On the Complexity of Graded Modal Logics with Converse

Bartosz Bednarczyk(✉), Emanuel Kieroński(✉), and Piotr Witkowski(✉)

Institute of Computer Science, University of Wrocław, Wrocław, Poland
{Bartosz.Bednarczyk,Emanuel.Kieronski,Piotr.Witkowski}@cs.uni.wroc.pl

Abstract. A complete classification of the complexity of the local and global satisfiability problems for graded modal language over traditional classes of frames has already been established. By "traditional" classes of frames we mean those characterized by any positive combination of reflexivity, seriality, symmetry, transitivity, and the Euclidean property. In this paper we fill the gaps remaining in an analogous classification of the graded modal language with graded converse modalities. In particular, we show its NExpTime-completeness over the class of Euclidean frames, demonstrating this way that over this class the considered language is harder than the language without graded modalities or without converse modalities. We also consider its variation disallowing graded converse modalities, but still admitting basic converse modalities. Our most important result for this variation is confirming an earlier conjecture that it is decidable over transitive frames. This contrasts with the undecidability of the language with graded converse modalities.

1 Introduction

Since many years modal logic has been an active topic in many academic disciplines, including philosophy, mathematics, linguistics, and computer science. Regarding applications in computer science, e.g., in knowledge representation or verification, some important variations are those involving graded and converse modalities. In this paper, we investigate their computational complexity.

By *a modal logic* we will mean a pair $(\mathcal{L}, \mathcal{F})$, represented usually as $\mathcal{F}(\mathcal{L}^*)$, where $\mathcal{L}$ is a *modal language*, $\mathcal{F}$ is a *class of frames*, and $\mathcal{L}^*$ is a short symbolic representation of $\mathcal{L}$ (see the next paragraph), characterizing the *modalities* of $\mathcal{L}$.

While we are mostly interested in languages with graded and converse modalities, to set the scene we need to mention languages without them. Overall, the following five languages are relevant: the basic *one-way modal language* ($\mathcal{L}^* = \Diamond$) containing only one, *forward*, modality $\Diamond$; *graded one-way modal language* ($\mathcal{L}^* = \Diamond_{\geq}$) extending the previous one by *graded* forward modalities, $\Diamond_{\geq n}$, for all $n \in \mathbb{N}$; *two-way modal language* ($\mathcal{L}^* = \Diamond, \Diamond^{-}$) containing basic forward modality and the *converse* modality $\Diamond^{-}$; *graded two-way modal language* ($\mathcal{L}^* = \Diamond_{\geq}, \Diamond^{-}_{\geq}$) containing the forward modality, the converse modality and their graded versions $\Diamond_{\geq n}$, $\Diamond^{-}_{\geq n}$, for all $n \in N$; and, additionally, a restriction of

F. Calimeri et al. (Eds.): JELIA 2019, LNAI 11468, pp. 642–658, 2019.
https://doi.org/10.1007/978-3-030-19570-0_42

the latter without graded converse modalities, but with basic converse modality ($\mathcal{L}^* = \Diamond_{\geq}, \diamondminus$).

The meaning of graded modalities is natural: $\Diamond_{\geq n}\varphi$ means "φ is true at no fewer than n successors of the current world", and $\diamondminus_{\geq}\varphi$ means "φ is true at no fewer than n predecessors of the current world". We also recall that $\Diamond\varphi$ means "φ is true at some successor of the current world" and $\diamondminus\varphi$—"φ is true at some predecessor of the current world". Thus, e.g., $\Diamond$ is simply $\Diamond_{\geq 1}$.

Our aim is to classify the complexity of the local ("in a world") and global ("in all worlds") satisfiability problems for all the logics obtained by combining any of the above languages with any class of frames from the so-called modal cube, that is a class of frames characterized by any positive combination of reflexivity (T), seriality (D), symmetry (B), transitivity (4), and the Euclidean property (5). See Fig. 1 for a visualization of the modal cube. Nodes of the depicted graph correspond to classes of frames and are labelled by letters denoting the above-mentioned properties, with S used in S4 and S5 for some historical reasons to denote reflexivity, and K denoting the class of all frames. Note that the modal cube contains only 15 classes, since some different combinations of the relevant properties lead to identical classes, e.g., seriality implies reflexivity, symmetry and transitivity imply the Euclideaness, and so on. A lot of work has been already done. The cases of basic one-way language and graded one-way language are completely understood. See Fig. 1. The results for the former can be established using some standard techniques, see, e.g., [3] and the classical paper [9]. The local satisfiability of the latter is systematically analysed in [7], with complexities turning out to lie between NP and NExpTime. As for its global satisfiability, some of the results follow from [7], some are given in [15], and the other can be easily obtained using again some standard techniques. In the case of non-graded two-way modal language, over most relevant classes of frames, tight complexity

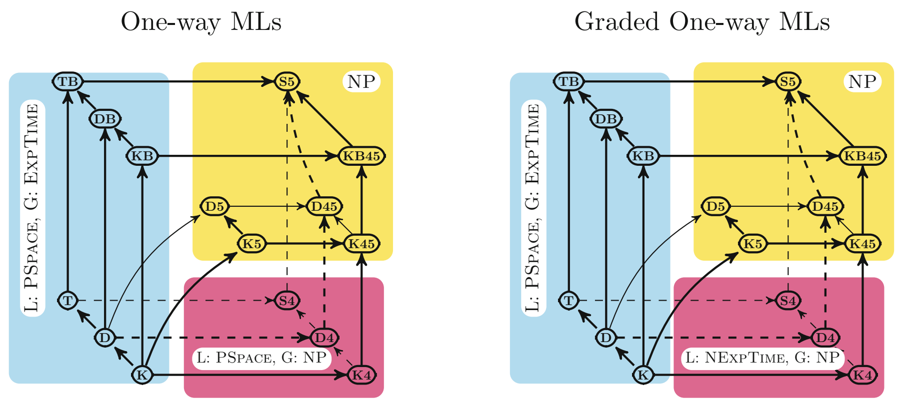

Fig. 1. Complexity of one-way modal logics. All bounds are tight. If local and global satisfiability differ in complexity then "L:" indicates the local and "G:"—the global satisfiability.

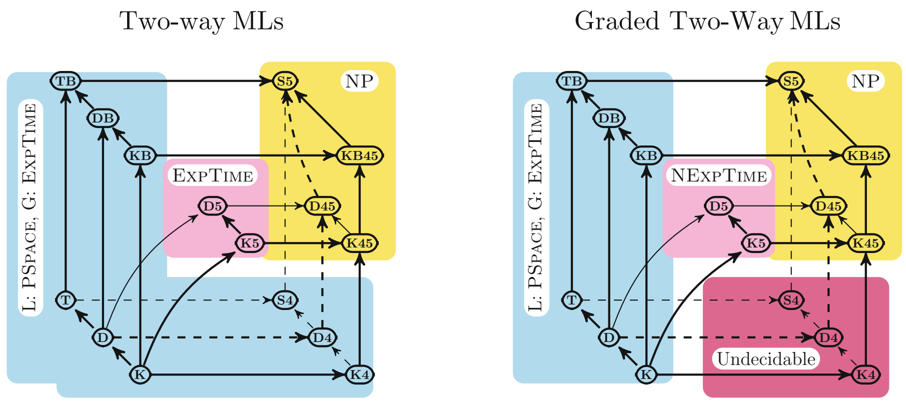

Fig. 2. Complexities of two-way modal logics. All bounds are tight.

bounds for local and global satisfiability are also known. However, according to the survey part of [15], for global satisfiability of the logics of transitive frames, K4($\Diamond, \bar{\Diamond}$), S4($\Diamond, \bar{\Diamond}$), D4($\Diamond, \bar{\Diamond}$), which is known to be in ExpTime (due to [5] or due to a translation to description logic $\mathcal{SI}$, whose satisfiability is in ExpTime [14]), the corresponding lower bound is missing. In the literature we were also not able find a tight lower bound for the logics of Euclidean frames, K5($\Diamond, \bar{\Diamond}$), D5($\Diamond, \bar{\Diamond}$). We provide both missing bounds in the full version of this paper [2], obtaining them by reductions from the acceptance problem for polynomially space bounded alternating Turing machines.[1] See the left part of Fig. 2 for a complete complexity map in this case.

Let us now turn our attention to the most expressive two-way graded modal language with both graded forward and graded converse modalities. Its local and global satisfiability problems over the class of all frames (K) are known to be, resp., PSpace-complete and ExpTime-complete (see the survey part of [15] and references therein). In Sect. 2.2, we explain how to obtain these bounds, as well as the same bounds in all cases involving neither transitivity nor Euclideaness. For the ExpTime-bound, we employ the so-called *standard translation*. Over K4, D4 and S4 the logics turn out to be undecidable [15]. We remark that these are the only undecidable members of the whole family of logics considered in this paper. What remains are the classes of frames involving the Euclidean property. We solve them in Sect. 3. We prove that the logics K5($\Diamond_{\geq}, \bar{\Diamond}_{\geq}$) and D5($\Diamond_{\geq}, \bar{\Diamond}_{\geq}$) are locally and globally NExpTime-complete. Interestingly, this is a higher complexity than the ExpTime-complexity of the language without graded modalities [5] and NP-complexity of the language without converse [7] over the same classes of frames. We also show that, when additionally transitivity is required, that is, for the logics K45($\Diamond_{\geq}, \bar{\Diamond}_{\geq}$) and D45($\Diamond_{\geq}, \bar{\Diamond}_{\geq}$), the complexity drops down to NP.

[1] As explained to the first author by Emil Jeřábek, the latter bound can be alternatively proved by a reduction from TB, whose ExpTime-hardness follows from [4].

Finally, we consider the above-mentioned intermediate language $(\Diamond^{-}, \Diamond_{\geq})$ in which we can count the successors, we have the basic converse modality, but we cannot count the predecessors. Our main result here, presented in Sect. 4, is the decidability of the corresponding logics of transitive frames K4, D4 and S4. The result is obtained by showing the finite model property of the logics. This way we confirm a conjecture stated in [8] (an analogous conjecture was also formulated in the description logic setting [6,15]). The logics of the remaining classes of frames retain their complexities from the graded two-way case.

Due to a large number of papers in which the complexity bounds from Figs. 1 and 2 are scattered, we have not referenced all of them in this introduction. A reader willing to find an appropriate reference is asked to use an online tool prepared by the first author (http://bartoszbednarczyk.com/mlnavigator). For missing proofs see [2].

Related formalisms. Graded modalities are examples of *counting quantifiers* which are present in various formalisms. In particular, counting quantifiers were introduced for first-order logic: $\exists^{\geq n} x\varphi$ means: "at least n elements x satisfy φ". The satisfiability problem for some fragments of first-order logic with counting quantifiers was shown to be decidable. In particular, the two-variable fragment is NExpTime-complete [11], the two-variable guarded fragment is ExpTime-complete [12], and the one-variable fragment is NP-complete [13]. Counting quantifiers are also present, in the form of the so-called *number restrictions*, in some description logics, DLs. As some standard DLs embed modal logics, some results on DLs with number restrictions may be used to infer upper bounds on the complexity of some graded modal logics.

2 Preliminaries

2.1 Languages, Kripke Structures and Satisfiability

Let us fix a countably infinite set Π of *propositional variables*. The *language* of graded two-way modal logic is defined inductively as the smallest set of formulas containing Π, closed under Boolean connectives and, for any formula φ, containing $\Diamond_{\geq n}\varphi$ and $\Diamond^{-}_{\geq n}\varphi$, for all $n \in \mathbb{N}$. Given a formula φ, we denote its *length* by $|\varphi|$, and measure it as the number of symbols required to write φ, with numbers in subscripts $_{\geq}n$ encoded in binary.

The basic modality $\Diamond$ can be defined in terms of graded modalities: $\Diamond\varphi := \Diamond_{\geq 1}\varphi$. Analogously, for the converse modality: $\Diamond^{-} := \Diamond^{-}_{\geq 1}$. Keeping this in mind, we may treat all languages mentioned in the introduction as fragments of the above defined graded two-way modal language. We remark that we may also introduce modalities $\Diamond_{\leq n}\varphi := \neg\Diamond_{\geq n+1}\varphi$, $\Diamond^{-}_{\leq n}\varphi := \neg\Diamond^{-}_{\geq n+1}\varphi$, $\Box\varphi := \neg\Diamond\neg\varphi$ and $\Box^{-}\varphi := \neg\Diamond^{-}\neg\varphi$.

The semantics is defined with respect to Kripke structures, that is, structures over the relational signature with unary predicates Π and a binary predicate R, represented as triples $\mathfrak{A} = \langle W, R, V\rangle$, where W is the universe, R is a binary relation on W, and V is a function $V : \Pi \to \mathcal{P}(W)$, called a *valuation*. The *satisfaction relation* is defined inductively as follows:

- $\mathfrak{A}, w \models p$ iff $w \in V(p)$, for $p \in \Pi$,
- $\mathfrak{A}, w \models \neg\varphi$ iff $\mathfrak{A}, w \not\models \varphi$ and similarly for the other Boolean connectives,
- $\mathfrak{A}, w \models \Diamond_{\geq n}\varphi$ iff there is at least n worlds $v \in W$ such that $\langle w, v\rangle \in R$ and $\mathfrak{A}, v \models \varphi$,
- $\mathfrak{A}, w \models \Diamondminus_{\geq n}\varphi$ iff there is at least n worlds $v \in W$ such that $\langle v, w\rangle \in R$ and $\mathfrak{A}, v \models \varphi$,

Given a structure $\mathfrak{A} = \langle W, R, V\rangle$ as above, we call the pair $\langle W, R\rangle$ its *frame*. For a class of frames $\mathcal{F}$, we define the local (global) satisfiability problem of a modal language $\mathcal{L}$ over $\mathcal{F}$ as follows. Given a formula φ of $\mathcal{L}$ verify if φ is satisfied at some world (all worlds) w of some structure $\mathfrak{A}$ whose frame belongs to $\mathcal{F}$. As said in the introduction, we are interested in all classes of frames characterized by any positive combination of reflexivity (T), seriality (D), symmetry (B), transitivity (4), and the Euclidean property (5).

2.2 Standard Translation

Modal logic can be seen as a fragment of first-order logic via the so-called *standard translation* (see, e.g., [3]). Here we present its variation suited for graded and converse modalities. We define functions $\mathbf{st}_z$ for $z \in \{x, y\}$. Let φ be a graded two-way modal logic formula. Below we explicitly show the definition of $\mathbf{st}_x$. The definition of $\mathbf{st}_y$ is symmetric.

$$\mathbf{st}_x(p) = p(x) \text{ for } p \in \Pi \tag{1}$$

$$\mathbf{st}_x(\varphi \wedge \psi) = \mathbf{st}_x(\varphi) \wedge \mathbf{st}_x(\psi) \text{ similarly for } \neg, \vee, \text{ etc.} \tag{2}$$

$$\mathbf{st}_x(\Diamond_{\geq C}\varphi) = \exists_{\geq C}.y(R(x, y) \wedge \mathbf{st}_y(\varphi)) \tag{3}$$

$$\mathbf{st}_x(\Diamondminus_{\geq C}\varphi) = \exists_{\geq C}.y(R(y, x) \wedge \mathbf{st}_y(\varphi)) \tag{4}$$

We note here that the obtained formula lies in the guarded two-variable fragment with counting quantifiers, GC^2, whose satisfiability is ExpTime-complete [12]. It is not difficult to see that φ is locally (globally) satisfiable iff $\exists x \mathbf{st}_x(\varphi)$ ($\forall x \mathbf{st}_x(\varphi)$) is satisfiable.

Since symmetry, seriality and reflexivity are trivially definable in GC^2, the standard translation can be used to provide a generic upper bound for the logics over all classes of frames from the modal cube involving neither transitivity nor Euclideaness. The global satisfiability for basic language $\Diamond$ is already ExpTime-hard [10] hence the following theorem holds.

Theorem 1. *The global satisfiability problem for $\mathcal{L}(\Diamond_{\geq}, \Diamondminus_{\geq})$ where $\mathcal{L}$ is any class of frames from the modal cube involving neither transitivity nor Euclideaness, is* ExpTime*-complete.*

In the case of local satisfiability, the complexity boils down to PSpace. For two-way graded language over K, D and T, we can adapt an existing tableaux algorithm by Tobies [14], yielding a tight PSpace bound. If the class of frames is symmetric, then the forward and converse modalities coincide and thus we may just apply the result for graded one-way language stated in [7]. Thus:

Theorem 2. *The local satisfiability problem for $\mathcal{L}(\Diamond_{\geq}, \Diamond^{-}_{\geq})$, where $\mathcal{L}$ is any class of frames from the modal cube involving neither transitivity nor Euclideaness, is* PSPACE*-complete.*

3 Euclidean Frames: Counting Successors and Predecessors

In this section, we consider the two-way graded modal language over frames from the modal cube satisfying the Euclidean property. We demonstrate an exponential gap (NEXPTIME vs NP) between the logics of Euclidean frames K5, D5 and the logics of transitive Euclidean frames K45, D45.

We note that for the two remaining Euclidean classes of frames, i.e., KB45 and S5, whose frames are additionally supposed to be symmetric, the obtained logics may be seen as one-way and thus their NP-completeness follows immediately from [7].

3.1 The Shape of Euclidean Frames

We begin by describing the shape of frames under consideration. Let $\mathfrak{A} = \langle W, R, V \rangle$ be a Kripke structure. A world $w \in W$ is called a *lantern* if $\langle w', w \rangle \notin R$, for every $w' \in W$. We say that lantern $l \in W$ *illuminates* world $w \in W$ if $\langle l, w \rangle \in R$. We say that l *illuminates a set of worlds* $I \subseteq W$ if l illuminates every world $w \in I$. We say that $w_1, w_2 \in W$ are *R-equivalent* (or simply *equivalent* if R is known from a context), if both $\langle w_1, w_2 \rangle \in R$ and $\langle w_2, w_1 \rangle \in R$. The *R-clique* for w_1 in $\mathfrak{A}$ is the set $Q_{\mathfrak{A}}(w_1) \subseteq W$ consisting of w_1 and all worlds R-equivalent to w_1. A world $w \in W$ is *reflexive* if $\langle w, w \rangle \in R$. We say that $\mathfrak{A}$ is *R-connected* if $\langle W, R \cup R^{-1} \rangle$ is a connected graph. By $L_{\mathfrak{A}}$ we denote the set of all lanterns in $\mathfrak{A}$. By $Q_{\mathfrak{A}}$ we denote $W \setminus L_{\mathfrak{A}}$. See Fig. 3.

Lemma 1. *Let $\mathfrak{A}$ be an R-connected structure over a Euclidean frame $\langle W, R \rangle$. All worlds in $Q_{\mathfrak{A}}$ are reflexive and $Q_{\mathfrak{A}}$ is an R-clique.*

Before we start proving complexity results for some more specific classes, we observe that global and local satisfiability are reducible to each other over any class involving Euclideaness. It follows from the fact that, as it usually happens for modal logics, we can restrict attention to R-connected frames and over such frames we can define a *universal modality* $\mathbf{U}$. Recall that $\mathbf{U}\varphi$ is true at a world w of a Kripke structure $\mathfrak{A}$ if and only if φ is true at all worlds of $\mathfrak{A}$. Once we understand how connected Euclidean structures look like, it is not hard to see that the universal modality can be defined by setting $\mathbf{U}\varphi := \Box\Box\boxminus\varphi$ and to prove the following lemma:

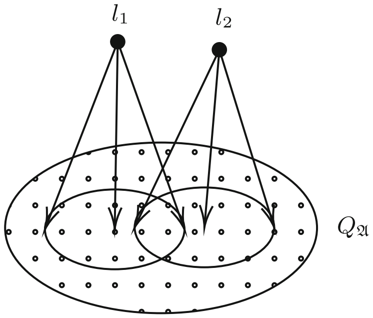

Fig. 3. A Euclidean structure $\mathfrak{A}$ with $L_{\mathfrak{A}} = \{l_1, l_2\}$

Lemma 2. *The universal modality* $\mathbf{U}$ *is definable in two-way modal language over connected Euclidean frames. Thus, for logics* $(\mathcal{L}, \mathcal{F})$*, where* $\mathcal{L}$ *contains the two-way modal language and* $\mathcal{F}$ *involves Euclideaness, the local and global satisfiability problems are polynomially interreducible.*

3.2 The Upper Bound for Graded Two-Way K5 and D5

Theorem 3. *The local and global satisfiability problems for* $\mathrm{K5}(\Diamond_{\geq}, \Diamond^{-}_{\geq})$ *and* $\mathrm{D5}(\Diamond_{\geq}, \Diamond^{-}_{\geq})$ *are in* NExpTime.

Proof. We start with the case of the class of all Euclidean frames K5. We translate a given modal formula φ to the two-variable logic with counting C^2, in which both graded modalities and the shape of connected Euclidean structures, as defined in Lemma 1, can be expressed. Since satisfiability of C^2 is in NExpTime [12], we obtain the desired conclusion. Recall the standard translation **st** from Sect. 2.2. Let $lantern(\cdot)$ be a new unary predicate and define $\varphi_{\mathbf{tr}}$ as

$$\mathbf{st}_x(\varphi) \wedge \forall x \forall y. (\neg lantern(x) \wedge \neg lantern(y) \rightarrow R(x,y)) \wedge (lantern(y) \rightarrow \neg R(x,y)).$$

Since $\mathbf{st}_x(\varphi)$ belongs to GC^2, $\varphi_{\mathbf{tr}}$ belongs to C^2 (but not to GC^2), Moreover, it features one free variable x. Let $\mathfrak{B}$ be a Kripke structure over a Euclidean frame. Expand $\mathfrak{B}$ to a structure $\mathfrak{B}^+$ by setting $lantern^{\mathfrak{B}^+} = \{w \in \mathfrak{B} \mid w \in L_{\mathfrak{B}}\}$. Taking into account Lemma 1 a structural induction on φ easily establishes the following condition

$$\mathfrak{B}, w_0 \models \varphi \text{ if and only if } \mathfrak{B}^+ \models \varphi_{\mathbf{tr}}[w_0/x] \text{ for every world } w_0 \in B.$$

Thus, a $\mathrm{K5}(\Diamond_{\geq}, \Diamond^{-}_{\geq})$ formula φ is locally satisfiable if and only if C^2 formula $\exists_{\geq 1} x. \varphi_{\mathbf{tr}}$ is satisfiable, yielding a NExpTime algorithm for $\mathrm{K5}(\Diamond_{\geq}, \Diamond^{-}_{\geq})$ local satisfiability. Membership of global satisfiability in NExpTime is implied by Lemma 2.

For the case of serial Euclidean frames, D5, it suffices to supplement the C^2 formula defined in the case of K5 with a conjunct $\exists x. (\neg lantern(x))$ expressing seriality. Correctness follows then from the simple observation that a Euclidean frame is serial if and only if it contains at least one non-lantern world (recall that all these worlds are reflexive).

3.3 Lower Bounds for Two-Way Graded K5 and D5

We now show a matching NExpTime-lower bound for the logics from the previous section. We concentrate on local satisfiability, but by Lemma 2 the results will hold also for global satisfiability. Actually, we obtain a stronger result, namely, we show that the two-way graded modal logics K5 and D5 remain NExpTime-hard even if counting in one-way (either backward or forward) is forbidden. In particular, we show hardness of the logics $K5(\Diamond_{\geq}, \bar{\Diamond})$ and $D5(\Diamond_{\geq}, \bar{\Diamond})$. We recall that this gives a higher complexity than the ExpTime-complexity of language $\Diamond, \bar{\Diamond}$ [5] and NP-complexity of language $\Diamond_{\geq}$ [7] over the same classes of frames. As a corollary, any adaptation of the translation to GF^2 from [5] fails when counting is allowed, unless ExpTime=NExpTime.

For proving our hardness result, we employ the *torus tiling problem*, where the goal is to decide whether there is a solution of tilings of an exponential torus.

Definition 1 (5.16 from [1]). *A* torus tiling problem $\mathcal{P}$ *is a tuple* $(\mathcal{T}, \mathcal{H}, \mathcal{V})$*, where* $\mathcal{T}$ *is a finite set of tile types and* $\mathcal{H}, \mathcal{V} \subseteq \mathcal{T} \times \mathcal{T}$ *represent the horizontal and vertical matching conditions. Let* $\mathcal{P}$ *be a tilling problem and* $c = t_0, t_1, \ldots, t_{n-1} \in \mathcal{T}^n$ *an initial condition. A mapping* $\tau : \{0, 1, \ldots, 2^n - 1\} \times \{0, 1, \ldots, 2^n - 1\} \to \mathcal{T}$ *is a* solution *for* $\mathcal{P}$ *and* c *if and only if, for all* $i, j < 2^n$*, the following holds* $(\tau(i, j), \tau(i \oplus_{2^n} 1, j)) \in \mathcal{H}, (\tau(i, j), \tau(i, j \oplus_{2^n} 1)) \in \mathcal{V}$ *and* $\tau(0, i) = t_i$ *for all* $i < n$*, where* $\oplus_i$ *denotes addition modulo* i*. It is well-known that there exists a* NExpTime*-complete torus tiling problem.*

Outline of the Proof. The proof is based on a polynomial time reduction from torus tiling problem as in Definition 1. Henceforward we assume that a NExpTime-complete torus tiling problem $\mathcal{P} = (\mathcal{T}, \mathcal{H}, \mathcal{V})$ is fixed. Let $c = t_0, t_1, \ldots, t_{n-1} \in \mathcal{T}^n$ be its initial condition. We write a formula which is (locally) satisfiable iff $(\mathcal{P}, c)$ has a solution. Each cell of the torus carries a *position* $(H, V) \in \{0, 1, \ldots, 2^n - 1\} \times \{0, 1, \ldots, 2^n - 1\}$, encoded in binary in a natural way by means of propositional letters $v_0, v_1, \ldots, v_{n-1}$ and $h_0, h_1, \ldots, h_{n-1}$, with h_0 and v_0 denoting the least significant bits. In the reduction, a single cell of the torus corresponds to a unique *inner*, i.e., non-lantern, world. Since there are exactly $2^n \cdot 2^n$ cells, we enforce that also the total number of inner worlds is equal to $2^n \cdot 2^n$. We make use of graded modalities to specify that every inner world has exactly $2^n \cdot 2^n$ successors. We stress here that this is the only place where we employ counting. Thus the proof works in the case where graded converse modalities are disallowed (but the basic converse modality will be necessary). Alternatively we could equivalently write that every inner world have exactly $2^n \cdot 2^n$ inner predecessors, and obtain hardness of the language with graded converse modalities, but without graded forward modalities.

Once we enforced a proper size of our torus, we must be sure that two distinct inner worlds carry different positions. We do it in two steps. We first write that a world with position $(0, 0)$ occurs in a model. For the second step, we assume that the grid is chessboard-like, i.e., all elements are coloured black or white in the same way as a chessboard is. Then, we say that every world is illuminated by four lanterns, where each of them propagates $\oplus_{2^n}1$ relation on the proper axis (from a black node to a white one and vice versa). Finally, having the torus prepared we encode a solution for a given tiling by simply labelling each inner world with some tile letter t and ensure (from the vantage point of lanterns) that any two horizontal or vertical neighbours do not violate the tiling constraints.

Encoding the Exponential Torus. Our goal is now to define a formula describing the exponential torus. The shape of the formula is following:

$$\varphi_{\text{torus}} \stackrel{\text{def}}{=} \varphi_{\text{firstCell}} \wedge \mathbf{U}\left(\varphi_{\text{partition}} \wedge \varphi_{\text{chessboard}} \wedge \varphi_{\text{torusSize}} \wedge \varphi_{\text{succ}}\right)$$

where $\mathbf{U}$ is the universal modality as in Lemma 2. The formula is going to say that: (i) the current world has position $(0, 0)$; (ii) every world is either a lantern or an inner world; (iii) the torus is chessboard-like, i.e., its cells are coloured *black* and *white* exactly as a real chessboard is; (iv) the overall size of the torus is equal to $2^n \cdot 2^n$; (v) each world of the torus has a proper vertical and a proper horizontal successor. The first four properties are straightforward to define:

$$\begin{aligned}
\varphi_{\text{firstCell}} &\stackrel{\text{def}}{=} \mathit{inner} \wedge \mathit{white} \wedge \bigwedge_{i=0}^{n-1} (\neg v_i \wedge \neg h_i) \\
\varphi_{\text{partition}} &\stackrel{\text{def}}{=} (\mathit{lantern} \leftrightarrow \neg \mathit{inner}) \wedge (\mathit{lantern} \leftrightarrow \neg \Diamond^{-} \top) \\
\varphi_{\text{chessboard}} &\stackrel{\text{def}}{=} (\mathit{white} \leftrightarrow \neg \mathit{black}) \wedge (\mathit{white} \leftrightarrow (v_0 \leftrightarrow h_0)) \\
\varphi_{\text{torusSize}} &\stackrel{\text{def}}{=} \mathit{inner} \rightarrow \Diamond_{=2^n \cdot 2^n} \top
\end{aligned}$$

The formula $\varphi_{\text{torusSize}}$ is valid, since the set of all inner worlds form a clique. The obtained formulae are of polynomial length since the number $2^n \cdot 2^n$ is encoded in binary.

What remains is to define φ_{succ}. For this, for every inner world we ensure that there exists a proper lantern responsible for establishing the appropriate successor relation. There will be four different types of such lanterns, denoted by propositional symbols: *vbw*, *hbw*, *vwb*, *hwb*. The intuition is the following: the first letter h or v indicates whether a lantern is responsible for H or V relation. The last two letters say whether a successor relation will be established between black and white worlds, or in the opposite way.

$$\begin{aligned}
\varphi_{\text{succ}} \stackrel{\text{def}}{=} (\mathit{lantern} \rightarrow & \bigvee_{\heartsuit \in \{vbw, hbw, vwb, hwb\}} (\heartsuit \wedge \varphi_{\heartsuit})) \wedge \\
& (\mathit{inner} \rightarrow \bigwedge_{\heartsuit \in \{vbw, hbw, vwb, hwb\}} \Diamond^{-}(\mathit{lantern} \wedge \varphi_{\heartsuit}))
\end{aligned}$$

Here we present φ_{vbw} only. The remaining formulas can be constructed in an analogous way and are explicitly shown in [2]. The formula below, intended to be interpreted at a lantern, consists of three parts: (i) the black and the white worlds illuminated by a lantern are pseudo-unique, i.e., all white (respectively, black) worlds illuminated by the same lantern carry the same position; uniqueness will follow later from $\varphi_{\text{torusSize}}$; (ii) all black worlds illuminated by a lantern have the same H-position as all white worlds illuminated by this lantern; (iii) if V_w (respectively, V_b) encodes a V-position of the white (respectively, black) worlds illuminated by a lantern, then $V_w = V_b \oplus_{2^n} 1$. Put $\varphi_{vbw} \stackrel{\text{def}}{=} \varphi_{\text{pseudoUniqueness}} \wedge \varphi_{\text{equalH}} \wedge \varphi_{V_w = V_b \oplus_{2^n} 1}$. The definitions of the first and the second part of φ_{vbw} are simple:

$$\varphi_{\text{pseudoUniqueness}} \stackrel{\text{def}}{=} \bigwedge_{c \in \{white, black\}} \bigwedge_{p \in \{v,h\}} \bigwedge_{i=0}^{n-1} \Diamond(c \wedge p_i) \rightarrow \Box(c \wedge p_i)$$

$$\varphi_{\text{equalH}} \stackrel{\text{def}}{=} \bigwedge_{i=0}^{n-1} \Diamond(black \wedge h_i) \leftrightarrow \Diamond(white \wedge h_i)$$

Finally, we need to encode the $\oplus_{2^n}$-operation as formula $\varphi_{V_w = V_b \oplus_{2^n} 1}$, but it is a standard implementation of binary addition. The following lemma says that the formula φ_{torus} indeed defines a proper torus. Its proof is routine.

Lemma 3. *Assume that the the formula φ_{torus} is locally satisfied at a world w of a Euclidean structure $\mathfrak{A} = \langle W, R, V\rangle$. Then, set $Q_{\mathfrak{A}}(w)$, i.e., the R-clique for w, contains exactly $2^n \cdot 2^n$ elements and each of them carries a different position (H, V), i.e., there are no two worlds v, v' satisfying exactly the same h_i- and v_i-predicates.*

Having defined a proper torus, it is quite easy to encode a solution to the torus tiling problem $\mathcal{P}$ with the initial condition c. Each inner node will be labelled with a single tile from $\mathcal{T}$ and using appropriate lanterns we enforce that two neighbouring worlds do not violate tiling rules $\mathcal{H}$ and $\mathcal{V}$. The whole process is again routine. Note that our intended modals are serial. Thus, the result holds also for the logic D5.

Theorem 4. *The local and global satisfiability problems for* $\mathrm{K5}(\Diamond_{\geq}, \Diamond^{-})$ *and* $\mathrm{D5}(\Diamond_{\geq}, \Diamond^{-})$ *are* NExpTime*-hard.*

Together with Theorem 3 this gives:

Theorem 5. *The local and global satisfiability problems for logics* $\mathrm{K5}(\Diamond_{\geq}, \Diamond^{-})$, $\mathrm{K5}(\Diamond_{\geq}, \Diamond^{-}_{\geq})$, $\mathrm{D5}(\Diamond_{\geq}, \Diamond^{-})$ *and* $\mathrm{D5}(\Diamond_{\geq}, \Diamond^{-}_{\geq})$ *are* NExpTime*-complete.*

3.4 Transitive Euclidean Frames

It turns out that the logics of transitive Euclidean frames have lower computational complexity. This is due to the following lemma.

Lemma 4. *Let $\mathfrak{A}$ be an R-connected structure over a transitive Euclidean frame $\langle W, R \rangle$. Then, every world $l \in L_{\mathfrak{A}}$ illuminates $Q_{\mathfrak{A}}$.*

A first-order formula stating that all non-lanterns are R-successors of all lanterns requires only two variables. Thus, as an immediate conclusion from Lemma 4, we can extend translation developed in the previous section to handle logic $\mathrm{K45}(\Diamond_{\geq}, \diamondminus_{\geq})$, and obtain NExpTime upper bound for satisfiability problem. In fact, the shape of transitive Euclidean structures is so simple that two variable logic is no longer necessary. Below we translate $\mathrm{K45}(\Diamond_{\geq}, \diamondminus_{\geq})$ and $\mathrm{D45}(\Diamond_{\geq}, \diamondminus_{\geq})$ to one-variable logic C^1, which is NP-complete [13].

Theorem 6. *The local and global satisfiability problems for* $\mathrm{K45}(\Diamond_{\geq}, \diamondminus_{\geq})$ *and* $\mathrm{D45}(\Diamond_{\geq}, \diamondminus_{\geq})$ *are in* NP.

Proof. The proof is similar in spirit to the proof of Lemma 3 in [7]. Let $\mathit{lantern}(\cdot)$ be a new unary predicate. We first define translation function $\mathbf{tr}$ that, given a $\mathrm{K45}(\Diamond_{\geq}, \diamondminus_{\geq})$ formula φ, produces an equisatisfiable C^1 formula $\mathbf{tr}(\varphi)$. We assume that all counting subscripts φ are non-zero.

$$\mathbf{tr}(p) = p(x) \text{ for all } p \in \Pi \tag{5}$$

$$\mathbf{tr}(\varphi \wedge \psi) = \mathbf{tr}(\varphi) \wedge \mathbf{tr}(\psi) \text{ similarly for } \neg, \vee, \text{ etc.} \tag{6}$$

$$\mathbf{tr}(\Diamond_{\geq C}\varphi) = \exists_{\geq C}.x(\neg \mathit{lantern}(x) \wedge \mathbf{tr}(\varphi)) \tag{7}$$

$$\mathbf{tr}(\Diamond_{\leq C}\varphi) = \exists_{\leq C}.x(\neg \mathit{lantern}(x) \wedge \mathbf{tr}(\varphi)) \tag{8}$$

$$\mathbf{tr}(\diamondminus_{\geq C}\varphi) = \neg \mathit{lantern}(x) \wedge \exists_{\geq C}.x(\mathbf{tr}(\varphi)) \tag{9}$$

$$\mathbf{tr}(\diamondminus_{\leq C}\varphi) = \mathit{lantern}(x) \vee \exists_{\leq C}.x(\mathbf{tr}(\varphi)) \tag{10}$$

Observe that $\mathbf{tr}(\varphi)$ is linear in the size of φ. Let $\mathfrak{B}$ be a Kripke structure over a transitive Euclidean frame. Expand $\mathfrak{B}$ to a structure $\mathfrak{B}^+$ by setting $\mathit{lantern}^{\mathfrak{B}^+} = \{w \in \mathfrak{B} \mid w \in L_{\mathfrak{B}}\}$. Taking into account Lemma 1 and Lemma 4, a structural induction on φ easily establishes the following condition

$$\mathfrak{B}, w_0 \models \varphi \text{ if and only if } \mathfrak{B}^+ \models \mathbf{tr}(\varphi)[w_0/x] \text{ for every world } w_0.$$

Thus, a $\mathrm{K45}(\Diamond_{\geq}, \diamondminus_{\geq})$ formula φ is locally satisfiable if and only if C^1 formula $\exists_{\geq 1}.x(\mathbf{tr}(x))$ is satisfiable, yielding an NP algorithm for $\mathrm{K45}(\Diamond_{\geq}, \diamondminus_{\geq})$ satisfiability. The algorithm for $\mathrm{D45}(\Diamond_{\geq}, \diamondminus_{\geq})$ is obtained by just a slight update to the one given above. It suffices to supplement the C^1 formula defined in the case of K45 with a conjunct $\exists x.(\neg \mathit{lantern}(x))$ expressing seriality (cf. the proof of Theorem 3).

4 Transitive Frames: Counting Successors, Accessing Predecessors

In this section, we consider the language $\Diamond_{\geq}, \Diamond^{-}$, that is the modal language in which we can count the successors, but cannot count the predecessors, having at our disposal only the basic converse modality. Over all classes of frames involving neither transitivity nor Euclideaness local satisfiability is PSpace-complete and global satisfiability is ExpTime-complete, as the tight lower and upper bounds can be transferred from, resp., the one-way non-graded language $\Diamond$ and the full two-way graded language. Over the classes of Euclidean frames K5 and D5, both problems are NExpTime-complete, as proved in Theorem 3. Over the classes of transitive Euclidean frames KB45, K45, D45, and S5 the problems are NP-complete, as the lower bound transfers from the language $\Diamond$, and the upper bound from the full two-way graded language (Theorem 6). So, over all the above-discussed classes of frames the complexities of $\Diamond_{\geq}, \Diamond^{-}$ and $\Diamond_{\geq}, \Diamond^{-}_{\geq}$ coincide. What is left are the classes of transitive frames K4, D4, and S4.

Recall that, in contrast to their one-way counterparts, the two-way graded logics of transitive frames K4($\Diamond_{\geq}, \Diamond^{-}_{\geq}$), D4($\Diamond_{\geq}, \Diamond^{-}_{\geq}$), and S4($\Diamond_{\geq}, \Diamond^{-}_{\geq}$) are undecidable [15]. Several papers [8][15][6] conjectured that decidability may possibly be regained if the restricted language $\Diamond_{\geq}, \Diamond^{-}$ is considered. Here we confirm this conjecture, demonstrating the finite model property for the obtained logics. We remark that we do not obtain tight complexity bounds in this case: The decision procedure arising is non-elementary, and the best lower bound is NExpTime.

In Lemma 5.5 from [15], it is shown that over the class of transitive structures global satisfiability and local satisfiability problems for the considered language are polynomially equivalent. The same can be easily shown when, additionally, reflexivity or seriality of structures are required. Thus, while below we explicitly deal with global satisfiability our results apply also to local satisfiability.

Let us concentrate on the class K4 of all transitive frames. The finite model construction we are going to present is the most complicated part of this paper. It begins similarly to the exponential model construction in the case of local satisfiability of K4($\Diamond_{\geq}$) from [7]: we introduce a Scott-type normal form (Lemma 5), and then generalize two pieces of model surgery used there (Lemma 6) to our setting: starting from any model, we first obtain a model with short *paths of cliques* and then we decrease the size of the cliques. Some adaptations of the constructions from [7] are necessary to properly deal with the converse modality. Having a model with short paths of cliques and small cliques, we develop some new machinery of *clique profiles* and *clique types* allowing us to decrease the overall size of the structure.

Lemma 5. *Given a formula φ of the language $(\Diamond_{\geq}, \Diamond\!\!\!\!\!-)$, we can compute in polynomial time a formula ψ of the form*

$$\eta \wedge \bigwedge_{1 \leq i \leq l} (p_i \rightarrow \Diamond_{\geq C_i} \pi_i) \wedge \bigwedge_{1 \leq i \leq m} (q_i \rightarrow \Diamond_{\leq D_i} \chi_i) \wedge \bigwedge_{1 \leq i \leq l'} (p'_i \rightarrow \overleftarrow{\Diamond} \pi'_i) \wedge \bigwedge_{1 \leq i \leq m'} (q'_i \rightarrow \overleftarrow{\Box} \neg \chi'_i) \quad (11)$$

where p_i, q_i, p'_i, q'_i are propositional variables, C_i, D_i are natural numbers, and η and π_i, χ_i, π'_i, χ'_i are propositional formulas, such that φ and ψ are globally satisfiable over exactly the same transitive frames.

Proof. A routine renaming process (cf. [7]).

Let us introduce some helpful terminology, copying it mostly from [7]. Let $\mathfrak{A} = \langle W, R, V \rangle$ be a transitive structure, and $w_1, w_2 \in W$. We say that w_2 is an *R-successor* of w_1 if $\langle w_1, w_2 \rangle \in R$; w_2 is a *strict R-successor* of w_1 if $\langle w_1, w_2 \rangle \in R$, but $\langle w_2, w_1 \rangle \notin R$; w_2 is a *direct R-successor* of w_1 if w_2 is a strict R-successor of w_1 and, for every $w \in W$ such that $\langle w_1, w \rangle \in R$ and $\langle w, w_2 \rangle \in R$ we have either $w \in Q_{\mathfrak{A}}(w_1)$ or $w \in Q_{\mathfrak{A}}(w_2)$. Recall that $Q_{\mathfrak{A}}(w)$ denotes the R-clique for w in $\mathfrak{A}$.

The *depth* of a structure $\mathfrak{A}$ is the maximum over all $k \geq 0$ for which there exists worlds $w_0, \ldots, w_k \in W$ such that w_i is a strict R-successor of w_{i-1} for every $1 \leq i \leq k$, or ∞ if no such a maximum exists. The *breadth* of $\mathfrak{A}$ is the maximum over all $k \geq 0$ for which there exists worlds $w, w_1, \ldots, w_k$ such that w_i is a direct R-successor of w for every $1 \leq i \leq k$, and the sets $Q_{\mathfrak{A}}(w_1), \ldots Q_{\mathfrak{A}}(w_k)$ are disjoint, or ∞ if no such a maximum exists. The *width* of $\mathfrak{A}$ is the smallest k such that $k \geq |Q_{\mathfrak{A}}(w)|$ for all $w \in W$, or ∞ if no such k exists.

Lemma 6. *Let φ be a normal form formula. If φ is globally satisfied in a transitive model $\mathfrak{A}$ then it is globally satisfied in a transitive model $\mathfrak{A}'$ with depth $d' \leq (\sum_{i=1}^{m} D_i) + m + m' + 1$ and width $c' \leq (\sum_{i=1}^{l} C_i) + l' + 1$.*

The above lemma can be proved by a construction being a minor modification of Stages 1 and 4 of the construction from the proof of Lemma 6 in [7], where the language without backward modalities is considered. Our adaptation just additionally takes care of backward witnesses and is rather straightforward. We remark here that also Stage 2 of the above mentioned construction could be adapted, giving a better bound on the depth of $\mathfrak{A}'$. We omit it here since such an improvement would not be crucial for our purposes. Stage 3 cannot be directly adapted.

To describe our next step, we need a few more definitions. Given a world w of a structure $\mathfrak{A}$, we define its *depth* as the maximum over all $k \geq 0$ for which there exist worlds $w = w_0, \ldots, w_k \in W$ such that w_i is a strict R-successor of w_{i-1} for every $1 \leq i \leq k$, or as ∞ if no such a maximum exists. For an R-clique Q we define its *depth* as the depth of w for any $w \in Q$; this definition is sound since for all $w_1 \in Q_{\mathfrak{A}}(w)$ the depth of w is equal to the depth of w_1.

From this point, we will mostly work on the level of cliques rather than individual worlds. We may view any structure $\mathfrak{A}$ as a partially ordered set of cliques. We write $\langle Q_1, Q_2 \rangle \in R$, and say that a clique Q_1 *sends* an edge to a clique Q_2 (or that Q_2 *receives* an edge from Q_1) if $\langle w_1, w_2 \rangle \in R$ for any (equivalently: for all) $w_1 \in Q_1$, $w_2 \in Q_2$.

A 1-type of a world w in $\mathfrak{A}$ is the set of all propositional variables p such that $\mathfrak{A} \models p$. We sometimes identify a 1-type with the conjunction of all its elements and negations of variables it does not contain. Given a natural number k, a structure $\mathfrak{A}$ and a clique Q in this structure $\mathfrak{A}$, we define a *k-profile* of Q (called just a *profile* if k is clear from the context) in $\mathfrak{A}$ as the tuple $prof^k_{\mathfrak{A}}(Q) = (\mathcal{H}, \mathcal{A}, \mathcal{B})$, where $\mathcal{H}$ is the multiset of 1-types in which the number of copies of each 1-type α equals $\min(k, |\{w \in Q : \mathfrak{A}, w \models \alpha\}|)$, $\mathcal{A}$ is the multiset of 1-types in which the number of copies of each 1-type α equals $\min(k, |\{w : \mathfrak{A}, w \models \alpha$ and w is a strict R-successor of a world from $Q\}|)$, and $\mathcal{B}$ is the set of 1-types of worlds for which a world from Q is its strict R-successor. Intuitively, $\mathcal{H}$ counts (up to k) realizations of 1-types *(H)ere* in Q, $\mathcal{A}$ counts (up to k) realizations 1-types *(A)bove* Q, and $\mathcal{B}$ says which 1-types appear *(B)elow* Q. Usually, given a normal form φ as in equation (11), we will be interested in M_φ-profiles of cliques, where $M_\varphi = \max(\{C_i\}_{i=1}^l \cup \{D_i + 1\}_{i=1}^m)$. Note that, given the M_φ-profiles of all cliques in a structure we are able to determine whether this structure is a global model of φ. The following observation is straightforward.

Lemma 7. *If $\mathfrak{A} \models \varphi$ for a normal form φ, and if in a structure $\mathfrak{A}'$ the M_φ-profile of every clique is equal to the M_φ-profile of some clique from $\mathfrak{A}$, then $\mathfrak{A}' \models \varphi$.*

We now prove the finite model property.

Lemma 8. *Let φ be a normal form formula. If φ is globally satisfied in a transitive model $\mathfrak{A}$ then it is globally satisfied in a finite transitive model $\mathfrak{A}'$.*

We assume that φ is as in (11). By Lemma 6, we may assume that $\mathfrak{A} = \langle W, R, V \rangle$ has depth $d \leq (\sum_{i=1}^m D_i) + m + m' + 1$ and width $c \leq (\sum_{i=1}^l C_i) + l' + 1$. Note that $\mathfrak{A}$ may be infinite due to possibly infinite breadth.

Let us split W into sets $U_0, \ldots, U_d$ with U_i consisting of all elements of W of depth i in $\mathfrak{A}$ (equivalently speaking: being the union of all cliques of depth i in $\mathfrak{A}$). They are called *layers*. Note that cliques from U_i may send R-edges only to cliques from U_j with $j < i$.

We now inductively define a sequence of models $\mathfrak{A} = \mathfrak{A}_{-1}, \mathfrak{A}_0, \ldots, \mathfrak{A}_d = \mathfrak{A}'$, $\mathfrak{A}_i = \langle W_i, R_i, V_i \rangle$ such that

- $W_i = U'_0 \cup \ldots \cup U'_i \cup U_{i+1} \cup \ldots \cup U_d$, where each U'_i is a finite union of some cliques from U_1,
- $V_i = V \restriction W_i$
- $\mathfrak{A}_i \restriction (U'_0 \cup \ldots \cup U'_i) = \mathfrak{A}_{i-1} \restriction (U'_0 \cup \ldots \cup U'_i)$,
- $\mathfrak{A}_i \restriction (U'_0 \cup \ldots \cup U'_{i-1} \cup U_{i+1} \cup \ldots \cup U_d) = \mathfrak{A}_{i-1} \restriction (U'_0 \cup \ldots \cup U'_{i-1} \cup U_{i+1} \cup \ldots \cup U_d)$
- in particular: $\mathfrak{A}_i \restriction (U_{i+1} \cup \ldots \cup U_d) = \mathfrak{A} \restriction (U_{i+1} \cup \ldots \cup U_d)$.

We obtain $\mathfrak{A}_i$ from $\mathfrak{A}_{i-1}$ by distinguishing a fragment U_i' of U_i, removing $U_i \setminus U_i'$ and adding some edges from $U_{i+1} \cup \ldots \cup U_d$ to U_i'; all the other edges remain untouched. We do it carefully, to avoid modifications of the profiles of the surviving cliques. Let us describe the process of constructing $\mathfrak{A}_i$ in details.

Assume $i \geq 0$. We first distinguish a finite subset U_i' of U_i. We define a *clique type* of every clique Q from U_i in $\mathfrak{A}_{i-1}$ as a triple $(\mathcal{H}, \mathcal{B}, S)$, where $\mathcal{H}$ and $\mathcal{B}$ are as in $prof^{M_\varphi}_{\mathfrak{A}_{i-1}}(Q)$ and S is the subset of cliques from $U_0' \cup \ldots \cup U_{i-1}'$, consisting of those cliques to which Q sends an R_{i-1}-edge (note that if $i = 0$, then this subset is empty). We stress that during the construction of $\mathfrak{A}_i$, the clique types of cliques are always computed in $\mathfrak{A}_{i-1}$.

For every clique type β realized in U_i, we mark M_φ cliques of this type, or all such cliques if there are less than M_φ of them. Let U_i' be the union of the marked cliques. We fix some arbitrary numbering of the marked cliques.

Now we define the relation R_i. As said before, for any pair of cliques Q_1, Q_2 both of which are contained in $U_0' \cup \ldots \cup U_{i-1}' \cup U_{i+1} \cup \ldots \cup U_d$ or in $U_0' \cup \ldots \cup U_i'$, we set $\langle Q_1, Q_2 \rangle \in R_i$ iff $\langle Q_1, Q_2 \rangle \in R_{i-1}$. It remains to define the R_i-edges from $U_{i+1} \cup \ldots \cup U_d$ to U_i'. For every clique Q from $U_{i+1} \cup \ldots \cup U_d$ and every clique type β realized in U_i', let $f(\beta)$ be the number of R_{i-1}-edges sent by Q to cliques of type β in U_i, if this number is not greater than M_φ, or, otherwise, let $f(\beta) = M_\varphi$. Let $f'(\beta)$ be the number of R_{i-1}-edges sent by Q to cliques of type β in U_i' (recall that this number is not greater than M_φ). We take all R_{i-1}-edges sent by Q to cliques of type β in U_i' to R_i. We send in $\mathfrak{A}_i$ $f(\beta) - f'(\beta)$ additional R_i-edges from Q to cliques of type β in U_i' using cliques to which Q does not send R_{i-1}-edges with minimal numbers in the fixed numbering. By the choice of U_i', we have enough such cliques in U_i'. We finish the construction of $\mathfrak{A}_i$ by removing all cliques from $U_i \setminus U_i'$.

Claim 7. *Each of the $\mathfrak{A}_i$ is a transitive structure.*

Claim 8. *The M_φ-profiles of every clique in $\mathfrak{A}_i$ is the same as its M_φ-profiles in $\mathfrak{A}$.*

The above claim and Lemma 7 imply that $\mathfrak{A}' = \mathfrak{A}_d$ is indeed a model of φ. As each of the U_i' contains a finite number of cliques and each of the cliques is finite, we get that $\mathfrak{A}'$ is finite. Let us estimate its size. To U_0' we take at most M_φ realizations of every clique type from U_0. M_φ is bounded exponentially, and the number of possible clique types in U_0 is bounded doubly exponentially in $|\varphi|$ (note that such cliques do not send any edges). Then, to construct U_i' we consider clique types distinguished, in particular, by the sets of cliques from $U_0' \cup \ldots U_{i-1}'$ to which a given clique sends edges. Thus, the number of cliques in U_i' may become exponentially larger than the number of cliques in U_{i-1}'. Thus, we can only estimate the number of cliques in our eventual finite model by a tower of exponents of height d (recall that our bound on d is exponential in $|\varphi|$, though a polynomial bound would not be difficult to obtain).

A careful inspection shows that our constructions respect reflexivity and seriality. Thus:

Theorem 9. *The logics* $K4(\Diamond_{\geq}, \Diamondminus)$, $D4(\Diamond_{\geq}, \Diamondminus)$, $S4(\Diamond_{\geq}, \Diamondminus)$ *have the finite model property. Their local and global satisfiability problems are decidable.*

Acknowledgements. We thank Evgeny Zolin for providing us a comprehensive list of gaps in the classification of the complexity of graded modal logics and for sharing with us his tikz files with modal cubes. We also thank Emil Jeřábek for his explanations concerning $K5(\Diamond, \Diamondminus)$. B.B. is supported by the Polish Ministry of Science and Higher Education program "Diamentowy Grant" no. DI2017 006447. E.K. and P.W. are supported by Polish National Science Centre grant no. 2016/21/B/ST6/01444.

References

1. Baader, F., Horrocks, I., Lutz, C., Sattler, U.: An Introduction to Description Logic. Cambridge University Press, Cambridge (2017). https://doi.org/10.1017/9781139025355
2. Bednarczyk, B., Kieronski, E., Witkowski, P.: On the complexity of graded modal logics with converse. CoRR abs/1812.04413 (2018). http://arxiv.org/abs/1812.04413
3. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, New York (2001). https://doi.org/10.1017/CBO9781107050884
4. Chen, C.-C., Lin, I.-P.: The complexity of propositional modal theories and the complexity of consistency of propositional modal theories. In: Nerode, A., Matiyasevich, Y.V. (eds.) LFCS 1994. LNCS, vol. 813, pp. 69–80. Springer, Heidelberg (1994). https://doi.org/10.1007/3-540-58140-5_8
5. Demri, S., de Nivelle, H.: Deciding regular grammar logics with converse through first-order logic. J. Logic Lang. Inf. **14**(3), 289–329 (2005). https://doi.org/10.1007/s10849-005-5788-9
6. Gutiérrez-Basulto, V., Ibáñez-García, Y.A., Jung, J.C.: Number restrictions on transitive roles in description logics with nominals. In: Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence, San Francisco, California, USA, 4–9 February 2017, pp. 1121–1127 (2017)
7. Kazakov, Y., Pratt-Hartmann, I.: A note on the complexity of the satisfiability problem for graded modal logics. In: Proceedings of the 24th Annual IEEE Symposium on Logic in Computer Science, LICS 2009, Los Angeles, CA, USA, 11–14 August 2009, pp. 407–416 (2009). https://doi.org/10.1109/LICS.2009.17
8. Kazakov, Y., Sattler, U., Zolin, E.: How many legs do I have? Non-simple roles in number restrictions revisited. In: 2007 Proceedings of 14th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2007, Yerevan, Armenia, 15–19 October, pp. 303–317 (2007). https://doi.org/10.1007/978-3-540-75560-9_23
9. Ladner, R.E.: The computational complexity of provability in systems of modal propositional logic. SIAM J. Comput. **6**(3), 467–480 (1977). https://doi.org/10.1137/0206033
10. Blackburn, P., van Benthem, J.: Handbook of Modal Logic, chapter Modal Logic: A Semantic Perspective, pp. 255–325. Elsevier (2006)
11. Pratt-Hartmann, I.: Complexity of the two-variable fragment with counting quantifiers. J. Logic Lang. Inf. **14**(3), 369–395 (2005). https://doi.org/10.1007/s10849-005-5791-1

12. Pratt-Hartmann, I.: Complexity of the guarded two-variable fragment with counting quantifiers. J. Log. Comput. **17**(1), 133–155 (2007). https://doi.org/10.1093/logcom/exl034
13. Pratt-Hartmann, I.: On the computational complexity of the numerically definite syllogistic and related logics. Bull. Symbolic Logic **14**(1), 1–28 (2008). https://doi.org/10.2178/bsl/1208358842
14. Tobies, S.: PSPACE reasoning for graded modal logics. J. Log. Comput. **11**(1), 85–106 (2001). https://doi.org/10.1093/logcom/11.1.85
15. Zolin, E.: Undecidability of the transitive graded modal logic with converse. J. Log. Comput. **27**(5), 1399–1420 (2017). https://doi.org/10.1093/logcom/exw026