Finite and Algorithmic Model Theory

Lecture 5 (Dresden 09.11.22, Long version with Errors)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

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Today's agenda

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Lecture based on

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Slides 29–33, 43–51 of [Montanari]

19:23-24:32 of lecture by [Anuj Dawar]

Slides 80–110 by [Diego Figueira]

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Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

Recap of Ehrenfeucht-Fraïssé games

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- Duration: *m rounds*.

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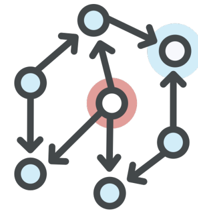
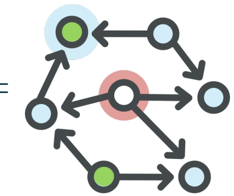
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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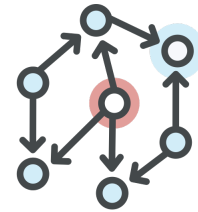
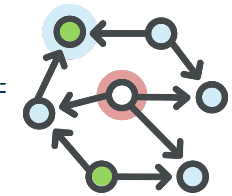
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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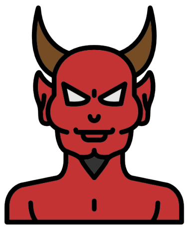
- Two players: Spoil∃r



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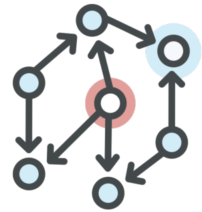
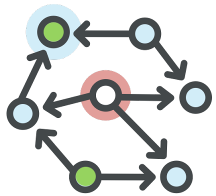
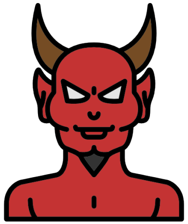
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 



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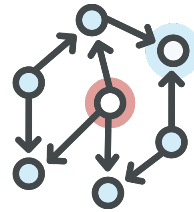
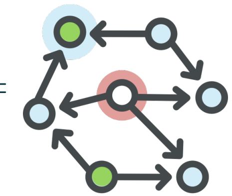
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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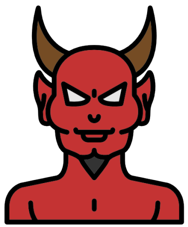

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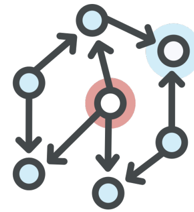
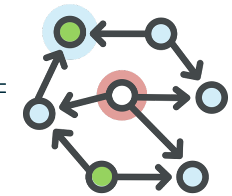
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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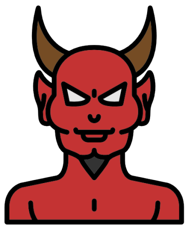

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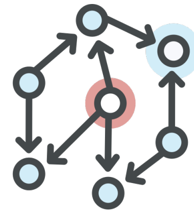
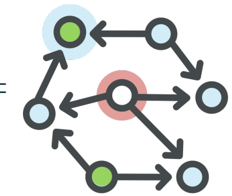
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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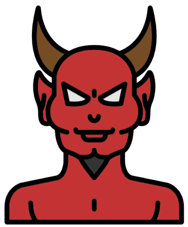

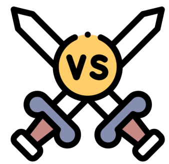

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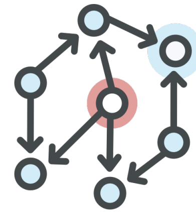
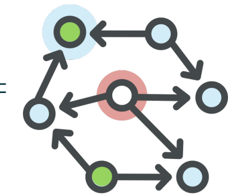
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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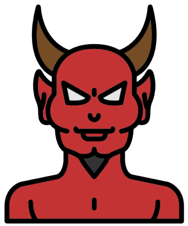

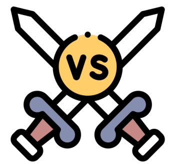

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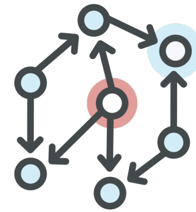
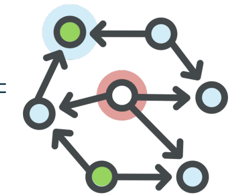
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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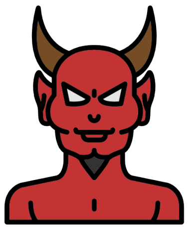

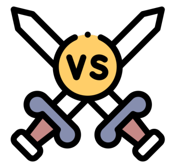

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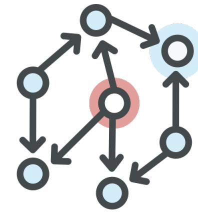
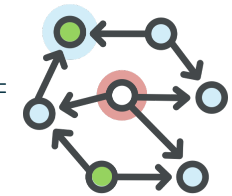
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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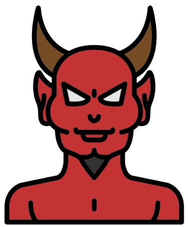

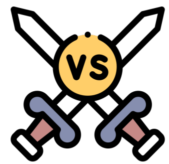

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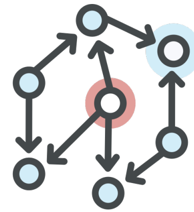
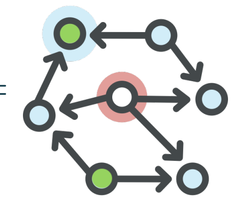
$\mathfrak{A} :=$  $\mathfrak{B} :=$ 

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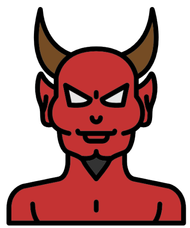
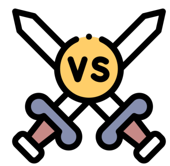

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1. ∃ selects a structure (say $\mathfrak{A}$) and picks an element (say $a_i \in A$)

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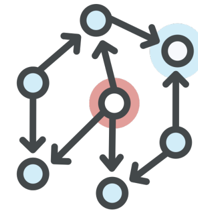
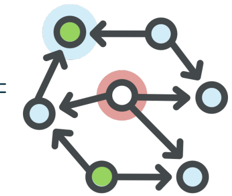
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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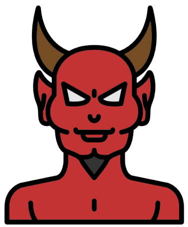

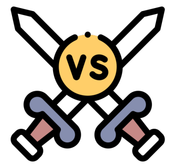

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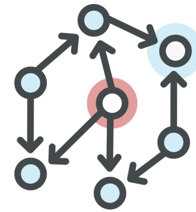
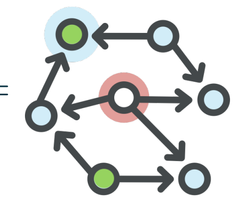
$\mathfrak{A} :=$  $\mathfrak{B} :=$ 

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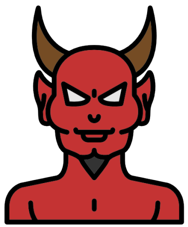 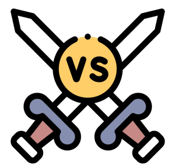

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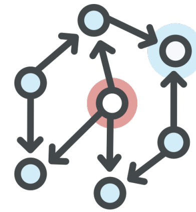
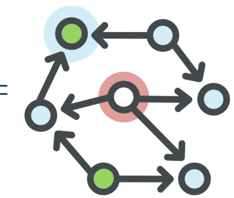
$\mathfrak{A} :=$  $\mathfrak{B} :=$ 

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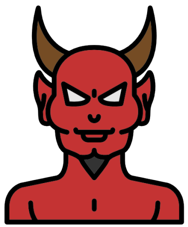 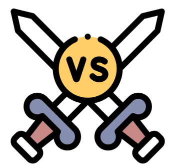

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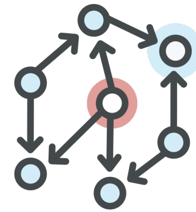
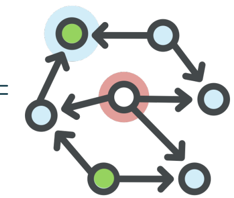
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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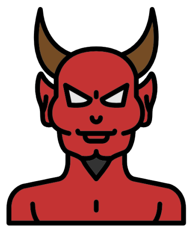 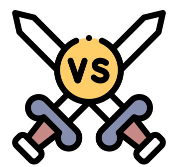

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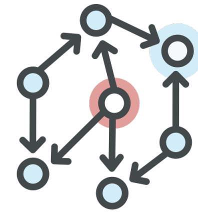
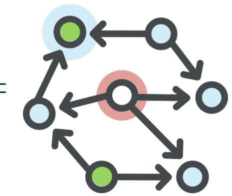
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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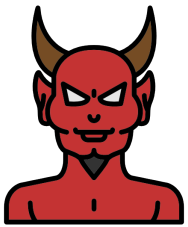

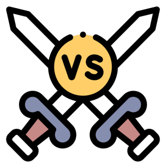

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Theorem (Fraïssé 1954 & Ehrenfeucht 1961)

$\forall$ has a winning strategy in m-round Ehrenfeucht-Fraïssé game on τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ iff $\mathfrak{A} \equiv^{\tau}_{m} \mathfrak{B}$.

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From m-round E-F Games to m-bisimulations

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- An abstract categorical and comonadic approaches: [Joyal et al.'1994] and [Abramsky'2022].

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From m-round E-F Games to m-bisimulations

Take $\mathcal{Z} := \left\{(\overline{a}_{1\ldots i}, \overline{b}_{1\ldots i}) \mid 1 \leq i \leq m, \text{ and } (\overline{a}, \overline{b}) \text{ is a history of the winning play of } \forall \text{ in } m\text{-round E-F game}\right\}$.

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Play as Duplicator, employing witnesses guaranteed by **(forth)** and **(back)** conditions.

Bisimulation as a more general concept

- One can define bisimulations $\simeq^{\mathsf{L}}_{\omega}$ (for ω rounds) for any logic L, e.g. Modal/Descr./Temporal logics.
- An abstract categorical and comonadic approaches: [Joyal et al.'1994] and [Abramsky'2022].
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Back and Forth Equivalence (a.k.a. Bisimulations)

We define an FO-m-bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$ as the relation $\mathcal{Z} \subseteq \bigcup_{i=0}^{m} A^i \times B^i$ with $(\varepsilon, \varepsilon) \in \mathcal{Z}$ fulfilling:

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m-Hintikka formulae

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Call $\varphi^m_{(\mathfrak{A},\varepsilon)}$ the m-Hintikka formula. Goal: $\mathfrak{B} \models \varphi^m_{(\mathfrak{A},\varepsilon)}$ iff there is an m-bisimulation $\mathcal{Z}$ between $\mathfrak{A}$ and $\mathfrak{B}$.

Proof ($\Leftarrow$) [We leave ($\Rightarrow$) as an exercise.]

Induction over k. Assumption: For any $(\bar{a},\bar{b}) \in \mathcal{Z}$ with $|\bar{a}|=|\bar{b}|=m-k$ we have $\mathfrak{B} \models \varphi^i_{(\mathfrak{A},\bar{a})}(\bar{b})$.

For $k=0$ we are done by **(atomic harmony)**. For $k>0$, take $(\bar{a},\bar{b}) \in \mathcal{Z}$ with $|\bar{a}| = |\bar{b}| = m-k-1$.

Take any $c \in A$. By **(forth)** there is $d \in B$ so that $(\bar{a}c, \bar{b}d) \in \mathcal{Z}$. By ind. ass. $\mathfrak{B} \models \varphi^i_{(\mathfrak{A},\bar{a}c)}(\bar{b}d)$.

m-Hintikka formulae

Goal: describe the m-isomorphism type of a τ-structure $\mathfrak{A}$ with an $\mathrm{FO}_m[\tau]$ formula.

Fix a structure $\mathfrak{A}$, a k-tuple $\bar{a}$ from A, and a k-tuple of variables $\bar{x}$. Define $\varphi^k_{(\mathfrak{A},\bar{a})}(\bar{x})$ inductively as

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Thus $\mathfrak{B} \models \exists x_i\ \varphi^k_{\bar{a}c}(\bar{b}, x_i)$.

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Thus $\mathfrak{B} \models \exists x_i\, \varphi^k_{\bar{a}c}(\bar{b}, x_i)$. By the choice of c, we conclude $\mathfrak{B} \models \bigwedge_{c\in A} \exists x_i\, \varphi^k_{\bar{a}c}(\bar{b}, x_i)$.

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Goal: describe the m-isomorphism type of a τ-structure $\mathfrak{A}$ with an $\mathrm{FO}_m[\tau]$ formula.

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Thus $\mathfrak{B} \models \exists x_i\; \varphi^k_{\bar{a}c}(\bar{b}, x_i)$. By the choice of c, we conclude $\mathfrak{B} \models \bigwedge_{c \in A} \exists x_i\; \varphi^k_{\bar{a}c}(\bar{b}, x_i)$.

By reasoning similarly and employing **(back)**, we conclude the satisfaction of the RHS of $\varphi^k_{\bar{a}c}(\bar{b})$. $\square$

Main theorem about Ehrenfeucht-Fraïssé games

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induction

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induction

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Lemma: For any τ-structures $\mathfrak{A}, \mathfrak{B}$ and $m \in \mathbb{N}$, the following are equivalent:

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2. There exists an *m*-bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$.

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Proof [**(2)** $\Rightarrow$ **(4)** by induction] Let $\mathcal{Z}$ be an *m*-bisimulation.

induction

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Proof [**(2)** $\Rightarrow$ **(4)** by induction] Let $\mathcal{Z}$ be an m-bisimulation. The case $m = 0 \rightsquigarrow$

induction

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induction atomic harmony

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Proof [**(2)** $\Rightarrow$ **(4)** by induction] Let $\mathcal{Z}$ be an m-bisimulation. The case $m = 0 \rightsquigarrow$ **(atomic harmony)**.

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Simplify $\mathrm{FO}_m[\tau]$

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induction atomic harmony Simplify $\mathrm{FO}_m[\tau]$

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induction   atomic harmony   Simplify $\mathrm{FO}_m[\tau]$ reduce

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induction atomic harmony Simplify $\mathrm{FO}_m[\tau]$ reduce

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induction atomic harmony Simplify $\mathrm{FO}_m[\tau]$ reduce intro

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induction atomic harmony Simplify $\mathrm{FO}_m[\tau]$ reduce intro

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induction atomic harmony Simplify $\mathrm{FO}_m[\tau]$ reduce intro witness

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induction   atomic harmony   Simplify $\mathrm{FO}_m[\tau]$ reduce   intro   witness

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We will now go through slides 78–110 from ESSLI 2016 by [Diego Figueira].

Another technique: Locality

Idea: First order logic can only express “local” properties

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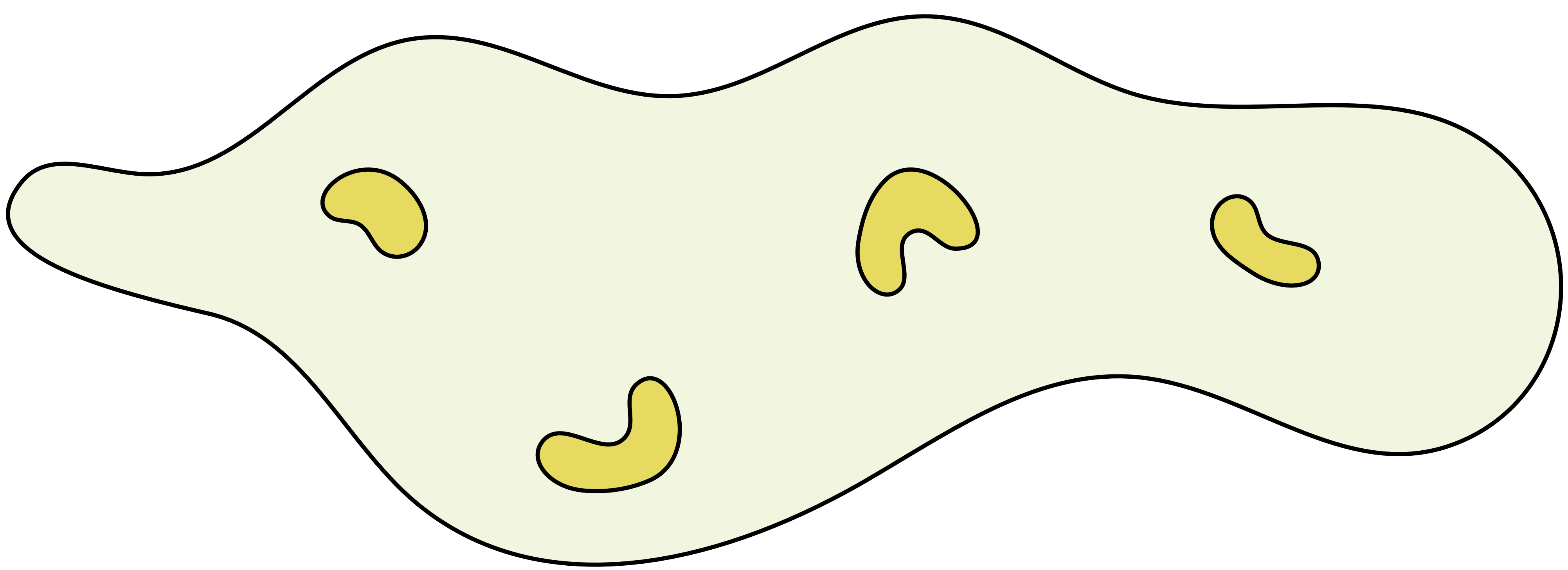
Local = properties of nodes which are close to one another



Hanf locality

Definition. The **Gaifman graph** of a structure $S = (V, R_1, \dots, R_m)$ is the **undirected** graph

$$G_S = (V, E) \quad \text{where} \quad E = \{ (u, v) \mid \exists\, (\dots, u, \dots, v, \dots) \in R_i \text{ for some } i \}$$

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200	Mr Smith	Cadillac
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Car	Country
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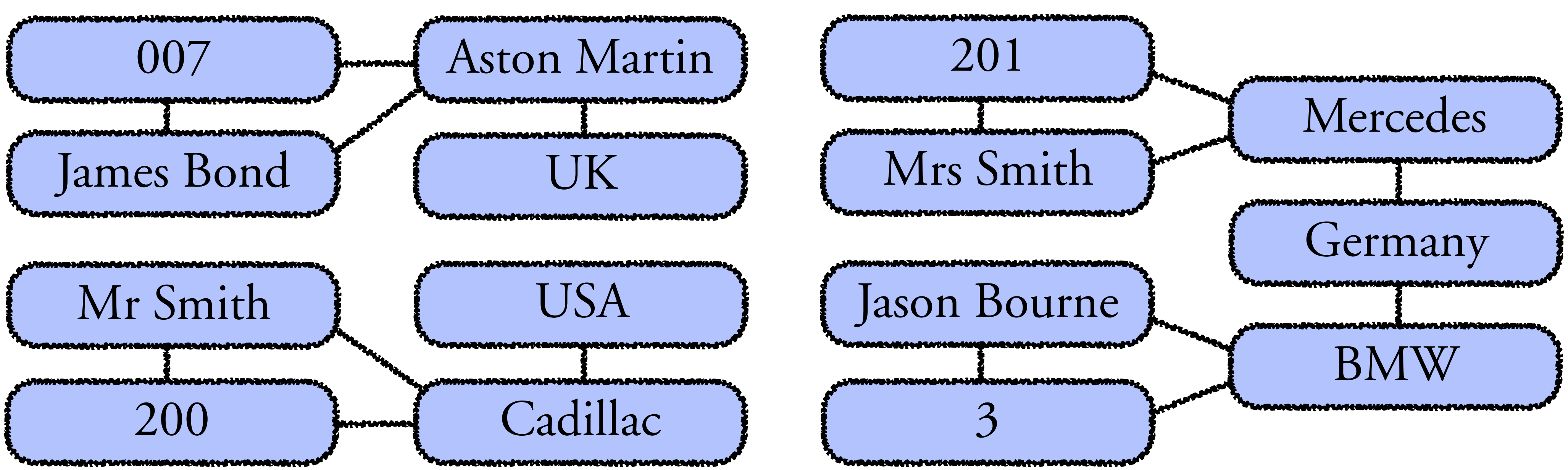

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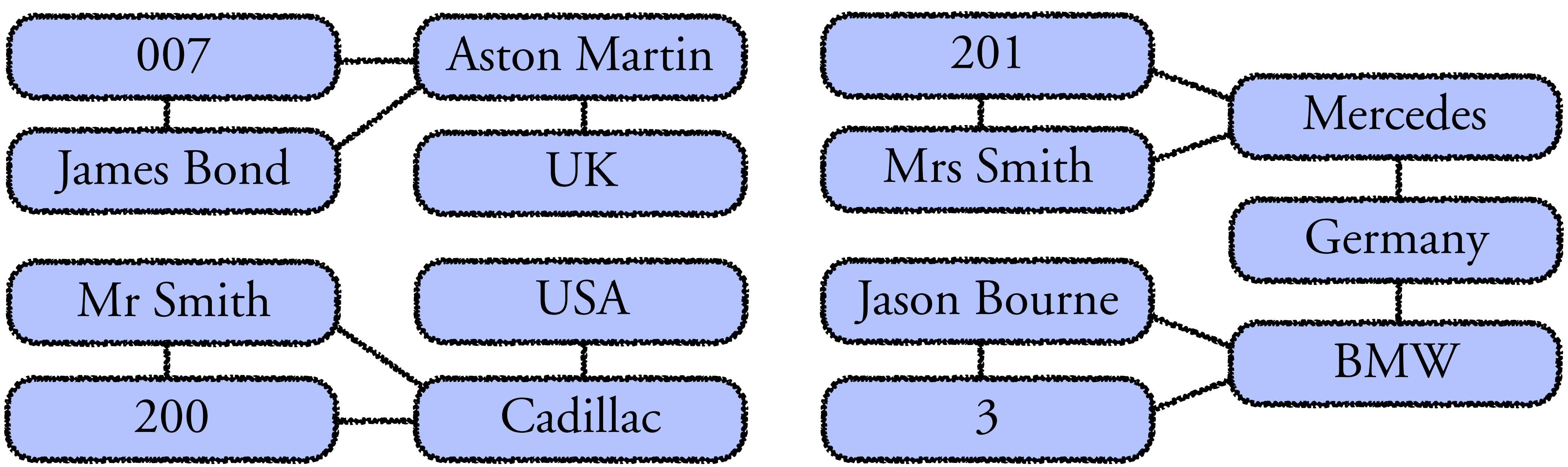
The Gaifman graph of a graph G is the underlying undirected graph.



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- $\text{dist}(u, v)$ = **distance** between u and v in the Gaifman graph
- $S[u, r]$ = sub-structure induced by $\{ v \mid \text{dist}(u, v) \leq r \}$ = **ball** around u of radius r

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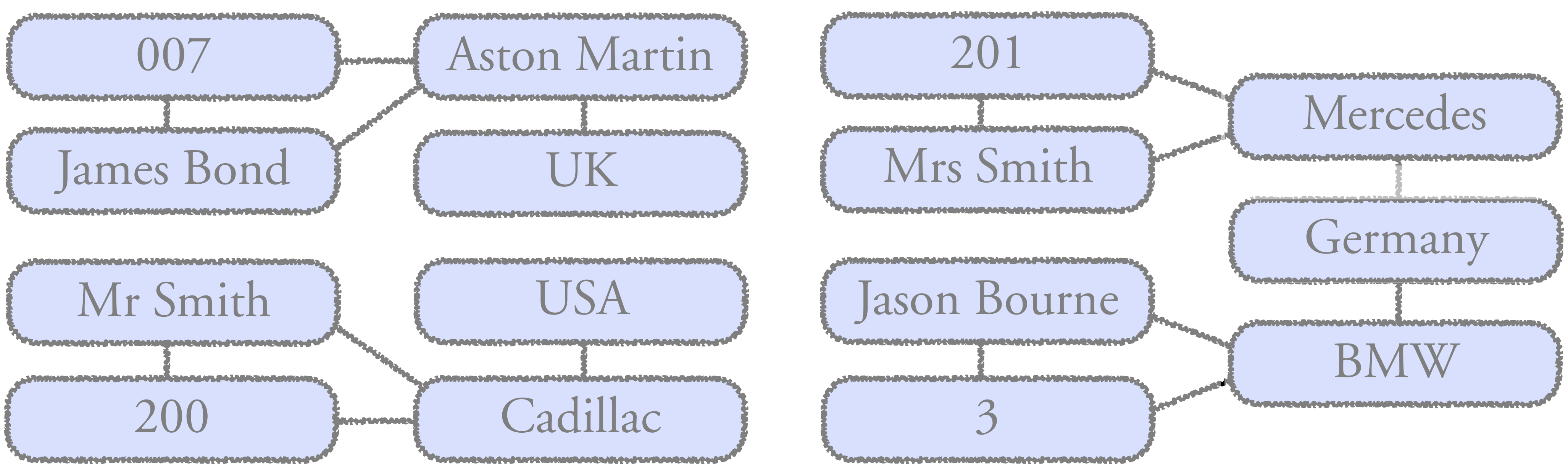

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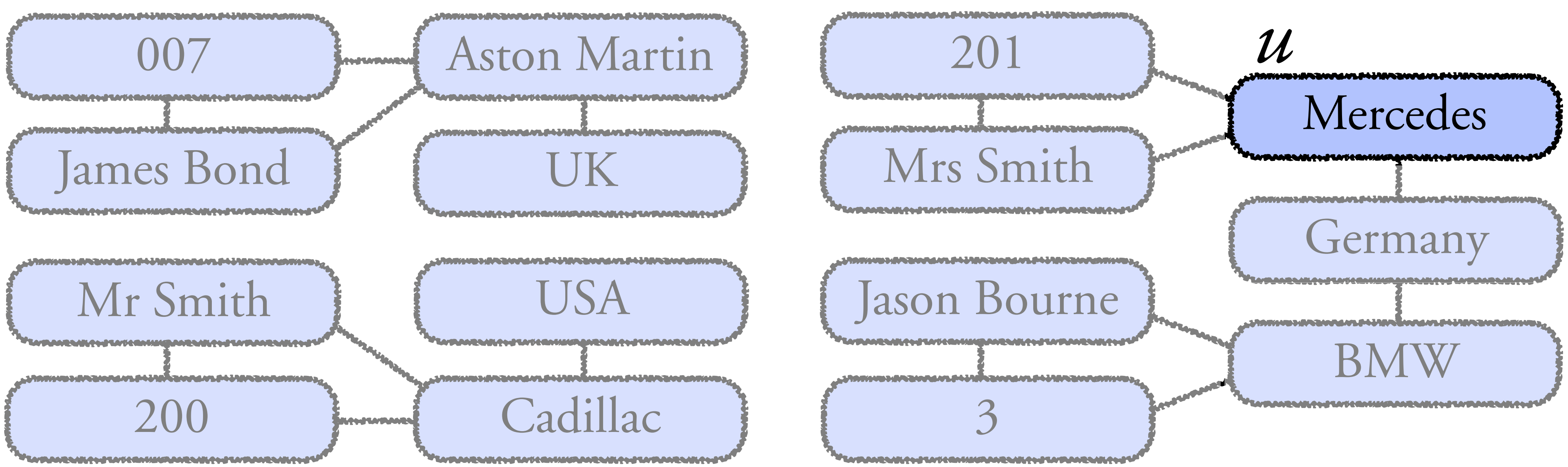

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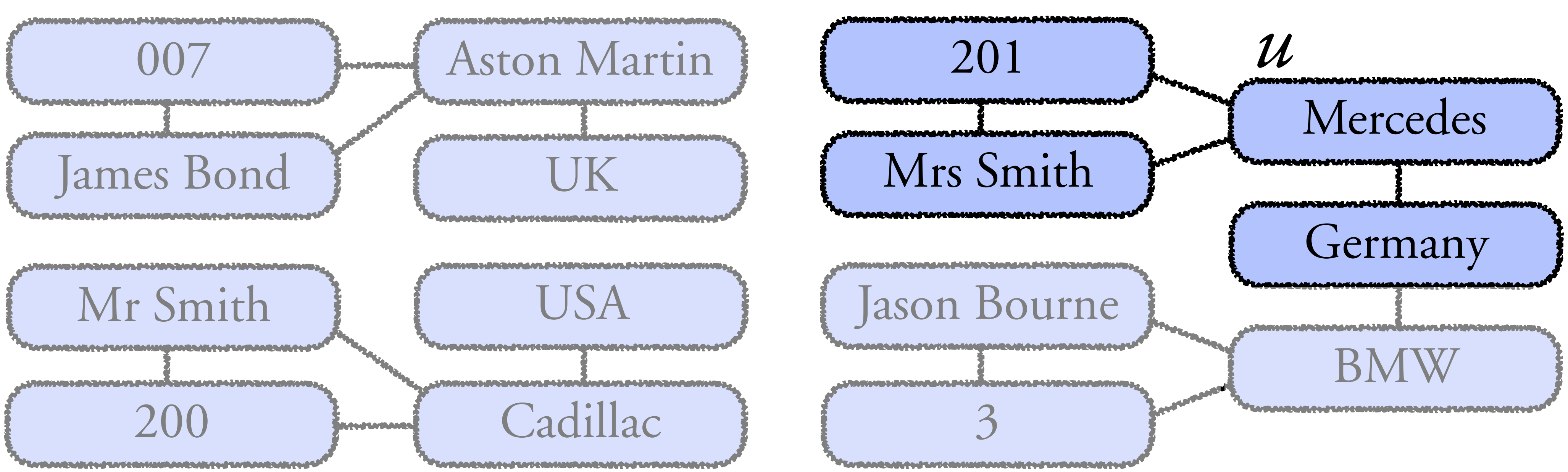

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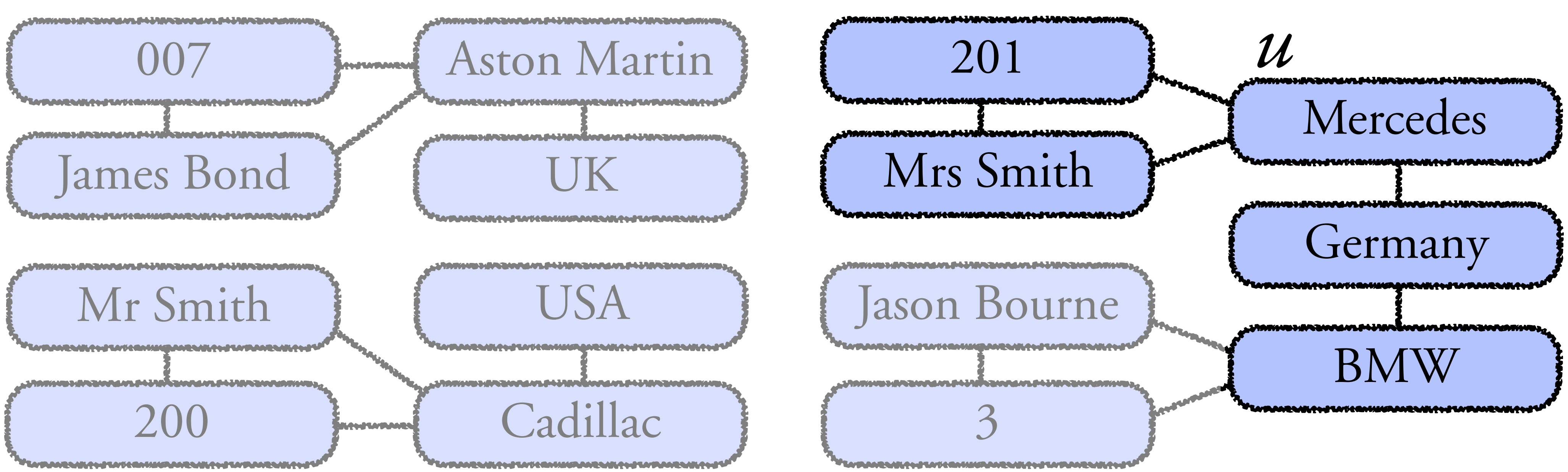

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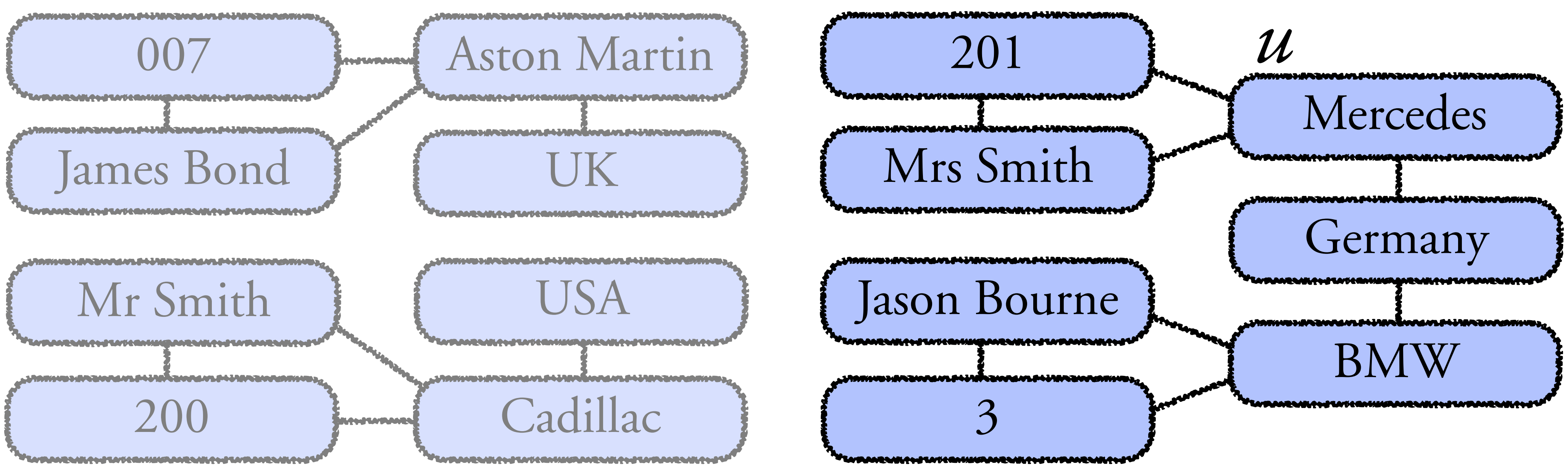

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Definition. Two structures S_1 and S_2 are **Hanf(r, t) - equivalent**

iff for each structure B, the two numbers

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are *either the same* or *both* $\geq t$.

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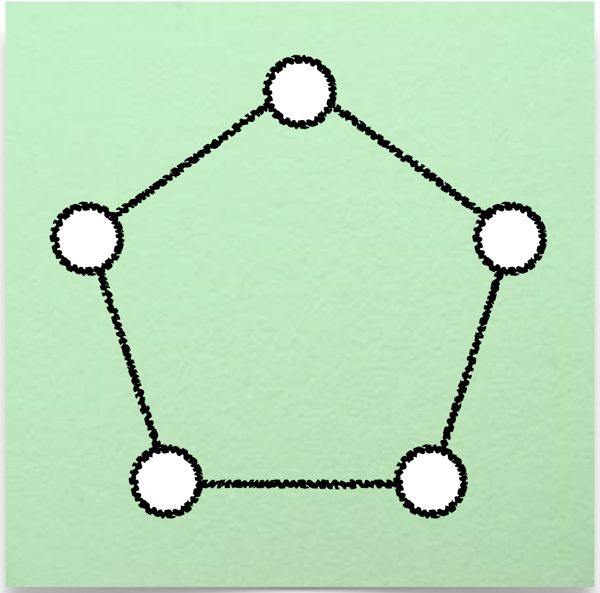
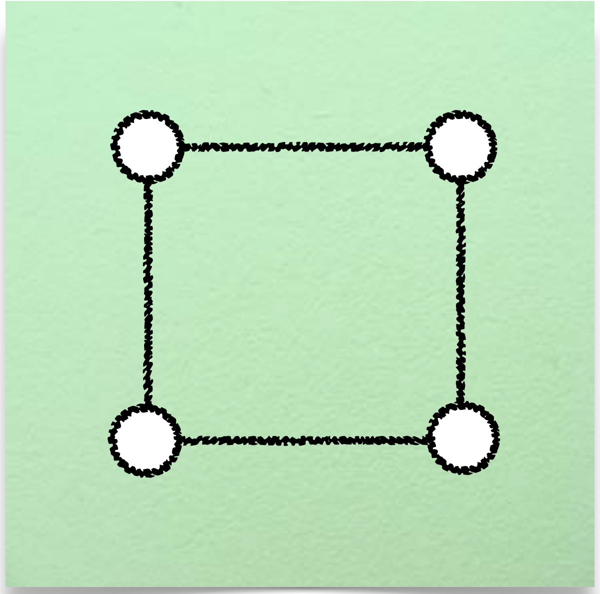
Example. S_1, S_2 are Hanf$(1,1)$ - equivalent iff they have the *same balls* of radius 1





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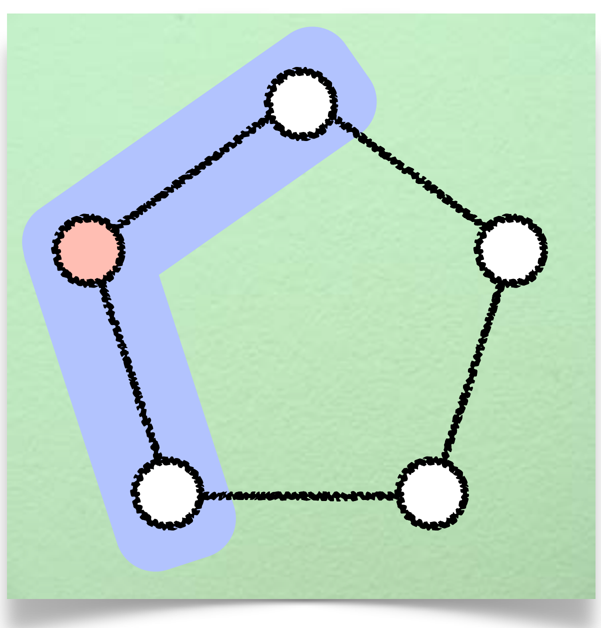

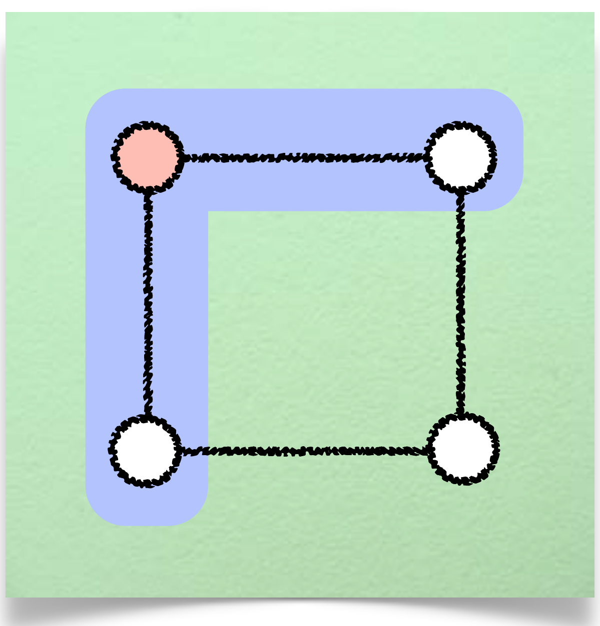

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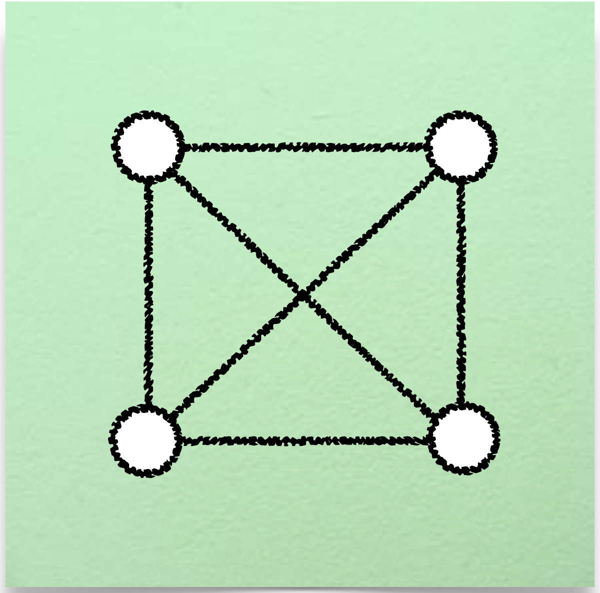

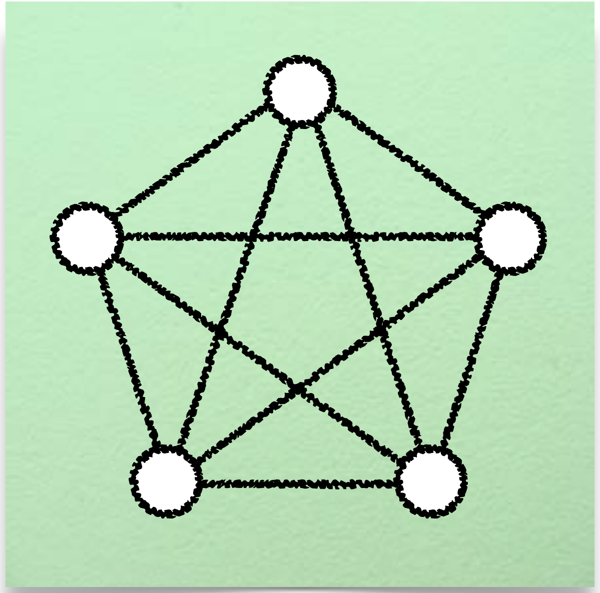

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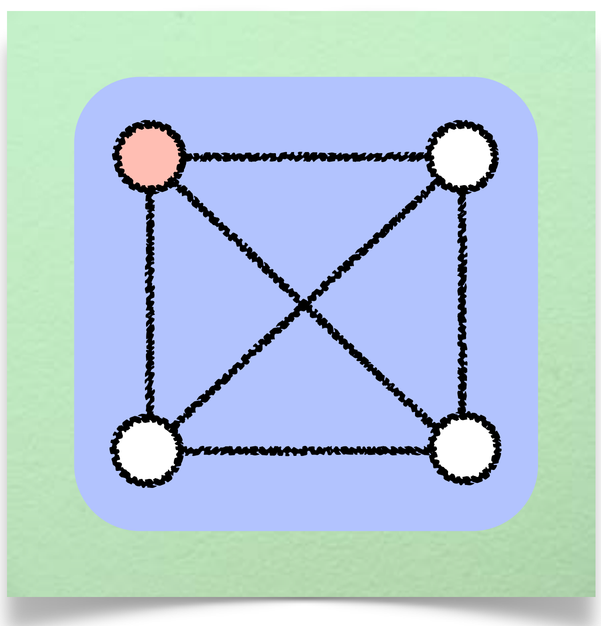

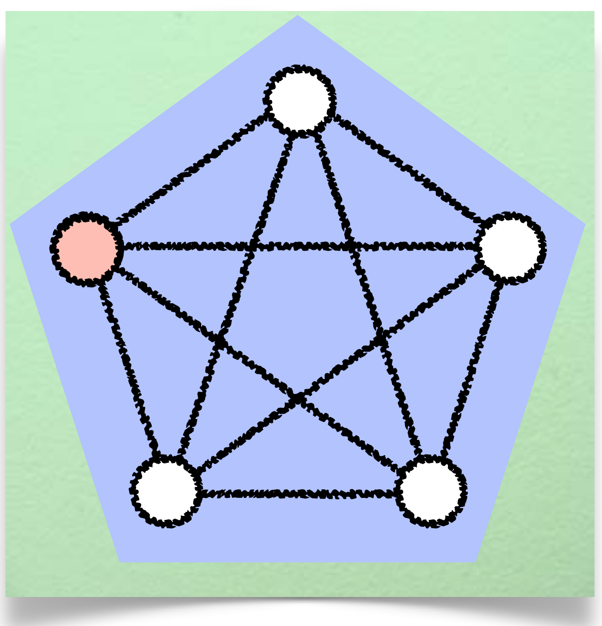

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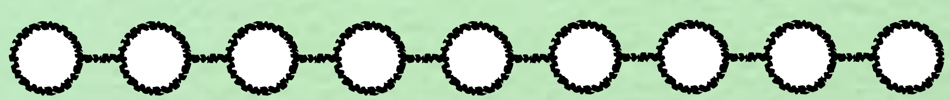

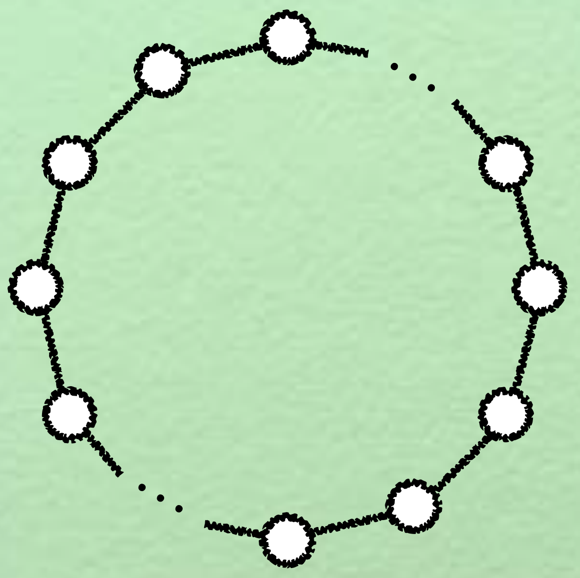

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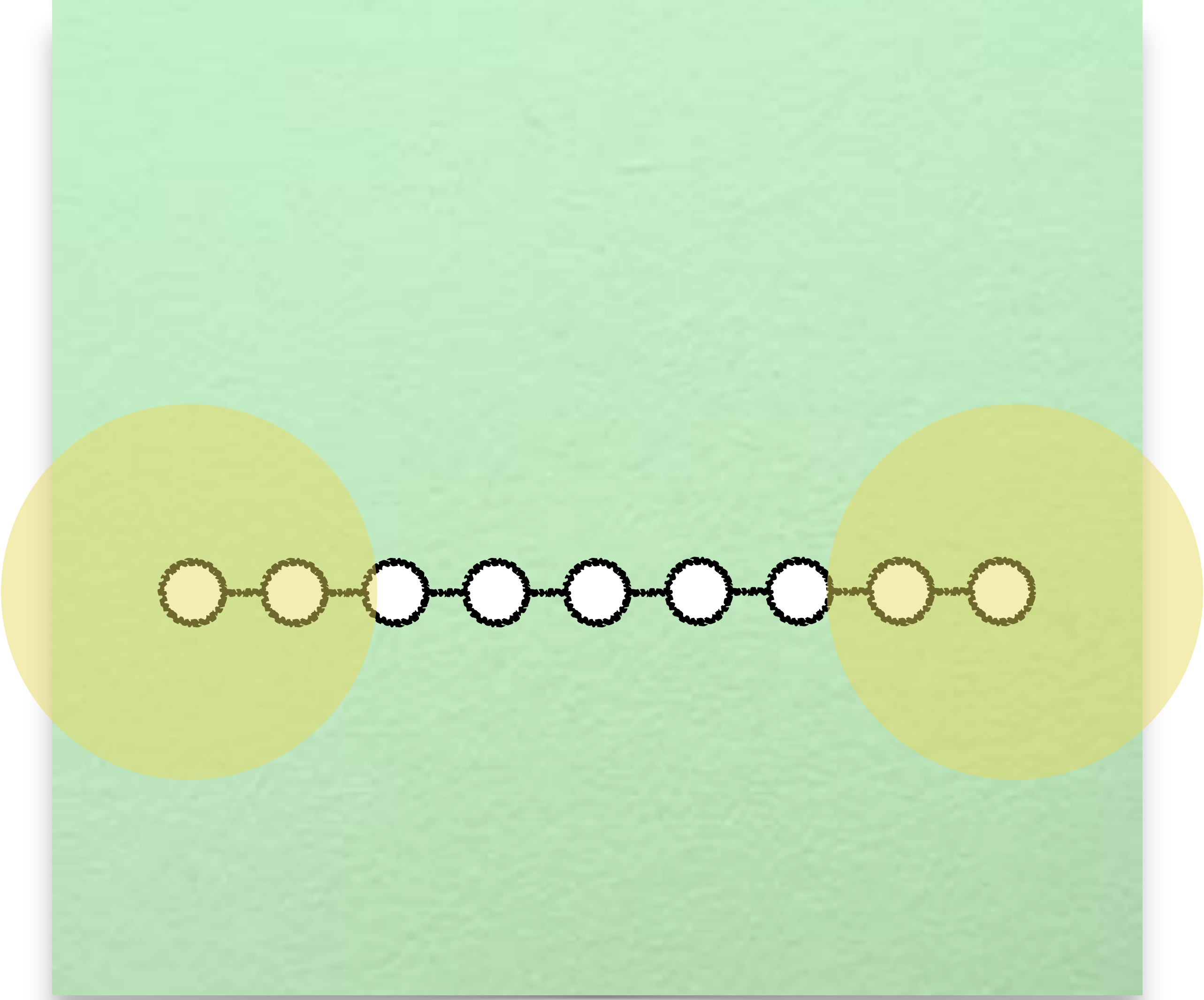

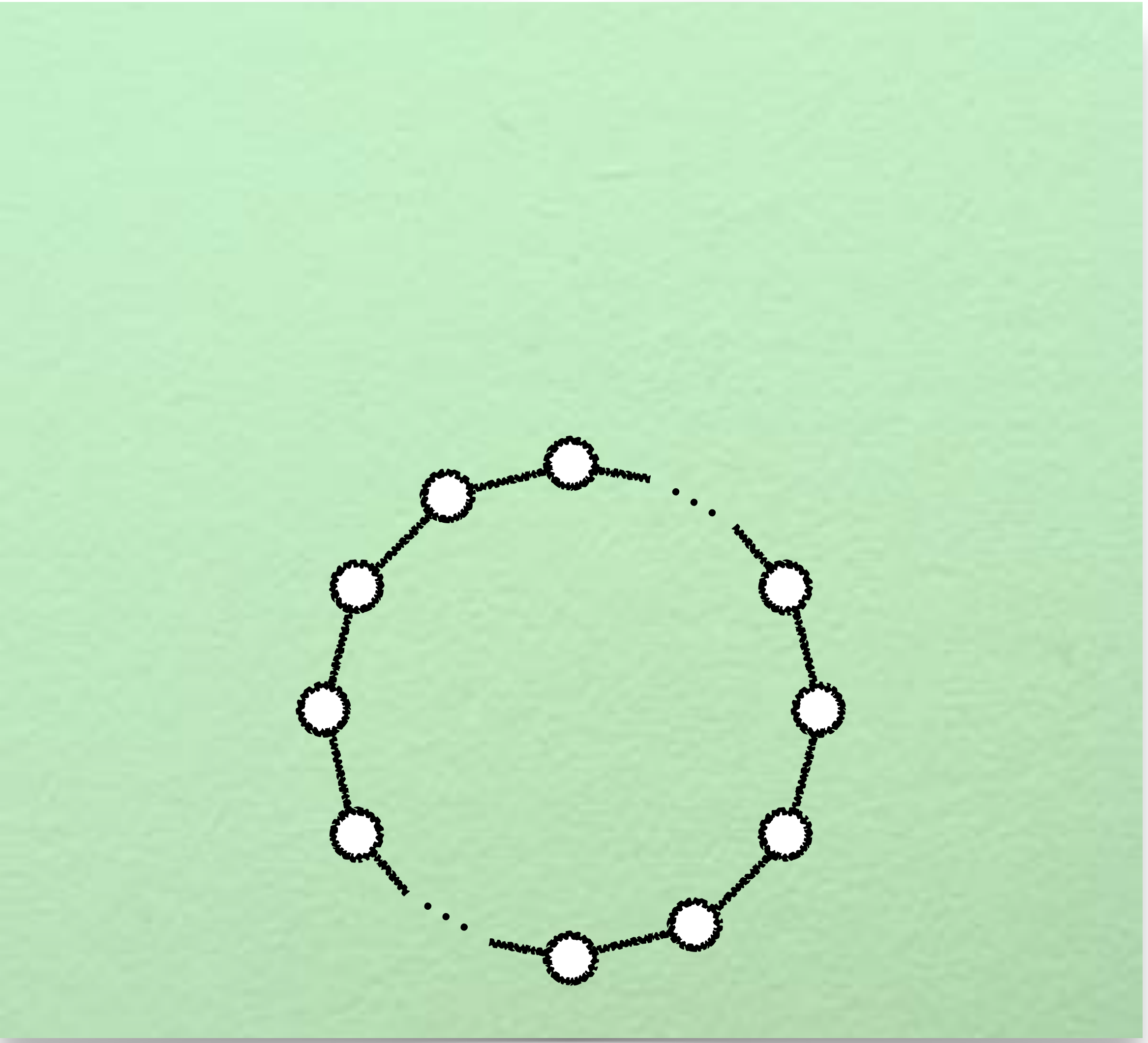

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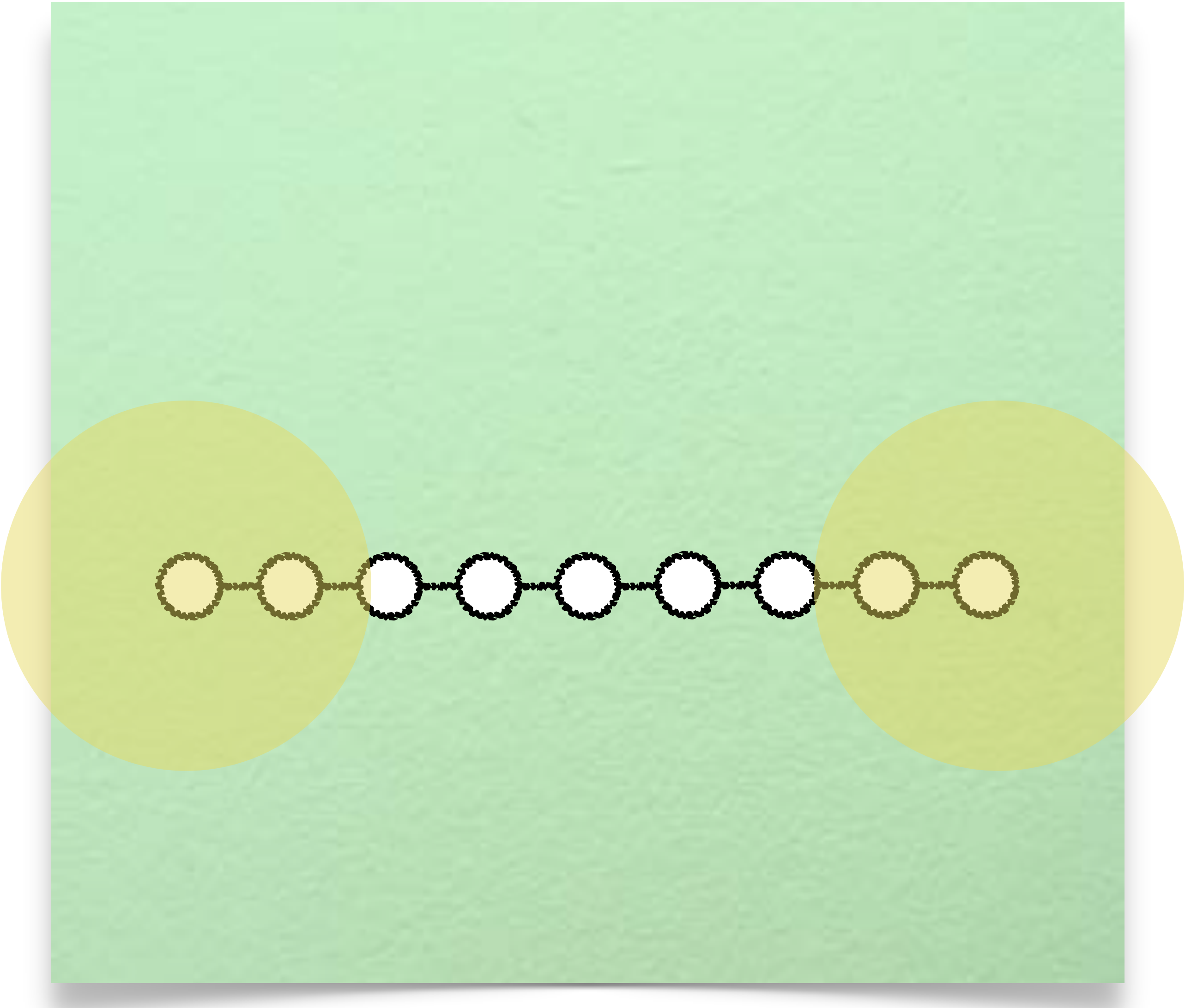

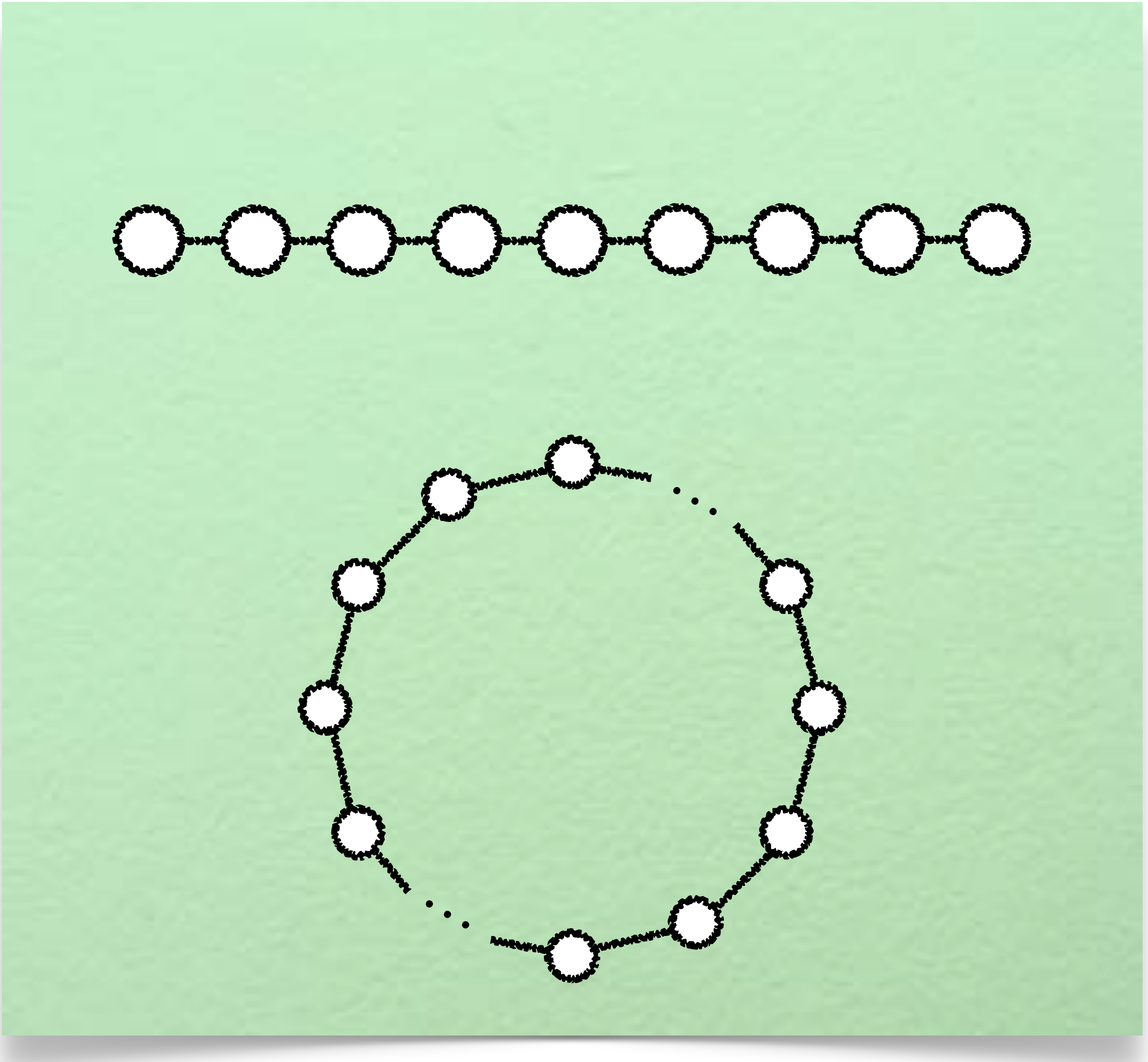

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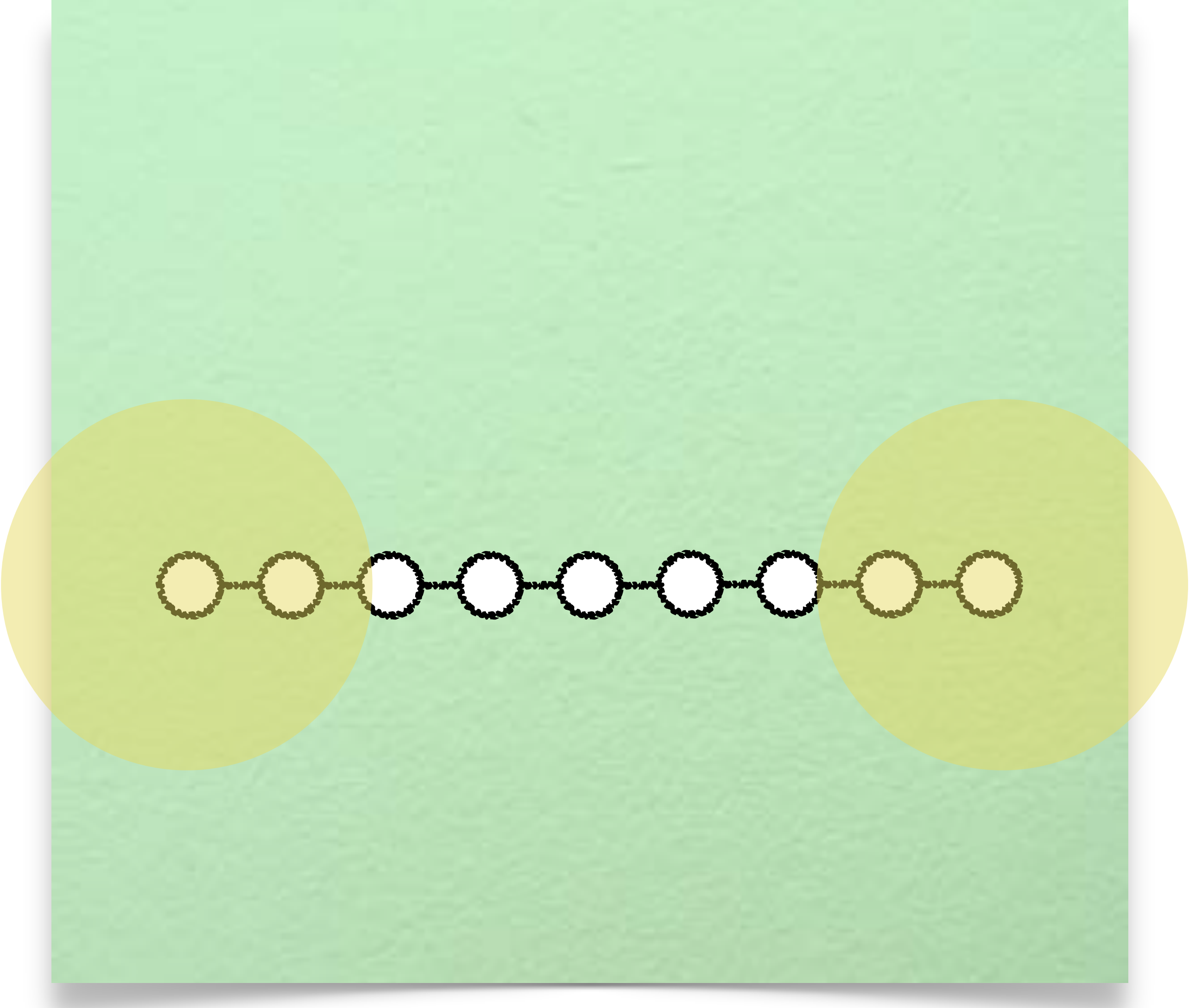

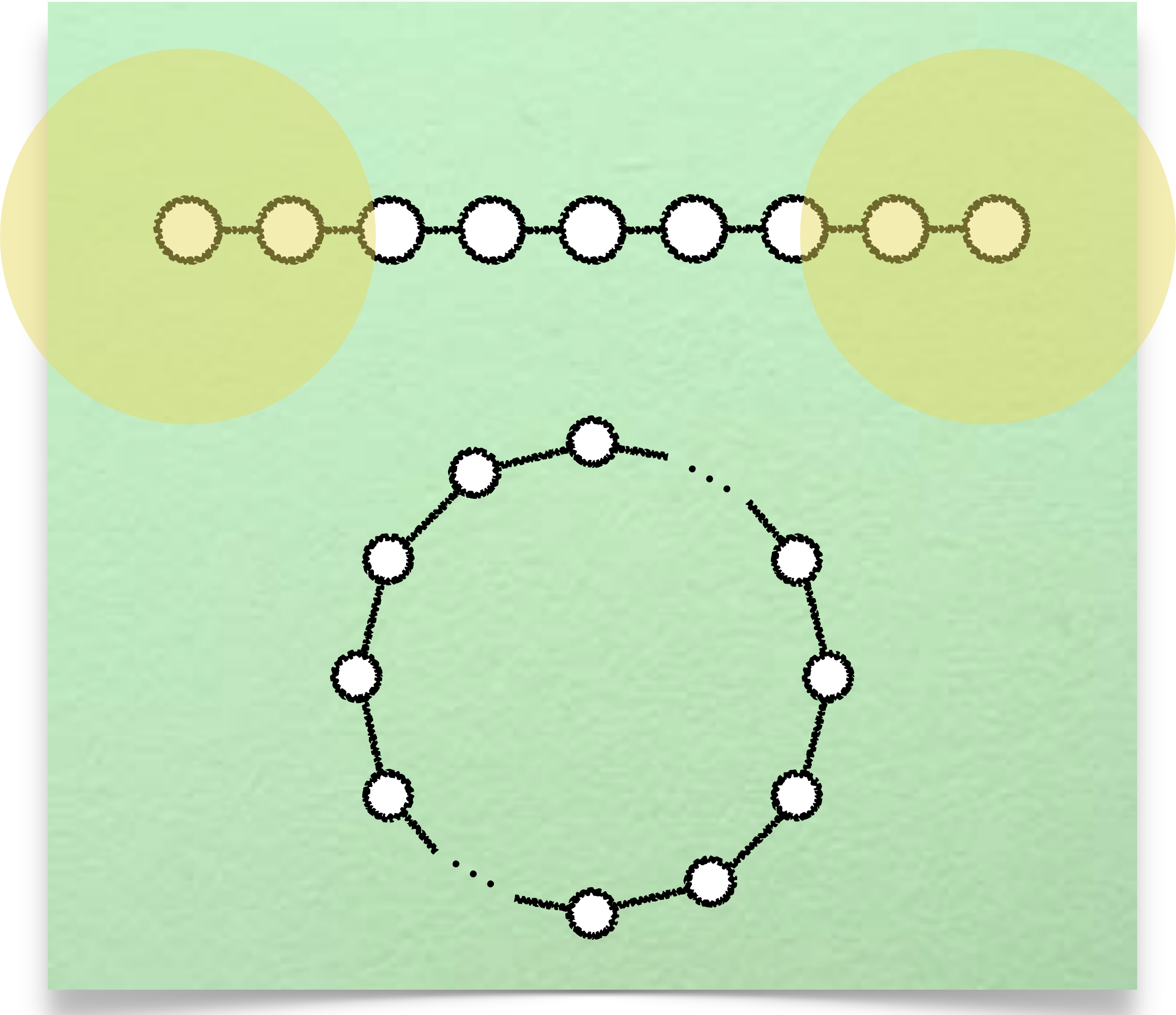

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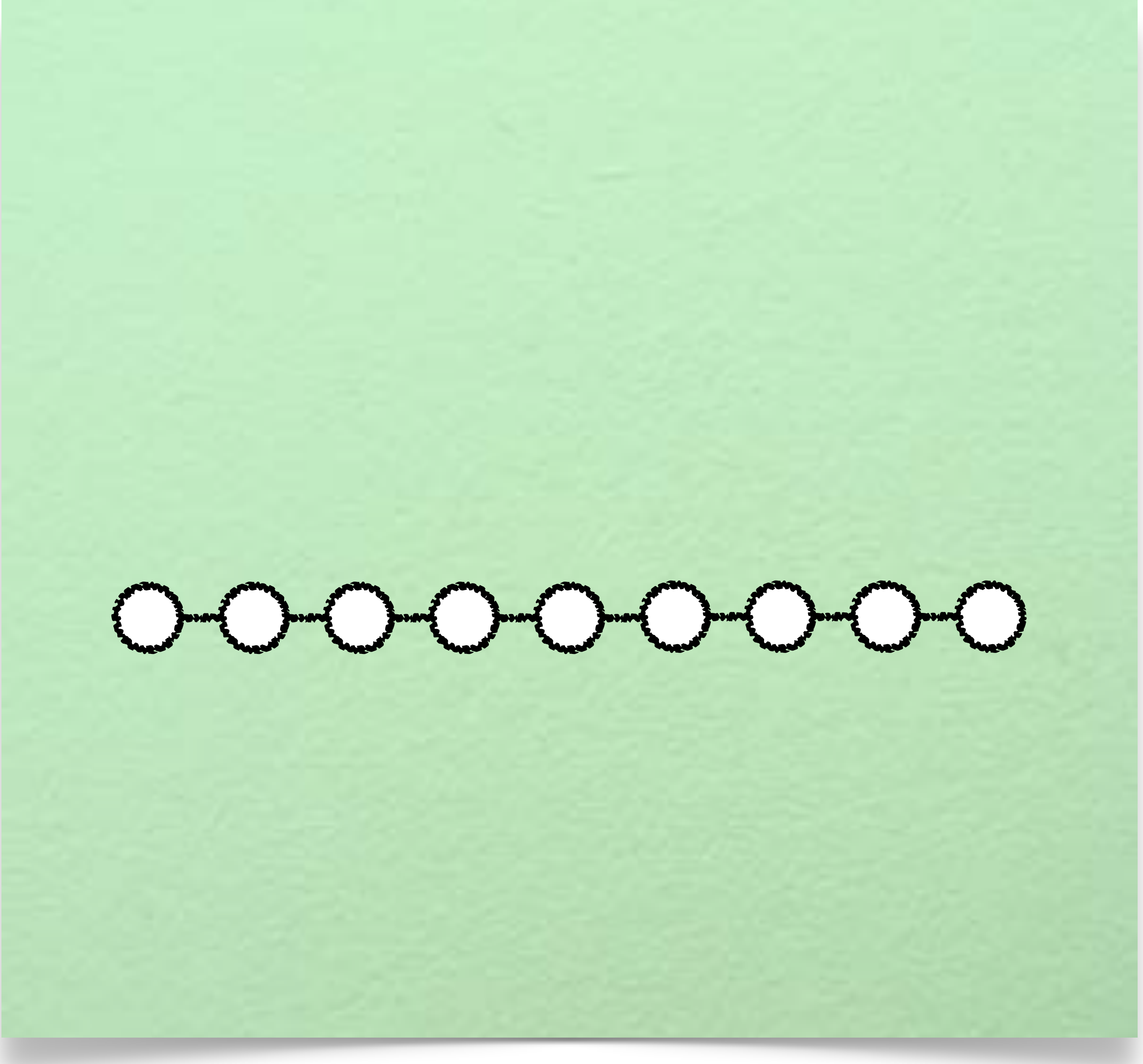

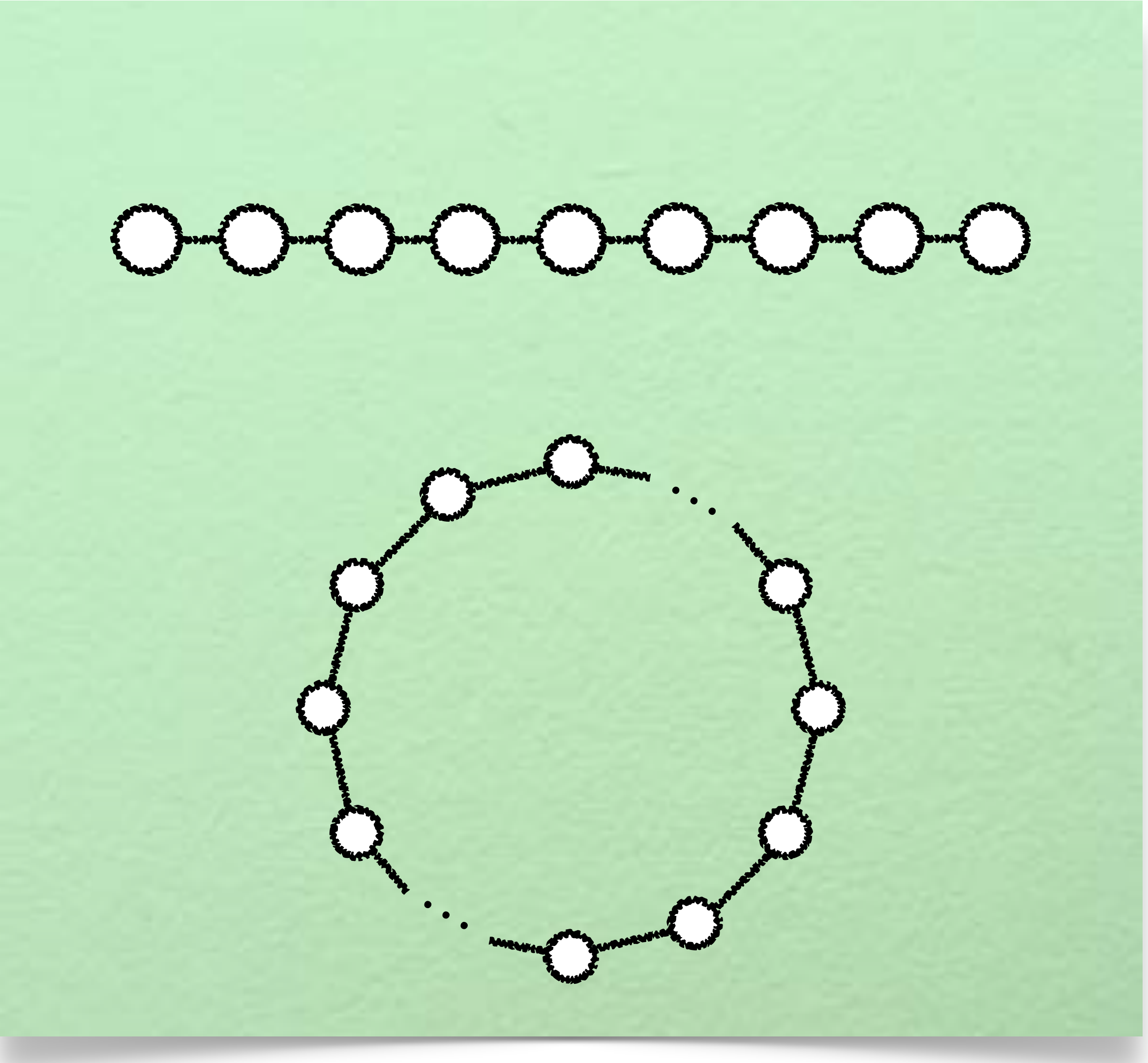

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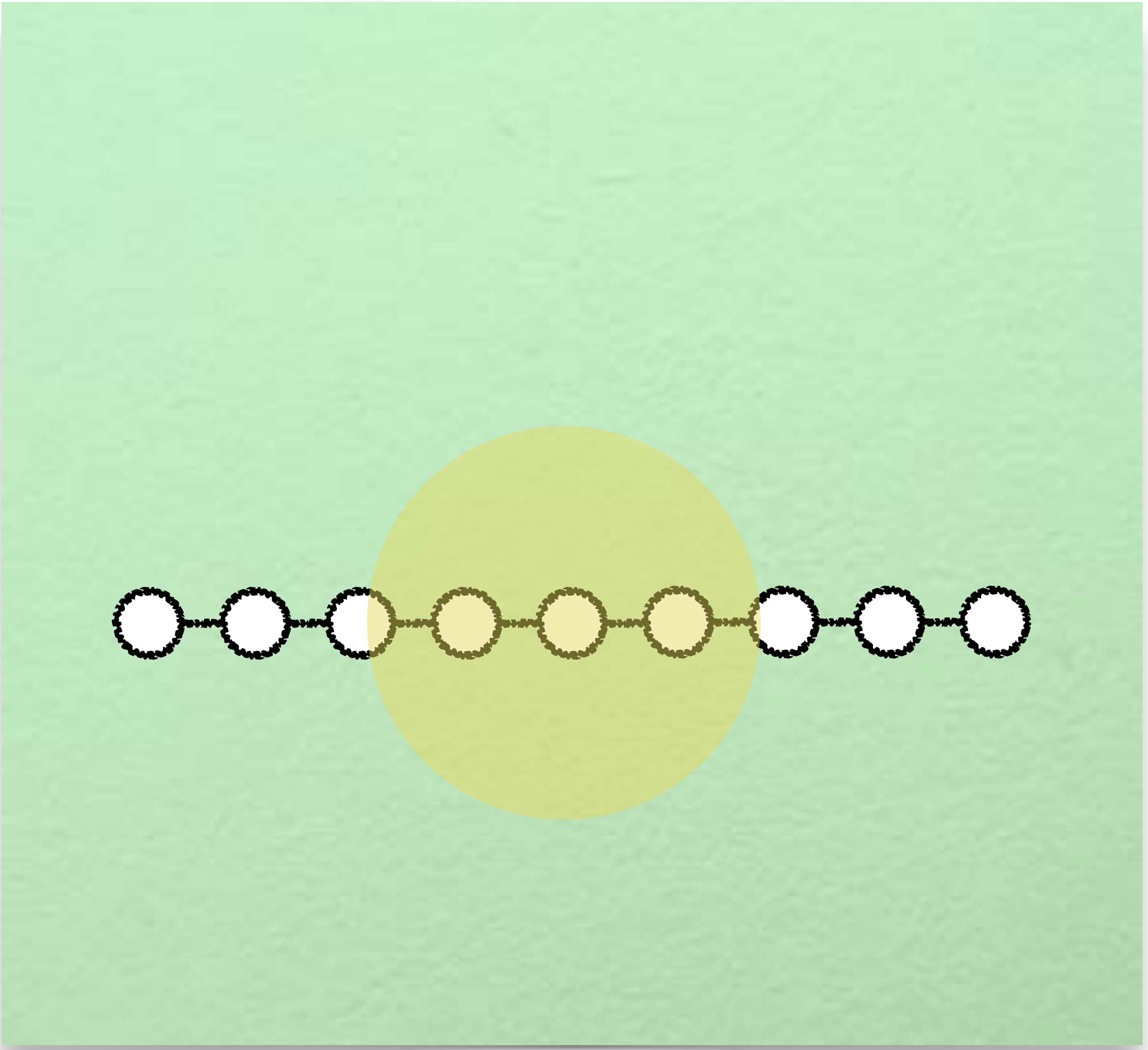

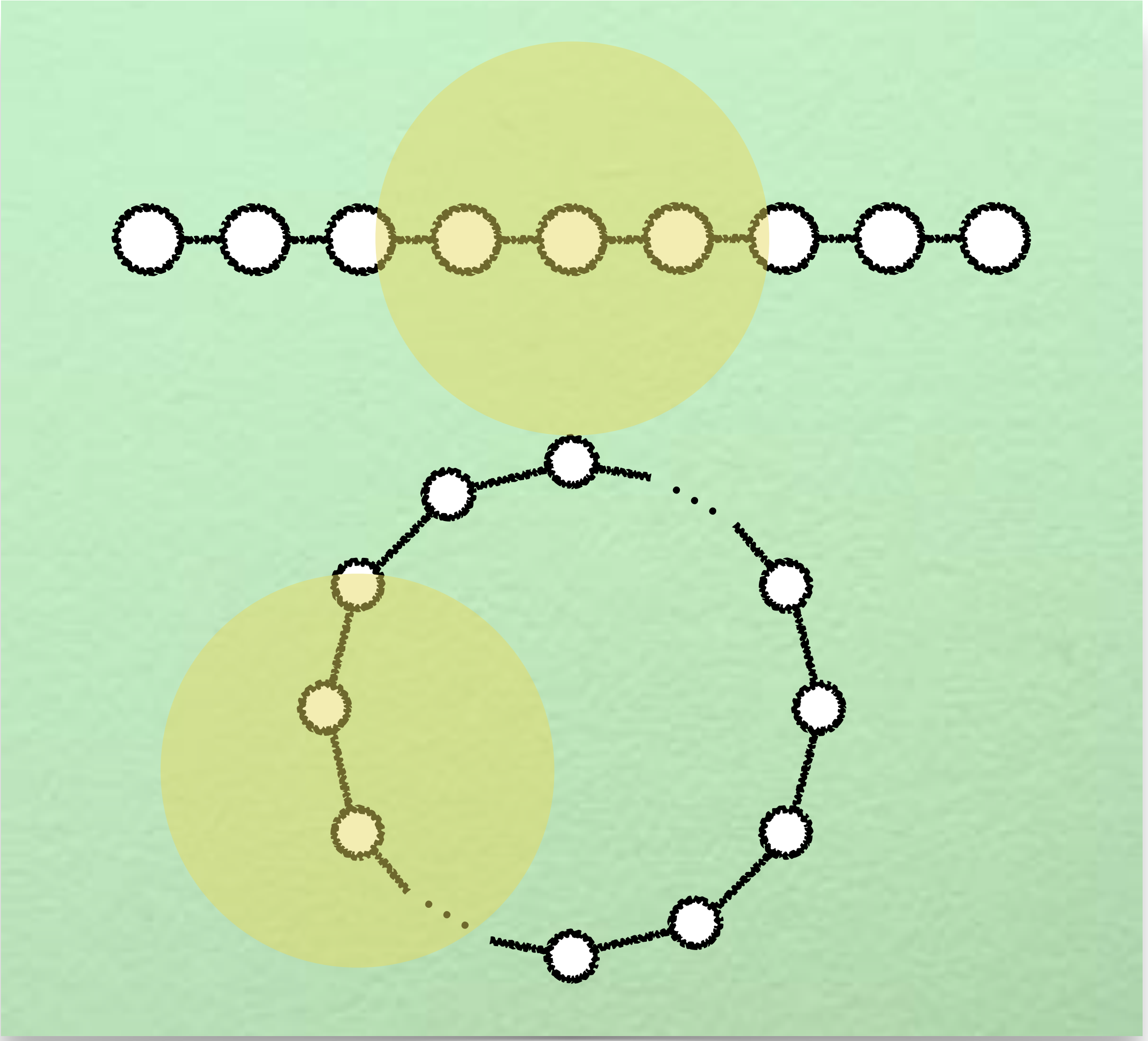

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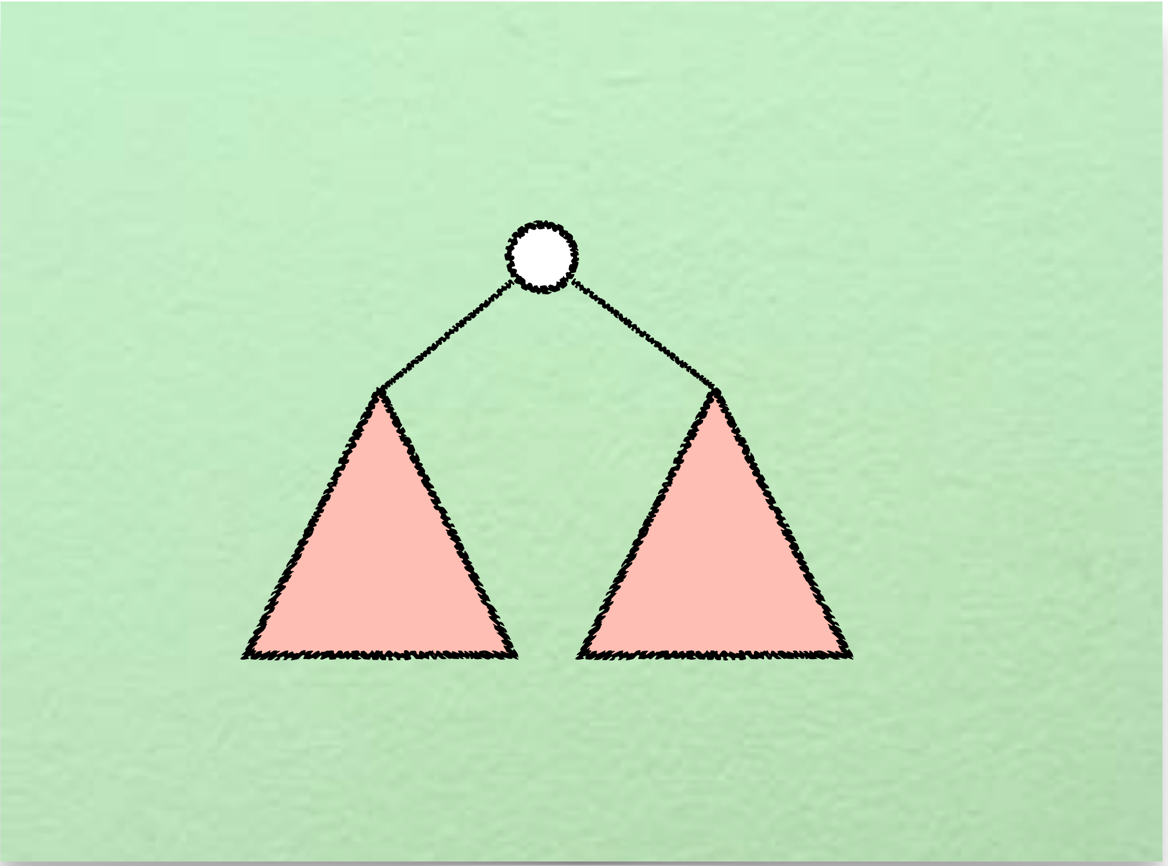

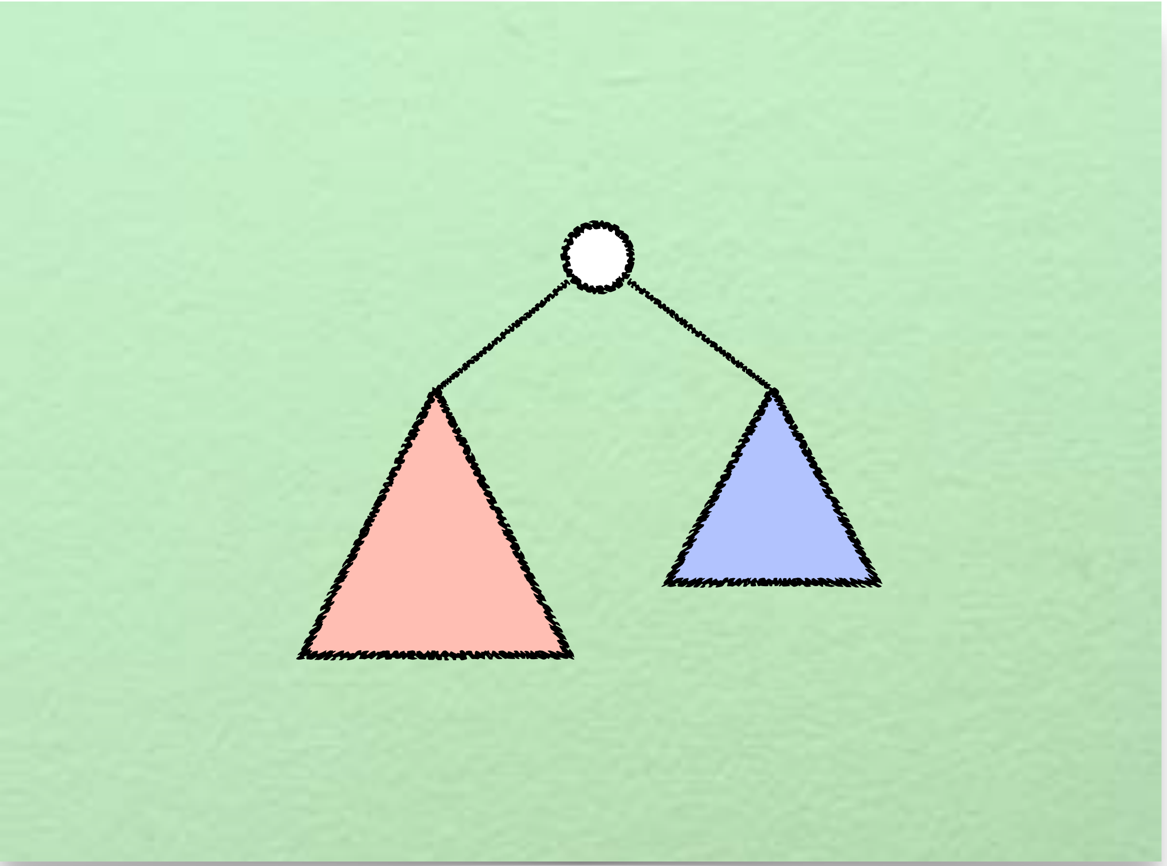

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Not (n+2)-equivalent yet they have the same 2^n-1 balls.

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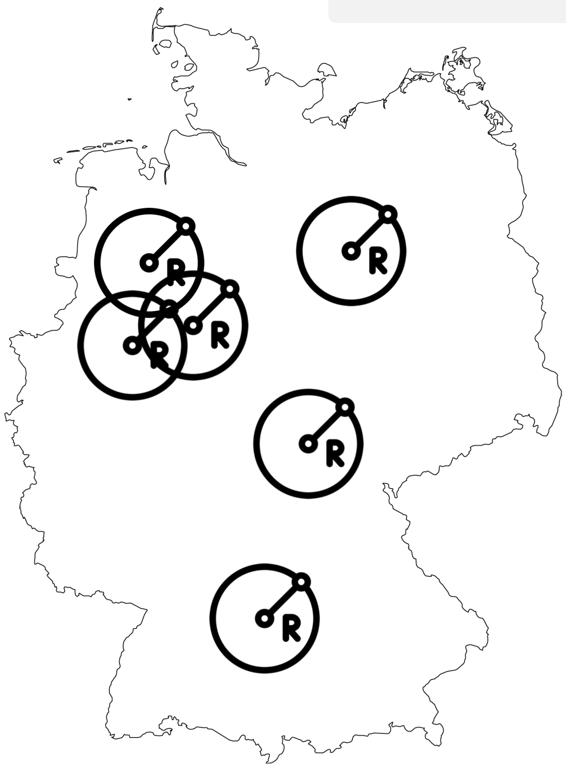
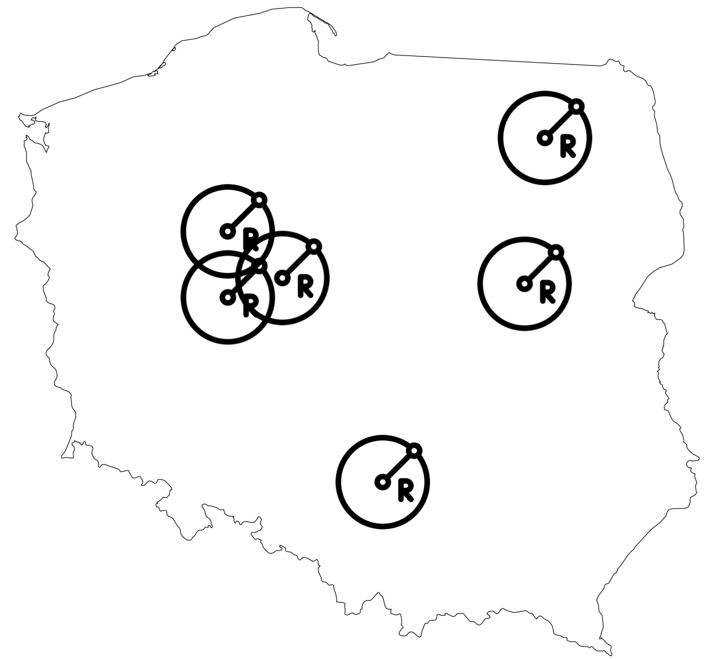
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

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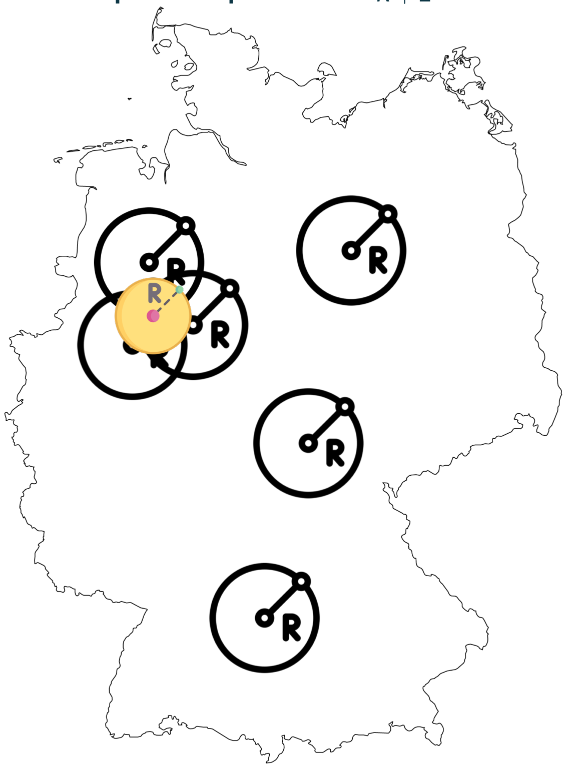

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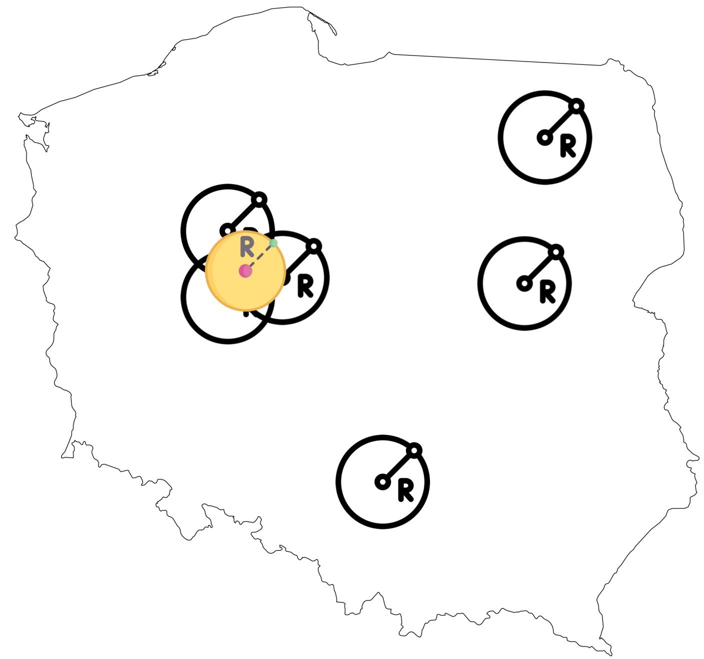

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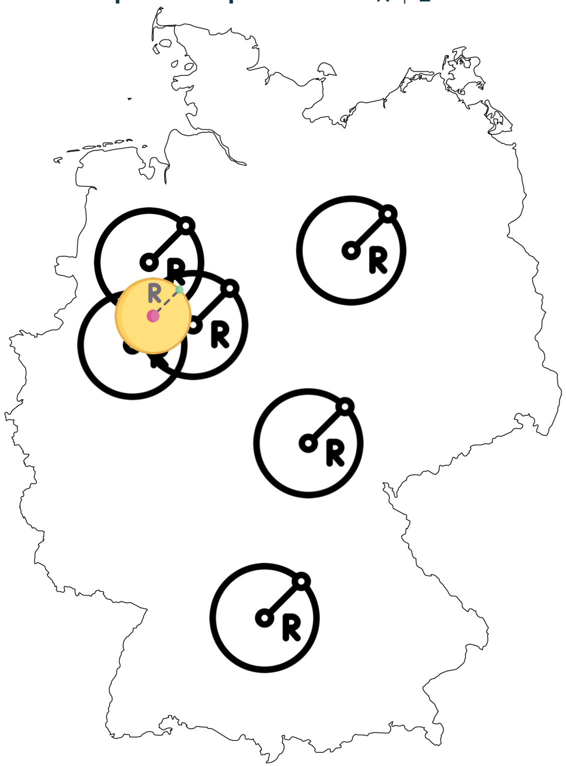

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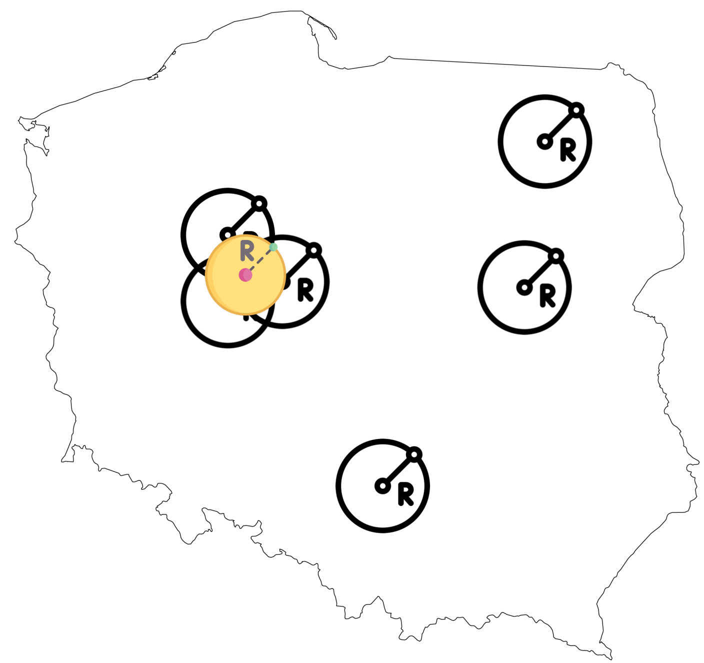

We know how to reply since $\mathfrak{A}[a_{k+1}, 3^{n-k-1}]$ is fully contained in some previously selected balls.

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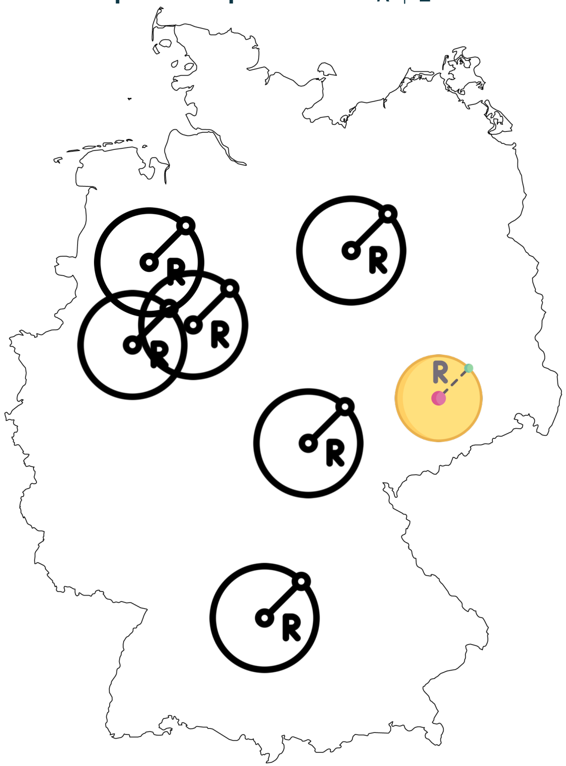

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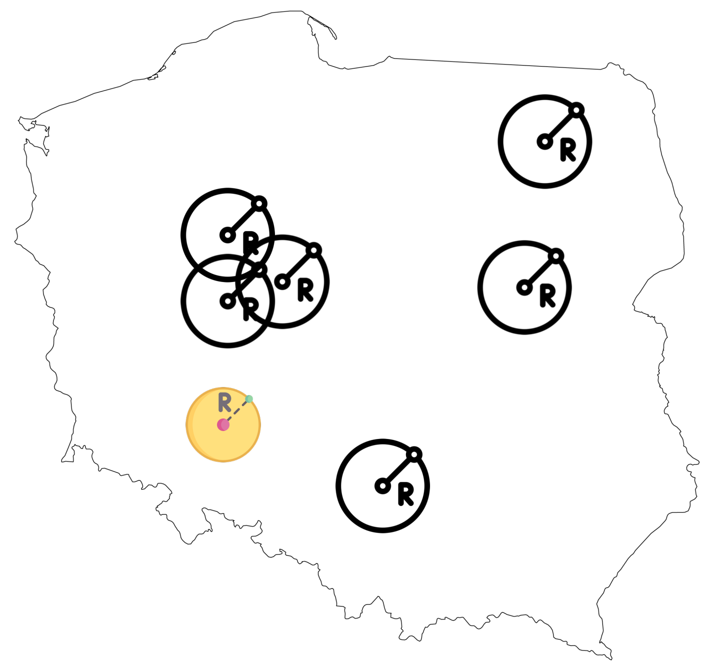

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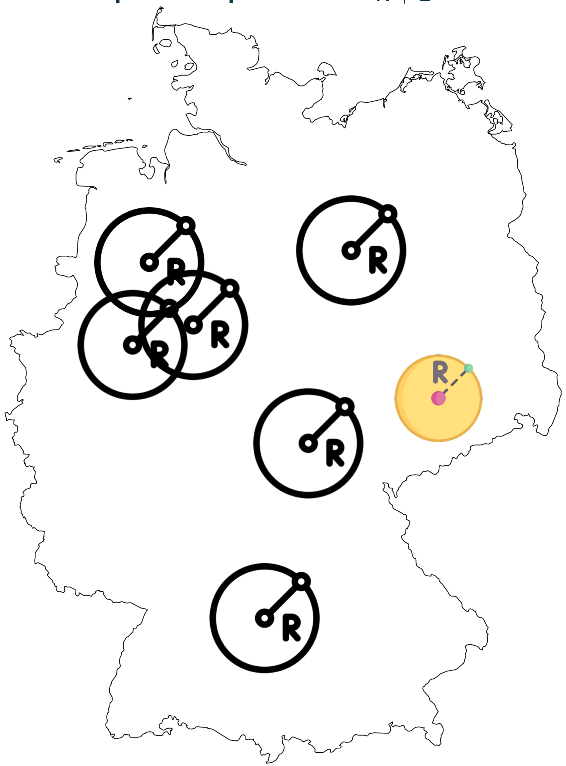
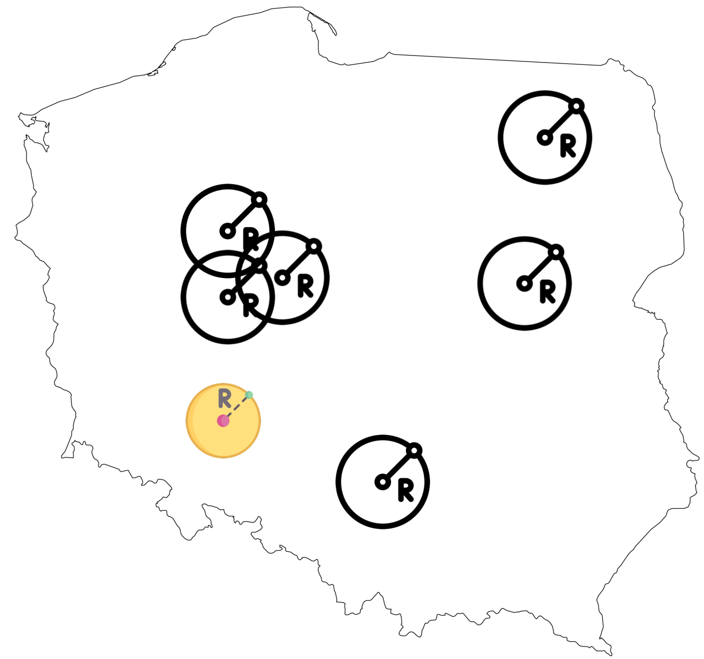
$\mathfrak{A} :=$ 

$\mathfrak{B} :=$ 

We know how to reply since we have sufficiently many realisations of $\mathfrak{A}[a_{k+1}, 3^{n-k-1}]$ in $\mathfrak{B}$.

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