

# COMPLEXITY THEORY

## Lecture 1: Introduction and Motivation

Markus Krötzsch, Stephan Mennicke, Lukas Gerlach  
Knowledge-Based Systems

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More recent versions of this slide deck might be available.  
For the most current version of this course, see  
[https://iccl.inf.tu-dresden.de/web/Complexity\\_Theory/en](https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en)

# Course Tutors



Markus Krötzsch  
Prof



Stephan Mennicke  
Exercises/Lectures



Lukas Gerlach  
Exercises/Lectures

# Organisation

## Lectures

Monday, DS 2 (9:20–10:50), REC C-118

Tuesday, DS 2 (9:20–10:50), APB E005

## Exercise Sessions (starting 18 October)

Wednesday, DS 3 (11:10–12:40), APB E005

## Web Page

[https://iccl.inf.tu-dresden.de/web/Complexity\\_Theory\\_\(WS2023/24\)](https://iccl.inf.tu-dresden.de/web/Complexity_Theory_(WS2023/24))

## Lecture Notes

Slides of current and past lectures will be online.

# Goals and Prerequisites

## Goals

- Introduce basic notions of **computational complexity theory**
- Introduce **commonly known complexity classes** ( $P$ ,  $NP$ ,  $PSPACE$ , ...) and discuss relationships between them
- Develop **tools to classify problems** into their corresponding complexity classes
- Introduce some **advanced topics of complexity theory** (e.g., circuits, probabilistic computation, quantum computing)

## (Non-)Prerequisites

- No particular prior courses needed
- Prior acquaintance with Turing Machines and basic topics in formal languages and complexity is helpful
- General mathematical and theoretical computer science skills necessary

# Reading List

- **Michael Sipser: Introduction to the Theory of Computation, International Edition; 3rd Edition; Cengage Learning 2013**
- Sanjeev Arora and Boaz Barak: **Computational Complexity: A Modern Approach**; Cambridge University Press 2009
- Michael R. Garey and David S. Johnson: **Computers and Intractability**; Bell Telephone Laboratories, Inc. 1979
- Erich Grädel: **Complexity Theory**; Lecture Notes, Winter Term 2009/10
- John E. Hopcroft and Jeffrey D. Ullman: **Introduction to Automata Theory, Languages, and Computation**; Addison Wesley Publishing Company 1979
- Christos H. Papadimitriou: **Computational Complexity**; 1995 Addison-Wesley Publishing Company, Inc

# Computational Problems are Everywhere

## Example 1.1:

- What are the factors of 54,623?
- What is the shortest route by car from Berlin to Hamburg?
- My program now runs for two weeks. Will it ever stop?
- Is this C++ program syntactically correct?

## Clear

Computational Problems are ubiquitous in our everyday life!

And, depending on what we want to do, those problems might be either **easily solvable** or **hardly solvable**.

Approach to problems:

[T]he way is to avoid what is strong, and strike at what is weak.

(Sun Tzu: The Art of War, Chapter 6: Weak Points and Strong)

# Examples

**Example 1.2 (Shortest Path Problem):** Given a weighted graph and two vertices  $s, t$ , find the shortest path between  $s$  and  $t$ .

Easily solvable using, e.g., Dijkstra's Algorithm.

**Example 1.3 (Longest Path Problem):** Given a weighted graph and two vertices  $s, t$ , find the **longest** path between  $s$  and  $t$ .

No efficient algorithm known, and believed to not exist (this problem is **NP-hard**)

## Observation

Difficulty of a problem is hard to assess

# Measuring the Difficulty of Problems

## Question

How can we measure the complexity of a problem?

## Approach

Estimate the resource requirements of the “best” algorithm that solves this problem.

Typical Resources:

- Running Time
- Memory Used

## Note

To assess the complexity of a problem, we need to consider **all possible algorithms** that solve this problem.



# Problems

## What actually is ... a Problem?

(Decision) Problems are **word problems** of particular languages.

**Example 1.4:** “Problem: Is a given graph connected?” will be modelled as the word problem of the language

$$\text{GCONN} := \{ \langle G \rangle \mid G \text{ is a connected graph} \}.$$

Then for a graph  $G$  we have

$$G \text{ is connected} \iff \langle G \rangle \in \text{GCONN}.$$

## Note

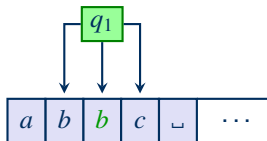
The notation  $\langle G \rangle$  denotes a suitable encoding of the graph  $G$  over some fixed alphabet (e.g.,  $\{0, 1\}$ ).

# Algorithms

## What actually is ... an Algorithm?

Different approaches to formalise the notion of an “algorithm”

- Turing Machines
- Lambda Calculus
- $\mu$ -Recursion
- ...



# Avoid What is Strong

Suppose we are given a language  $\mathcal{L}$  and a word  $w$ .

## Question

Does there need to exist **any** algorithm that decides whether  $w \in \mathcal{L}$ ?

## Answer

No. Some problems are **undecidable**.

### Example 1.5:

- The Halting Problem of Turing machines
- The Entscheidungsproblem (Is a first-order logical statement true?)
- Finding the lowest air fare between two cities ( $\rightarrow$  Reference)
- Deciding syntactic validity of C++ programs ( $\rightarrow$  Reference)

**Avoid:** We will focus mostly on decidable problems in this course.

# Time and Space

## Difficulty

Measuring running time and memory requirements depends highly on the **machine**, and not so much on the **problem**.

## Resort

Measure time and space only **asymptotically** using **Big-O**-Notation:

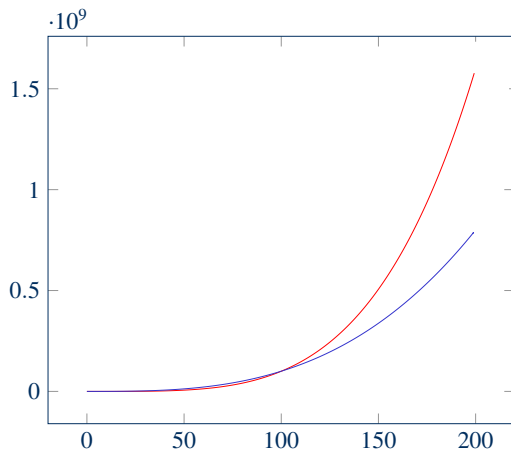
$$f(n) = O(g(n)) \iff f(n) \text{ "asymptotically bounded by" } g(n)$$

More formally:

$$f(n) = O(g(n)) \iff \exists c > 0 \exists n_0 \in \mathbb{N} \forall n > n_0: f(n) \leq c \cdot g(n).$$

# Big- $\mathcal{O}$ -Notation: Example

$$100n^3 + 1729n = \mathcal{O}(n^4):$$



# Complexity of Problems

## Approach

The **time (space) complexity** of a problem is the asymptotic running time of a fastest (least memory consumptive) algorithm that solves the problem.

## Problem

Still too difficult . . .

**Example 1.6 (Travelling Salesman Problem):** Given a weighted graph, find the shortest simple path visiting every node.

- Best known algorithm runs in time  $O(n^2 2^n)$   
(Bellman-Held-Karp algorithm)
- Best known lower bound is  $O(n \log n)$
- Exact complexity of TSP **unknown**

# Even more abstraction

## Approach

Divide decision problems into the “quality” of their fastest algorithms:

- P is the class of problems **solvable in polynomial time**
- PSpace is the class of problems **solvable in polynomial space**
- ExpTime is the class of problems **solvable in exponential time**
- L is the class of problems **solvable in logarithmic space**  
(apart from the input)
- NP is the class of problems **verifiable in polynomial time**
- NL is the class of problems **verifiable in logarithmic space**

And *many* more!

$\oplus P$ ,  $\#P$ , AC,  $AC^0$ , ACC0, AM, AP, APSpace, BPL, BPP, BQP, coNP, E, FP, IP, MA, MIP, NC, NExpTime, P/poly, PH, PP, RL, RP,  $\Sigma_i^P$ , TISP( $T(n)$ ,  $S(n)$ ), ZPP, ...

# Strike at What is Weak

## Approach (cf. Cobham–Edmonds Thesis)

The problems in  $P$  are “tractable” or “efficiently solvable” (and those outside are not)

**Example 1.7:** The following problems are in  $P$ :

- Shortest Path Problem
- Satisfiability of Horn-Formulas
- Linear Programming
- Primality

## Note

The Cobham-Edmonds-Thesis is only a **rule of thumb**: there are (practically) tractable problems outside of  $P$ , and (practically) intractable problems in  $P$ .



# Friend or Foe?

## Caveat

It is not known how big  $P$  is.

In particular, it is unknown whether  $P \neq NP$  or not.

## Approach

Try to find out which problems in a class are at least as hard as others.

**Complete** problems are then the hardest problems of a class.

**Example 1.8:** Satisfiability of propositional formulas is **NP-complete**: if we can efficiently decide whether a propositional formula is satisfiable, we can solve **any** problem in NP efficiently.

**But:** we still do not know whether we can or cannot solve satisfiability efficiently. We only know it will be difficult to find out . . .

# Learning Goals

- Get an overview over the foundations of Complexity Theory
- Gain insights into advanced techniques and results in Complexity Theory
- Understand what it means to “compute” something, and what the strengths and limits of different computing approaches are
- Get a feeling of how hard certain problems are, and where this hardness comes from
- Appreciate how very little we actually know about the computational complexity of many problems

# Lecture Outline (1)

- **Turing Machines** (Revision)  
Definition of Turing Machines; Variants; Computational Equivalence; Decidability and Recognizability; Enumeration; Oracles
- **Undecidability**  
Examples of Undecidable Problems; Mapping Reductions; Rice's Theorem; Recursion Theorem
- **Time Complexity**  
Measuring Time Complexity; Many-One Reductions; Cook-Levin Theorem; Time Complexity Classes (P, NP, ExpTime); NP-completeness; pseudo-NP-complete problems
- **Space Complexity**  
Space Complexity Classes (PSpace, L, NL); Savitch's Theorem; PSpace-completeness; NL-completeness; NL = coNL

## Lecture Outline (2)

- **Diagonalisation**

Hierarchy Theorems (det. Time, non-det. Time, Space); Gap Theorem; Ladner's Theorem; Relativisation; Baker-Gill-Solovay Theorem

- **Alternation**

Alternating Turing Machines;  $\text{APTime} = \text{PSPACE}$ ;  $\text{APSPACE} = \text{ExpTime}$ ; Polynomial Hierarchy

- **Circuit Complexity**

Boolean Circuits; Alternative Proof of Cook-Levin Theorem; Parallel Computation (NC); P-completeness; P/poly; (Karp-Lipton Theorem, Meyer's Theorem)

- **Probabilistic Computation**

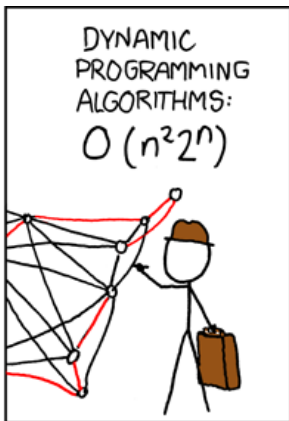
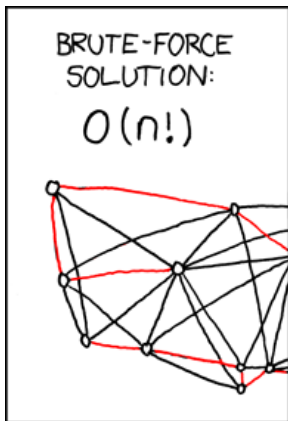
Randomised Complexity Classes (RP, PP, BPP, ZPP); Sipser-Gács-Lautemann Theorem

- **Quantum Computing**

Quantum mechanics for computer scientists, entanglement, quantum circuits, BQP

# Avoid what is Strong, and Strike at what is Weak

Sometimes the best way to solve a problem is to avoid it ...



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