

A systematic proof theory for several modal logics

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Abstract. The family of normal propositional modal logic systems are given a highly systematic organisation by their model theory. This model theory is generally given using Kripkean frame semantics, and it is systematic in the sense that for the most important systems we have a clean, exact correspondence between their constitutive axioms as they are usually given in a Hilbert style and conditions on the accessibility relation on frames. By contrast, the usual structural proof theory of modal logic, as given in Gentzen systems, is ad-hoc. While we can formulate several modal logics in the sequent calculus that enjoy cut-elimination, their formalisation arises through system-by-system fine tuning to ensure that the cut-elimination holds, and the correspondence to the formulation in the Hilbert systems becomes opaque.

We introduce a systematic presentation for the systems **K**, **D**, **M**, and **S4** in the Calculus of structures, a structural proof theory that enjoys the *deep inference* property. Because of this property, it is possible to axiomatise the modal logics in a manner directly analogous to the Hilbert axiomatisation. We show that the calculus possesses a cut-elimination property directly analogous to cut-elimination for the sequent calculus for these systems, and we discuss the extension to several other modal logics.

1 Introduction

Propositional modal and tense logics provide a rich source of formalisms applied in computer science, which are useful in declarative approaches to programming, knowledge representation and the specification of computing systems due to a number of reasons:

- They offer increased expressiveness to formal languages in a way that is useful for the specification of a range of useful properties; for example, two important such applications arose first in specifications for mobile systems [8], and secondly in an extension of an important complexity result in the theory of the λ -calculus [1], both taken from the proceedings of a recent *ACM Principles of Programming Languages* conference.

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- While many propositional modal logics can be encoded in first-order predicate logic in a standard way, for many applications the modal systems are more intuitive, being closer to specification in natural language, and more tractable, since most normal propositional modal logics are decidable;
- The normal modal logics possess an elementary model theory in the form of frame semantics (due to Kripke, Hintikka and Kanger, see [7]) that provides a systematic correspondence between their constitutive axioms as they are usually given in Hilbert style and conditions on the accessibility relation on frames¹. Thus we can say the model theory of the normal modal logics is systematic.

By contrast, the usual structural proof theory of modal logic, as given in the sequent calculus and natural deduction, is ad-hoc. While we can formulate several modal logics in the sequent calculus that enjoy cut-elimination, their formalisation arises through system-by-system fine tuning to ensure that the cut-elimination holds, and the correspondence to the formulation in the Hilbert systems becomes opaque. This, no doubt, is responsible for the relatively low profile of the proof theory of modal logic: for example the textbook of Blackburn, de Rijke and Venema [3] does not cover the proof theory.

In this paper we will provide a new proof theoretic basis for modal logic using the calculus of structures introduced by Guglielmi [11]. This proof calculus provides a structural proof theory in the sense of Gentzen for logical systems because it possesses a notion of cut-free proof directly analogous to that for the sequent calculus. It possesses several properties that give proof theory carried out in the calculus of structures important advantages over the traditional proof theory carried out in structural proof theory: these we call *atomicity* of inference for cut, contraction and weakening, which depends upon the systems formulation using *deep inference*², *symmetric inference*³, and absence of a distinction between logical and extra-logical structure⁴. The immediate benefits of these characteristics are that the whole calculus enjoys *locality* and *finiteness*.

¹ See Garson [9], and Blackburn, de Rijke and Venema [3] for readable accounts of this relationship.

² The first proof theories making use of deep inference appear to be due to *Schuette*, but these systems lack cut-elimination [14]. A weak form of deep inference is possessed by Belnap's *Display Logic*, which possesses the most sophisticated set of cut-elimination results for modal logic. The calculus is fundamentally asymmetric, and possess a subtle and complex extra-logical machinery; relating the formal results in that calculus to those developed here is a substantial task in need of further work [2, 19].

³ Symmetric inference is possessed by the single-sided sequent systems developed by Schuette and Tait, see for example [15].

⁴ This distinction exists in a hidden form in natural deduction, in the rules governing assumptions, and more explicitly in the need to formulate the extra-logical notion of sequents and their management by means of structural rules in the sequent calculus. By contrast, the calculus of structures does not need to distinguish between formulae and the structural context in which the formulae occur in proofs.

2 Proof theory of system M

Let us begin by establishing some terminology. A *logical system* is defined over a formal language by an inference system that determines its *theory*, a subset of the sentences of the language. Each formula in the theory of a logical system is called a *theorem*. Logical systems are *schematic*, that is they allow schematic letters that may stand in for arbitrary propositions, so the theory of a logical system for a more restrictive language can naturally be projected to richer languages by application of substitution. Two logical systems are *equivalent* if their theories are the same, they are *distinct* otherwise. If the theory of one logical system strictly contains the theory of another, then we call the former system the stronger system.

We begin by showing how modal logic is defined using Hilbert Systems, as these proof calculi give the oldest and most common axiomatisations, and so we may consider these axiomatisations as standard. The sentences of classical propositional logic contain an inexhaustible supply of schematic letters, letters for truth and falsity, and are closed under the binary operators \supset , \leftrightarrow , \wedge and \vee and the unary operator \neg . The Hilbert-style axiomatisation of this logical system proceeds in the standard way, with two rules, modus ponens and the rule *uniform substitution* allowing the schematic letters to be replaced by any proposition.

The weakest normal modal logic **K** is a logical system over this language extended with two modal operators \Box and \Diamond , which are unary operators. The system is axiomatised by extending classical propositional logic with the axioms and rules:

- Axiom **K**: $\Box(p \supset q) \supset (\Box p \supset \Box q)$;
Axiom **DM**: $\neg \Box p \leftrightarrow \Diamond \neg p$;
Necessitation rule: if $\vdash p$ then $\vdash \Box p$.

Naturally, this corresponds to the class of frame models where there are no conditions imposed on the frame accessibility relation. The stronger modal logics are obtained by adding further axioms. All of the most studied systems of modal logic arise by adding one or more of the following axioms:

Definition 1.

1. *Axiom D*: $\Box p \supset \Diamond p$
2. *Axiom T*: $\Box p \supset p$
3. *Axiom 4*: $\Box p \supset \Box \Box p$
4. *Axiom B*: $p \supset \Box \Diamond p$
5. *Axiom 5*: $\Diamond p \supset \Box \Diamond p$

which correspond to constraining the accessibility relation to be serial (ie. there are always further frames accessible from any frame), reflexive, transitive, symmetric and Euclidean respectively. There are a total of fifteen distinct logical systems obtainable, which we may organise in a lattice according to the ‘stronger than’ relation, with the most important of these being:

- D** Obtained by extending system **K** with rule **D**;
- M** Obtained by extending system **K** with rule **M**: note this system is often named **T**, a nomenclature we avoid to prevent confusion with Goedel's system **T**.
- S4** Obtained by extending system **M** by rule **4**;
- B** Obtained by extending system **M** by rule **B**;
- S5** Obtained by extending system **M** by rule **5**, or equivalently by extending system **M** by rules **B** and **4**.

The strongest system from these modal logics that is perfectly straightforward to formulate in a sequent system and to prove cut-free is system **G-M** (for Gentzen system **M**): we formulate this using Tait's approach of one-sided sequents. We make use of a *dualising function* in formulating the syntax of the calculus: $\bar{\phi}$ is defined recursively to be the De Morgan dual of a function:

$$\begin{aligned} \overline{\neg A} &= A & \overline{\bar{p}} &= \neg p \\ \overline{A \wedge B} &= \overline{A} \vee \overline{B} & \overline{A \vee B} &= \overline{A} \wedge \overline{B} \\ \overline{\mathbf{tt}} &= \mathbf{ff} & \overline{\mathbf{ff}} &= \mathbf{tt} \end{aligned}$$

where p is a schematic letter, and $A \supset B$ and $A \leftrightarrow B$ are treated as derived connectives using the standard encodings. The inferences of the system are given by trees generated by the following inference rules, where the nodes are multisets of formulae (indicated by Γ, Δ , with the notations $\square\Gamma, \diamond\Delta$ indicating the result of prefixing each formula of the multiset appropriately):

Axiom and cut:

$$\frac{}{\Gamma, A, \overline{A}} \text{ ax} \quad \frac{\Gamma, A \quad \Delta, \overline{A}}{\Gamma, \Delta} \text{ cut}$$

Contraction and weakening:

$$\frac{\Gamma, A, A}{\Gamma, A} \text{ contr} \quad \frac{\Gamma}{\Gamma, A} \text{ wk}$$

Truth:

$$\frac{}{\mathbf{tt}} \mathbf{tt}$$

Logical connectives and modal operators:

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \vee \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \wedge$$

$$\frac{\Gamma, \overline{A}}{\Gamma, \neg A} \neg$$

$$\frac{\Gamma, A}{\diamond\Gamma, \square A} \square_1 \quad \frac{\Gamma, A}{\Gamma, \diamond A} \diamond_1$$

The theorems of this system then are given by the concluding formulae of inferences where the concluding sequent contains exactly one formula. It is easily shown that the sequent formulation contains all the theorems of the Hilbert formulation, and with a little ingenuity, modelling the multisets by disjunctions, the reverse containment can also be demonstrated. We can also show the following theorem, where an inference rule or set of inference rules are *admissible* if all theorems of the system may be proven without their use:

Theorem 1. *The cut rule is admissible in the sequent calculus formalisation of system \mathbf{M} .*

This cut-elimination theorem is not really more complex to prove than that for its subsystem of classical logic, though there is a novelty: since the inference rules for the modal operators introduce ‘side-effects’; the rule \Box_1 adds \diamond modalities to side assumptions, which can then be used in cuts. The new rule doesn’t interfere with permutability of cuts, however.

3 The calculus of structures

Now we introduce modal logic for the calculus of structures for this system \mathbf{M} . As for the single-sided sequent calculus, we make use of De Morgan dualities between connectives in the formulation of the system. By convention, the calculus has a rather different notation for formulae than is used in Hilbert and Gentzen proof calculi: the calculus of structures uses a bracket notation for the logical connectives to suggest the associativity and commutativity properties: the calculus treats *structures* that are equivalence classes of the syntactic representations quotiented over these relations, and also quotiented over rules for units. We also introduce a dualising operator in the same way as for the single-sided sequent calculus.

So the *formulae* of the calculus of structures are built up from the units \mathbf{t} , \mathbf{ff} , the schematic letters; for each schematic letter a we admit the complement \bar{a} as a formula; and whenever R_1, \dots, R_n are formulae, so are $[R_1, \dots, R_n]$ and (R_1, \dots, R_n) . A *formula context* $S\{-\}$ is obtained from a formula by replacing any leaf (eg. a schematic letter) by ‘-’; then $S\{R\}$ is the formula obtained by replacing this - by the formula R .

The *duals* of formulae are defined recursively, where the dual of each schematic letter is its complement and vica versa, and

$$\begin{aligned} \overline{(R_1, \dots, R_n)} &= [\bar{R}_1, \dots, \bar{R}_n] \\ \overline{[R_1, \dots, R_n]} &= (\bar{R}_1, \dots, \bar{R}_n) \\ \bar{\mathbf{t}} &= \mathbf{ff} \quad \bar{\mathbf{ff}} = \mathbf{t} \end{aligned}$$

so for any formula R , $R = \overline{\bar{R}}$.

The *structures* are the equivalence classes of formulae obtained by quotienting over

1. (Associativity)

$$[R_1, \dots, R_i, [T_1, \dots, T_j], U_1, \dots, U_k] = [R_1, \dots, R_i, T_1, \dots, T_j, U_1, \dots, U_k]$$

and

$$(R_1, \dots, R_i, (T_1, \dots, T_j), U_1, \dots, U_k) = (R_1, \dots, R_i, T_1, \dots, T_j, U_1, \dots, U_k)$$

2. (Congruence) If $R_1 = R_2$ then $S\{R_1\} = S\{R_2\}$, for any formula context $S\{-\}$;
3. (Commutativity) $[R_1, R_2] = [R_2, R_1]$ and $(R_1, R_2) = (R_2, R_1)$;
4. (Identity) $R = [R, \mathbf{ff}]$ and $R = (R, \mathbf{tt})$.

Inferences are chains of applications of inference rules, where each inference rule has one premise and one conclusion, which is a structure rather than a formula. All of the inference rules are *deep*, which means that each rule is given by a pair of formulae specifying the premise and conclusion that are both given with the same formula context. The inference rules for classical logic, the system **SKS** are given:

Interaction and cut rules:

$$i \downarrow \frac{S\{\mathbf{tt}\}}{S[R, \bar{R}]} \quad i \uparrow \frac{S(R, \bar{R})}{S\{\mathbf{ff}\}}$$

The switch rule:

$$s \frac{S([R, T], U)}{S[(R, U), T]}$$

The medial rule:

$$m \frac{S[(R, U), (T, V)]}{S([R, T], [U, V])}$$

The weakening rules:

$$w \downarrow \frac{S\{\mathbf{ff}\}}{S\{R\}} \quad w \uparrow \frac{S\{R, \bar{R}\}}{S\{\mathbf{tt}\}}$$

And the contraction rules:

$$c \downarrow \frac{S[R, R]}{S\{R\}} \quad c \uparrow \frac{S\{R\}}{S(R, R)}$$

The theorems of this system are the formulas belonging to the structures that occur as conclusions of inferences whose premise is the structure \mathbf{tt} .

This system has several desirable properties, as discussed in [5]:

1. The system is self dual: each rule labelled with \uparrow takes form:

$$* \uparrow \frac{S\{R\}}{S\{T\}}$$

matching by De Morgan duality the form of the corresponding rule labelled with \downarrow takes form:

$$* \downarrow \frac{S\{\overline{T}\}}{S\{\overline{R}\}}$$

where the two rules not labelled with \uparrow or \downarrow are self-dual.

2. The entire *up-fragment*, ie. the rules labelled with \uparrow , is admissible. This is shown, in op. cit. and discussed in more detail below, by means of a translation from cut-free proofs of the sequent calculus into proofs of system **KS**, that is, system **SKS** without the rules of the up-fragment. Since this is so closely related to cut-elimination in the sequent calculus, we may call this result cut-elimination for the calculus of structures.
3. We can restrict the interaction, cut, weakening and contraction rules to atoms, by which we mean the applications of the rules using formulae R can be restricted to the case where R is an atoms (ie. a or \bar{a}) both for system **SKS** and the cut-free system **KS**. Furthermore, these restrictions are achieved by simple local transformations on inferences, as opposed to the complex global transformation that we associate with cut-elimination. The system with these restrictions we call *atomic SKS* (analogously atomic **KS**), or just **aSKS** (analogously **aKS**). An analogous restriction to atoms can be achieved for the sequent calculus by similarly local transformations in the case of the axiom and weakening rules, but for fundamental reasons of the shallowness of inference cannot be achieved at all for contraction, and only by means of global transformations in the case of cut.
4. System **aSKS** enjoys important computational properties: it is *local*, and so to is its subsystem **aKS**, in the sense that looking at the inferences going either up or down, structure is rearranged, or atoms introduced, abandoned or duplicated, but arbitrarily large substructures are never introduced, abandoned or duplicated. Bruennler also discusses an important advantage corollary to locality and atomicity of cut: we obtain *finiteness* for a slight restriction of **aKS**, which offers exceptionally perspicacious consistency proofs for predicate logic. It is fair to point out that these benefits do carry a cost: by contrast to the sequent calculus there are a great many possible rules that one may apply during proof search, so one cannot read off a tableaux algorithm from **KS** in the way one can for cut-free **LK**. Devising efficient proof search algorithms that can take advantage of these properties is an important goal of the program of research in the calculus of structures.

We can extend this calculus to obtain the system **M** by allowing formulae of the form $\Box R$, and $\Diamond R$, extending the equivalences defining structures with $\mathbf{tt} = \Box \mathbf{tt}$ and $\mathbf{ff} = \Diamond \mathbf{ff}$ and extending the set of inferences as follows:

$$k \downarrow \frac{S\{\Box[R, T]\}}{S[\Box R, \Diamond T]}$$

$$j \downarrow \frac{S\{\Box(R, T)\}}{S(\Box R, \Box T)}$$

$$t \downarrow \frac{S\{R\}}{S\{\diamond R\}}$$

We call the system that extends system **KS** with the above down rules **KS-M**, and the symmetric system extending system **SKS** with both the down and up rules **SKS-M**.

We show the equivalence of system **SKS-M** to **M** in two steps. Firstly we show we can map inferences of **SKS-M** onto inferences of **M**:

Definition 2. We define the map $-^s$ from formulae of **M** onto structures of **SKS-M** recursively:

$$\begin{aligned} p^s &= p \text{ for } p \text{ one of } \mathbf{t}, \mathbf{ff}, \text{ or a schematic letter;} \\ (\neg A)^s &= \overline{A^s} \\ (A \supset B)^s &= [\overline{A^s}, B^s] \\ (A \leftrightarrow B)^s &= ((A \supset B) \wedge (B \supset A))^s \\ (A \wedge B)^s &= (A^s, B^s) \\ (A \vee B)^s &= [A^s, B^s] \\ (\Box A)^s &= \Box A^s \\ (\diamond A)^s &= \diamond A^s \end{aligned}$$

- Proposition 1.** 1. There is a map $-^h$ from structures of **SKS-M** to formulae of **M** with the properties that $-^{hs}$ is the identity map on structures, and for all formulae A, B , if A is a subformula of B then either A^{sh} is a subformula of B^{sh} or $(\neg A)^{sh}$ is a subformula of $(\neg B)^{sh}$.
2. If $A^s = B^s$ then $\vdash A \leftrightarrow B$ is a theorem of **M**;
3. If $R = A^s$ and A is a theorem of **M**, then R is a theorem of **SKS-M**;

Proof. Part 1 follows from defining the obvious recursive map from the formulae of **SKS-M**, and observing that for any structure we may choose a lexically simplest representative formula (given some total order on atoms and their complements), to obtain the reverse map which is easily verified to have the desired properties.

Part 2 follows easily from the observation that $-^s$ is bijective when restricted to disjunctive normal forms.

To prove part 3, we must first prove that each axiom of **M** maps onto a theorem of **SKS-M**, which is a straightforward exercise in the construction of inferences in the calculus of structures: for didactic reasons we recommend the reader works through at least two cases, for example $(A \supset B \supset C) \supset (A \supset C) \supset B \supset C$, and rule **K**.

We then must show that inferences of the Hilbert calculus map onto inferences of **SKS-M** by considering the three inference rules. To see that inferences making use of uniform substitutivity are modelled in **SKS-M** we need simply observe that the inference rules of **SKS-M** are schematic. The other two rules we need an inductive argument: for modus ponens and necessitation, assume the premisses map onto theorems of **SKS-M**:

1. Modus ponens: observe that with deep inference, since $\mathbf{tt} = (\mathbf{tt}, \mathbf{tt})$, if R and T are theorems, so is (R, T) . Thus by the induction hypothesis, $((A \supset B)^s, A^s) = ([\overline{A^s}, B^s], A^s)$ is a theorem, and so we obtain:

$$i \uparrow \frac{s \frac{([\overline{A^s}, B^s], A^s)}{([\overline{A^s}, A^s], B^s)}}{[\mathbf{ff}, B^s]}$$

from which the conclusion follows by observing that $[\mathbf{ff}, B^s] = B^s$.

2. Necessitation: since $\mathbf{tt} = \Box \mathbf{tt}$, deep inference allows us to infer that $\Box R$ is a theorem if R is, from which the result immediately follows.

Theorem 2. *M and SK-M are equivalent.*

Proof. For each inference rule of SK-M, which takes the general form:

$$* \frac{S\{R\}}{S\{T\}}$$

we show that $R^h \supset T^h$ is a theorem of **M**. The dual rules map onto these from the theorem of classical logic: $(A \supset B) \leftrightarrow (\neg B \supset \neg A)$. The following lemma is a straightforward exercise in modal logic provability:

Lemma 1. *If $A \supset B$ is a theorem of **M**, then so are:*

1. $A \wedge C \supset B \wedge C$;
2. $A \vee C \supset B \vee C$;
3. $\Box A \supset \Box B$;
4. $\Diamond A \supset \Diamond B$.

from which, by an induction on the makeup of formula contexts $S\{-\}$, we obtain for each inference rule $(S\{R\})^h \supset (S\{T\})^h$. Then inferences of **SK-M** can be mapped onto chains of applications of modus ponens, starting from the theorem \mathbf{tt} .

Theorem 3. *The up-fragment of SKS-M is admissible.*

Proof. We show this by mapping cut-free proofs of **G-M** onto proofs of **KS-M**, following just the same technique as [5], that is, for each rule of **G-M**

we must find an inference of **KS-M** with premise $\Box^i(\Gamma_1^s, \dots, \Gamma_n^s)$ for some i and conclusion Δ^s , where we extend the mapping $-^s$ from formulae to sequents by considering sequents to be disjunctions of their constituent formulae, and where the notation $\Box^i A$ indicates the formulae obtained from A by prefixing \Box to it i times. These proofs are straightforward; the only case where we need i to be nonzero is mapping \Box_1 ; the rule then maps onto a number of applications of the $k \downarrow$ rule equal to the number of formulae in Γ . We can then use these inferences to construct our map on cut-free proofs recursively.

4 Systems **K**, **D**, **S4**, and **S5**

We can extend the account of the structural proof theory that we have given of **M** to the systems **K**, **D**, and **S4** quite analogously. First we consider the axiomatisation of these systems in the sequent calculus: define two new inference rules:

$$\frac{\diamond\Gamma, A}{\diamond\Gamma, \Box A} \Box_2 \qquad \frac{\Gamma, A}{\Gamma, \diamond A} \diamond_2$$

Then we obtain our axiomatisations as follows:

1. **G-K** is obtained by dropping rule \diamond_1 from **G-M**;
2. **G-D** is obtained from **G-M** by substituting rule \diamond_2 for \diamond_1 ;
3. **G-S4** is obtained from **G-M** by substituting rule \Box_2 for \Box_1 ;

Theorem 4. *Each of these systems is equivalent to the matching Hilbert system, and for each, the cut rule is admissible.*

Proof. For **G-K** and **G-D** the proofs are quite analogous to that for **G-M**. **G-S4** is similarly shown to be equivalent to the matching system **S4**; the cut-elimination proof is more subtle because many of the usual permutations on cuts fail, a fact that is unfortunately rather glossed over in the literature. Cf. Ohnishi and Matsumoto, and Valenti: [13, ?,?].

Our axiomatisation in the calculus of structures proceeds by defining the inference rules:

$$d \downarrow \frac{S\{\Box \mathbf{ff}\}}{S\{\mathbf{ff}\}}$$

$$4 \downarrow \frac{S\{\diamond \diamond R\}}{S\{\diamond R\}}$$

Then we obtain the equivalent systems:

1. **KS-K** is **KS-T** without the $t \downarrow$ rule;
2. **KS-D** is **KS-K** together with the $d \downarrow$ rule;
3. **KS-S4** is **KS-K** and the $t \downarrow$ and $4 \downarrow$ rules;
4. **SKS-K**, **SKS-D**, and **SKS-S4** are the symmetric versions of the above.

Theorem 5. *The analogs of proposition 1 and theorem 2 hold for each of **SKS-K**, **SKS-D**, and **SKS-S4**.*

The axiomatisation of systems **S5** and **B** in the one-sided sequent calculus present serious difficulties, however. For example, it is possible to axiomatise **S5** by replacing the rule \Box_1 of **G-M** by any one of the following three rules⁵:

⁵ The first of these three alternatives, using \Box_3 is the one-sided analogue of the two-sided sequent formulation of **S5** due to Ohnishi and Matsumoto, [13].

$$\frac{\diamond\Gamma, \Box\Delta, A}{\diamond\Gamma, \Box\Delta, \Box A} \Box_3$$

$$\frac{\Gamma, \Box\Delta, A}{\diamond\Gamma, \Box\Delta, \Box A} \Box_4$$

$$\frac{\diamond\Gamma, \Delta, A}{\diamond\Gamma, \Box\Delta, \Box A} \Box_5$$

This system can be mapped onto a reasonable axiomatisation of **KS-S5** which we axiomatise as **SKS-S4** plus the following rule:

$$B \downarrow \frac{S\{R\}}{S\{\Box \diamond R\}}$$

We can show that the theories of the systems **S5**, the three possible formulations of **G-S5** and **SKS-S5** are equivalent, and that cut-free proofs of **G-S5** map onto **KS-S5**, but unfortunately cut-elimination for all three variants of **G-S5** fails:

- Proposition 2.** 1. *The axiomatisations of **G-S5** formulated using rules \Box_3 and \Box_5 have no cut-free proof of $\neg A \vee \Box \diamond A$, a theorem of **S5**.*
2. *The axiomatisation of **G-S5** formulated using rule \Box_4 has no cut-free proof of $\neg \Box \Box A \vee \diamond A$, a theorem of **S5**.*

Proof. The cut-bearing proofs for part 1 have cut formula $\Box A$ that c, and for part 2 cut formula $\Box \diamond A$: in both cases all the side formulae have \Box as their main operator, which in each case is the principal conclusion of the appropriate \Box_i rule.

While cut-free sequent systems for **S5** have appeared in the literature (see, for example, [4]), these formulations do not take the simple form of the sequent systems treated in this paper, possessing either more complex conditions on side-formulae, or being given in a two-sided formulation that is not easily symmetrisable. Furthermore, we do not know of any cut-free sequent formulation of **B**.

5 Further directions

The most serious defect of the account given here from the point of view of formal aesthetics is that our proofs are entirely parasitic: we show cut-elimination for our systems in the calculus of structures by means of translations into a quite separate theory, so though we could pedantically claim to have provided a syntactic proof of cut-elimination, the proof is quite as external as so-called semantical proofs of cut-elimination. Furthermore, we are left without resources to deal with the cases where we do not have appropriate cut-free axiomatisations, such as for system **B**. Guglielmi [11] provides a technique, called splitting, that

may be regarded as the preferred way to give internal, syntactic proofs of cut-elimination in the calculus of structures, and in joint work with one of the authors (in preparation) has shown how this technique may be extended to deal with the case of SKS. Clearly the most pressing direction for future work is to extend this technique to the systems described here.

Given ‘good’ proofs of cut-elimination for these systems, what next? We can axiomatise system **B** in the calculus of structure by adding $B \downarrow$ to **KS-M** to obtain **KS-B**, and likewise **SKS-B**. Here our inference rules dealing with the modal operators find themselves in a one-one relationship to four of the five axioms given in the lattice of Hilbert systems described in definition 1; we can extend this to rule **5** with the inference rule:

$$5 \downarrow \frac{S\{\diamond R\}}{S\{\square \diamond R\}}$$

- Conjecture 1.*
1. For the thirteen asymmetric systems corresponding to those systems from the lattice of modal Hilbert systems that can be formulated either without rule $5 \downarrow$ or with rule $4 \downarrow$, the up-fragment is admissible.
 2. For the systems **KS-K5**, and **KS-D5** obtained by extending systems **KS-K** and **KS-D** with the rule $5 \downarrow$, cut is inadmissible.

If true, a further problem is to find cut-free formulations **KS-K5**, and **KS-D5**, ideally in a manner that conserves the attractive correspondence to the lattice of Hilbert systems.

Finally, in our discussion of Bruennler’s work on classical logic in the calculus of structures, we saw how the calculus of structures possesses important advantages over the sequent calculus, namely *atomicity* for cut, contraction and weakening; and *locality* and *finiteness* for the whole system. Atomicity of cut we obtain immediately by duality, however the other advantages do not hold for our system in its current formulation. The close formal similarities to the modal operators of linear logic makes the work on locality of Strassburger [10, 16] in the context of non-commutative linear logic relevant here, although there are special difficulties that face modal systems without rule $t \downarrow$ that need to be dealt with separately. Success in obtaining locality here may allow us to leverage the work of Bruscoli tying proof search in noncommutative linear logic to the evolution of processes in CCS-like algebras [6, 12]

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