Exercise 3: Denotational and Operational Semantics

Concurrency Theory — Summer 2024

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Exercise 2.3

Consider again the requirements on fixed points (from slide 20 on, lecture 4). We left option B (local looping) as an exercise. Study this option and carry out an example in the style of the ones given in options A and C. Do we learn anything about the fixed point requirements?

If

$$\texttt{while}\ b\ \texttt{do}\ S$$

does not terminate *locally*, then starting from s_0, there are states $s_1, \ldots, s_n$ such that

$$\mathcal{B}[\![b]\!] = \texttt{tt} \text{ for } i \leq n$$

and

$$\mathcal{S}_{\text{ds}}[\![S]\!]\, s_i = \begin{cases} s_{i+1} & \text{for } i < n \\ \texttt{undef} & \text{for } i = n \end{cases}$$

$$\texttt{while}\ (x \geqq 0)\ \texttt{do if}\ (x \equiv 0)\ \texttt{then while true do skip else}\ x\texttt{:=}x \ominus 1$$

Exercise 2.3

Let g be any fixed point of the associated function F. In case $i < n$, we get $g_0\, s_i = s_{i+1}$.

In case $i = n$,

$$\begin{aligned} g_0\, s_n &= (F\, g_0) s_n \\ &= \mathsf{cond}(\, \mathcal{B}[\![b]\!]\, , g_0 \circ \mathcal{S}_{\mathrm{ds}}[\![S]\!]\, , \mathrm{id}\,) s_n \\ &= (g_0 \circ \mathcal{S}_{\mathrm{ds}}[\![S]\!]) s_n \\ &= \mathtt{undef} \end{aligned}$$

Thus, any fixed point g of F will have the property $g_0\, s_0 = \mathtt{undef}$.

Exercise 2.5

Let S be a nonempty set and define $\mathcal{P}_{\text{fin}}(S) = \{K \mid K \text{ if finite and } K \subseteq S\}$.

1. Show that $(\mathcal{P}_{\text{fin}}(S), \subseteq)$ as well as $(\mathcal{P}_{\text{fin}}(S), \supseteq)$ are po-sets.

   - $\subseteq$ is reflexive, transitive, and antisymmetric (borrowing from set theory);
   - for all po-sets $\mathfrak{A} = (D, \preccurlyeq)$, $\mathfrak{B} = (D, \preccurlyeq^{-1})$ is also a po-set:

     reflexivity for $d \in D$, we have $d \preccurlyeq d$ (since $\mathfrak{A}$ is a po-set and, thereby, $\preccurlyeq$ is reflexive); hence, $d \preccurlyeq^{-1} d$

     transitivity let $d_1, d_2, d_3 \in D$, such that $d_1 \preccurlyeq^{-1} d_2$ and $d_2 \preccurlyeq^{-1} d_3$. Thus, $d_3 \preccurlyeq d_2$ and $d_2 \preccurlyeq d_1$. Therefore, $d_3 \preccurlyeq d_1$ showing that $d_1 \preccurlyeq^{-1} d_3$ holds.

     antisymmetry let $d_1, d_2 \in D$ such that $d_1 \preccurlyeq^{-1} d_2$ and $d_2 \preccurlyeq^{-1} d_1$. Hence, $d_1 \preccurlyeq d_2$ and $d_2 \preccurlyeq d_1$ which implies $d_1 = d_2$ since $\mathfrak{A}$ is a po-set.

     Hence, $\mathfrak{B}$ is a po-set as well.
   - concluding, $(\mathcal{P}_{\text{fin}}(S), \supseteq)$ is a po-set.

2. Do both po-sets have a least element for all choices of S?

Exercise 2.5

- $\subseteq$: $\emptyset$ for all choices of S:
  - Let K be a finite subset of S; then $\emptyset \subseteq K$ since $\emptyset$ is a subset of every set.
- $\supseteq$:
  - if S is finite, then S is the least element.
  - if S is infinite, there is no least element:
    - suppose there was a least element S_0
    - that is, $\forall K \underset{\text{fin}}{\supseteq} S : S_0 \supseteq K$
    - as S_0 itself is finite, S_0 has finite cardinality, i.e. $|S_0| = k$ for $k \in \mathbb{N}$
    - since S is infinite, there is an element $s_{\lightning} \in S \setminus S_0$ such that $S_{\lightning} = S_0 \cup \{s_{\lightning}\}$ is a finite subset of S
    - but $S_{\lightning} \supsetneq S_0$ (i.e., is strictly smaller), meaning S_0 cannot be the least element

3. Prove or disprove that every subset of $\mathcal{P}_{\text{fin}}(S)$ has a least upper bound w.r.t. $\subseteq$.

- consider $S = \mathbb{N}$:

Exercise 2.5

    - $Y = \{\{n\} \mid n \in \mathbb{N}\} \subseteq \mathcal{P}_{\text{fin}}(\mathbb{N})$ is a counterexample
    - the only upper bound of Y is $\mathbb{N}$
    - but $\mathbb{N} \notin \mathcal{P}_{\text{fin}}(\mathbb{N})$
- for any infinite set S, $\mathcal{P}_{\text{fin}}(S)$ is a counterexample.
- considering finite $Y \subseteq \mathcal{P}_{\text{fin}}(S)$, we get $\bigcup Y$ as the least upper bound
  1. $\bigcup Y$ is an upper bound of Y: follows by definition of $\bigcup$
  2. $\bigcup Y$ is the least upper bound of Y:
     - suppose $\Upsilon \subsetneq \bigcup Y$ is an upper bound of Y;
     - there is some set $M \in Y$ such that there is an $x \in M$ and $x \notin \Upsilon$
     - then $M \not\subseteq \Upsilon$, meaning Υ is not an upper bound of Y, contradicting our assumption that Υ is an upper bound of Y. $\lightning$

4. Provide a set S such that $(\mathcal{P}_{\text{fin}}(S), \subseteq)$ has a chain with no upper bound and, therefore, no least upper bound.

- we pick $S = \mathbb{N}$ and let $\mathbb{N}^n = \{i \mid 0 \leq i \leq n\}$ (note, $|\mathbb{N}^n| = n + 1$)

- then $\Upsilon = \{\mathbb{N}^j \mid j \in \mathbb{N}\}$ forms a chain:
  - let $\mathbb{N}^m$ and $\mathbb{N}^n$ be two elements of the chain such that (w.l.o.g.) $m \leq n$
  - then $\mathbb{N}^m \subseteq \mathbb{N}^n$ since:
    - for $x \in \mathbb{N}^m$, we get $x \leq m$
    - since $m \leq n$, we get $x \leq n$ (as $\leq$ is transitive)
    - hence, $x \in \mathbb{N}^n$
- Υ has no upper bound: **by contradiction**
  - suppose, there was an upper bound N of Υ
  - then N is a finite subset of $\mathbb{N}$
  - hence, $|N| = k$ for $k \in \mathbb{N}$
  - since $\mathbb{N}^k \in \Upsilon$ and $|\mathbb{N}^k| = k + 1$, $\mathbb{N}^k \not\subseteq N$
  - so N is not an upper bound. ↯

5. Is any of the aforementioned po-sets a complete lattice? ccpo?

- if S is finite, then $(\mathcal{P}_{\text{fin}}(S), \subseteq / \supseteq)$ forms a complete lattices, also ccpo

Exercise 2.5

- if S is infinite:
  - $(\mathcal{P}_{\text{fin}}(S), \subseteq)$ is not a ccpo (thus, not a complete lattice) by the counterexample we gave above
  - $(\mathcal{P}_{\text{fin}}(S), \supseteq)$ is not a complete lattice because not all subsets $Y \subseteq \mathcal{P}_{\text{fin}}(S)$ have a least upper bound:
    - if $Y \neq \emptyset$, then $\bigcap Y$[1] is the least upper bound (proof similar to the least upper bound finitely many subsets and $\bigcup$)
    - if $Y = \emptyset$, then $\bigcap Y = \bigcap \emptyset = S$ is infinite and, therefore, not an upper bound.

6. Analyze $(\mathcal{P}(S), \subseteq)$ where $\mathcal{P}(S) = \{K \mid K \subseteq S\}$, whether it forms a complete lattice? How about ccpo?

- it is a complete lattice, and, therefore, also a ccpo-set
- it has $\emptyset$ as its least element
- for any arbitrary subset $Y \subseteq \mathcal{P}(S)$, $\bigcup Y \in \mathcal{P}(S)$ and forms the least upper bound of Y

7. Construct a subset Y of $\mathbf{State} \hookrightarrow \mathbf{State}$ such that Y has no upper bound.

   - let $s_1, s_2 \in \mathbf{State}$ such that $s_1 \neq s_2$
   - then $Y = \{g_1, \mathrm{id}\}$ with

$$g_1\ s = \begin{cases} s_2 & \text{if } s = s_1 \\ \texttt{undef} & \text{otherwise} \end{cases}$$

   is a non-empty subset of $\mathbf{State} \hookrightarrow \mathbf{State}$ with no upper bound.

[1] $\bigcap Y := \{x \mid \forall X \in Y : x \in X\}$

Exercise 2.7

Assume $(D, \preccurlyeq)$ and $(D', \preccurlyeq')$ are ccpo's, and assume function $f : D \to D'$ satisfies

$$\bigsqcup{}'\{f\,d \mid d \in Y\} = f(\bigsqcup Y)$$

for all non-empty chain Y. Show that f is monotone.

Proof: Let $d_1, d_2 \in D$ with $d_1 \preccurlyeq d_2$. Then d_2 is an upper bound of the necessarily non-empty chain $Y = \{d_1, d_2\}$. It is even the least upper bound, i.e. $d_2 = \bigsqcup Y$.

$\bigsqcup{}'(f(Y)) = f(\bigsqcup Y) = f\,d_2$. Since $f\,d_1 \in f(Y)$, $f\,d_1 \preccurlyeq' f\,d_2$. ■