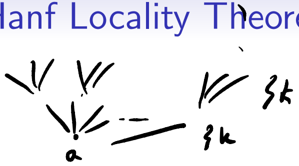


We say $\mathbb{A}$ and $\mathbb{B}$ are *Hanf equivalent* with radius r and threshold q ($\mathbb{A} \simeq_{r,q} \mathbb{B}$) if, for every $a \in A$ the two sets

$$\{a' \in A \mid N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{A}}(a')\} \quad \text{and} \quad \{b \in B \mid N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{B}}(b)\}$$

either have the same size or both have size greater than q;
and, similarly for every $b \in B$.

Hanf Locality Theorem



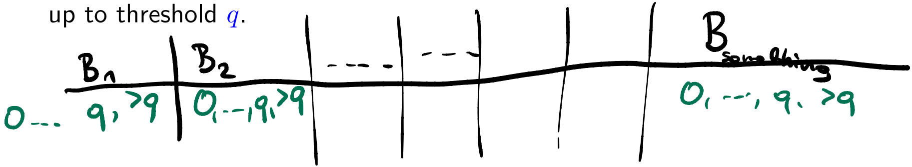
Theorem (Hanf)

For every vocabulary σ and every p there qre r and q such that for any σ-structures $\mathbb{A}$ and $\mathbb{B}$: if $\mathbb{A} \simeq_{r,q} \mathbb{B}$ then $\mathbb{A} \equiv_p \mathbb{B}$.

For $\mathbb{A} \in \mathcal{D}_k$:

> $N_r^{\mathbb{A}}(a)$ has at most $k^r + 1$ elements
>
> each $\simeq_{r,q}$ has finite index.

Each $\simeq_{r,q}$-class t can be characterised by a finite table, I_t, giving isomorphism types of neighbourhoods and numbers of their occurrences up to threshold q.



Satisfaction on $\mathcal{D}_k$

For a sentence φ of FO, we can compute a set of tables $\{I_1, \ldots, I_s\}$ describing $\simeq_{r,q}$-classes consistent with it.
This computation is independent of any structure $\mathbb{A}$. !

Given a structure $\mathbb{A} \in \mathcal{D}_k$,

for each a, determine the isomorphism type of $N_r^{\mathbb{A}}(a)$

construct the table describing the $\simeq_{r,q}$-class of $\mathbb{A}$.

compare against $\{I_1, \ldots, I_s\}$ to determine whether $\mathbb{A} \models \varphi$.

For fixed k, r, q, this requires time *linear* in the size of $\mathbb{A}$.

$|\mathbb{A}|$

Note: evaluation for FO is in $O(f(l, k)n)$.

$|\varphi|$ bound on degree

$\forall x_1 \forall x_2 \exists x_3 \; R(x_1, x_2, x_1)$

FO

GNFO

2Exp - compl.

$\alpha(x_1, \dots, x_n) \wedge \neg \varphi(x_1, \dots, x_n)$

Prenex classes

$\forall \exists \forall \varphi$

$\exists^* \varphi$

$\forall^* \exists^* \varphi$

$\exists \forall \exists \varphi$

FO^k

k-variable fragment

GF

$\forall x_1, \dots, x_n$

$\alpha(x_1, \dots, x_n) \rightarrow \varphi$

FO^4 is undec.

FO^3 — " —

FO^2 NExp - complete

FO^1 NP - complete

FO^1 over purely relational signatures with symbols of arity ≤ 1

$\boxed{\text{NP-compl}}$

Example: $\forall x \left(R(x) \rightarrow (\exists x \; B(x)) \right) = \varphi$

$$\mathfrak{A} = \overset{R, \neg B}{(0)} \quad \overset{\neg R, B}{(1)} \models \varphi$$

$$\psi = \exists x \; B(x) \; \wedge \; \forall x \; \neg B(x)$$

$C^1 = FO^1 + \text{counting}$

$\in \mathbb{N}$, written in binary

$$\varphi ::= R(x) \mid \varphi \wedge \varphi \mid \neg \varphi \mid \underbrace{\exists^{\le k}_x \varphi}_{\text{there are at most } k \text{ elements in our structure satisfying } \varphi} \mid \exists^{\ge k}_x \varphi$$

$\vee$, $\rightarrow$, $\leftrightarrow$

rel. symbol

$$\exists x\, \varphi \equiv \exists^{\ge 1}_x \varphi$$

$$\forall x\, \varphi \equiv \neg \exists x \neg \varphi \equiv \neg \exists^{\ge 1}_x \neg \varphi \equiv \exists^{\le 0}_x \neg \varphi$$

$$\exists^{\ge 2^n}_x A(x) = \varphi$$

$|\varphi| \approx n+1$

$$\underbrace{A \quad A \quad A \quad \dots \quad A}_{2^n \approx 2^{|\varphi|}}$$

$\boxed{\text{MODELS CAN HAVE EXP SIZE}}$

Def: A C^1 formula φ is in normal form if it looks like

$$\varphi = \bigwedge_{i=1}^{n} \exists_x^{\bowtie_i C_i} \psi_i$$

$\bowtie_i$: $\leq, \geq$

C_i: $\in \mathbb{N}$

ψ_i: without quantifiers

Lemma: There is a procedure that works in NP and from φ produces φ' in normal form s.t. φ is sat iff φ' is sat and $|\varphi'| = O(|\varphi|)$.

Take the most nested quantifier from φ.

$$\varphi = (\text{-----} (\exists_x^{\bowtie C} \psi) \text{---}) \text{---})$$

Guess whether $\exists_x^{\bowtie C} \psi$ is satisfied. If yes, replace with $\top$. If no, replace with $\bot$.

Return $\underbrace{\varphi[\exists_x^{\bowtie C} \psi / \top/\bot]}_{\text{replaced formula}} \wedge^{\pm} (\exists_x^{\bowtie C} \psi)$.

negation if replaced by $\bot$.