Finite and Algorithmic Model Theory

Lecture 2 (Dresden 19.10.22, Long version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

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Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

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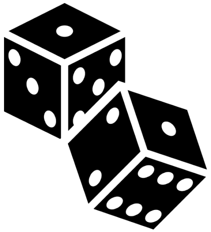

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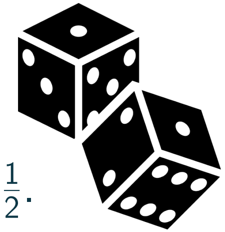

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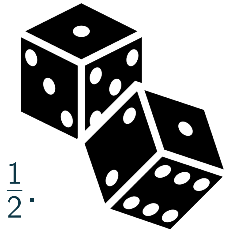

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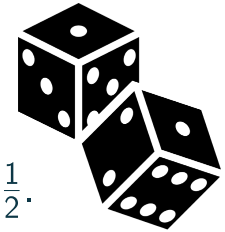

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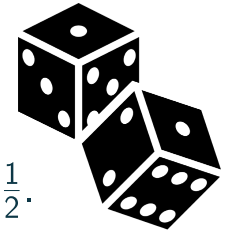

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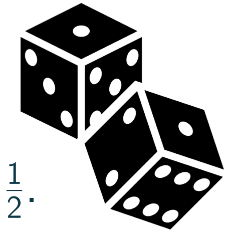

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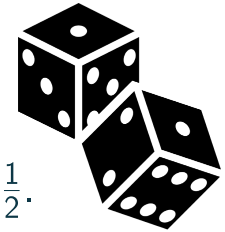

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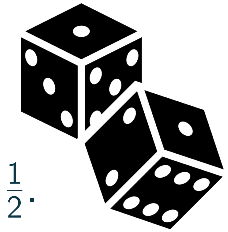

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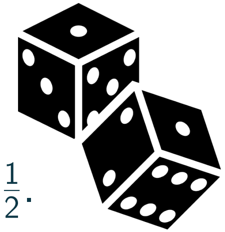

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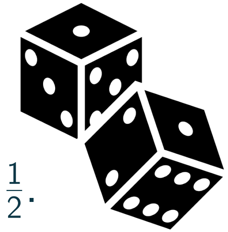

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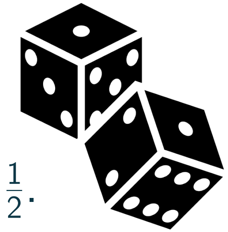

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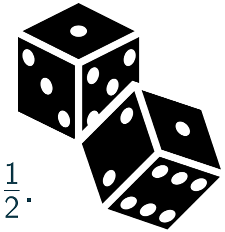

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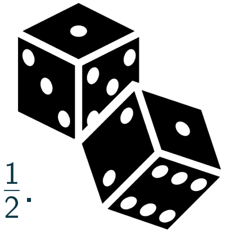

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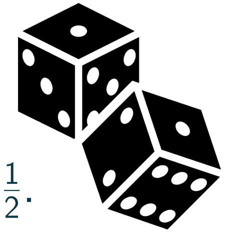

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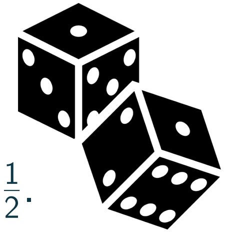

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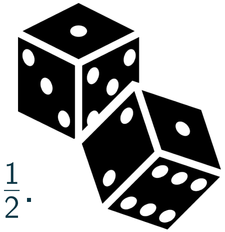

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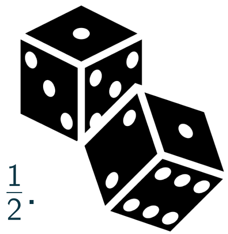

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2. Take $\mathcal{P}$:= "the graph has a triangle". $\mu_3(\mathcal{P}) = \frac{1}{8}$.

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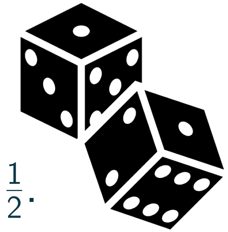

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2. Take $\mathcal{P}$:= "the graph has a triangle". $\mu_3(\mathcal{P}) = \frac{1}{8}$. Since $\mu_{3n}(\mathcal{P}) \geq 1 - (1-\frac{1}{8})^n$, we get

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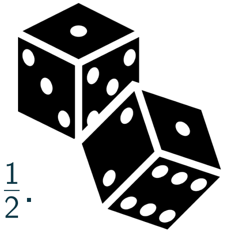

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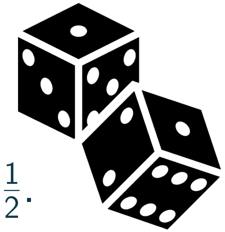

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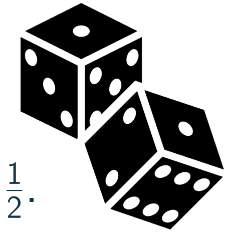

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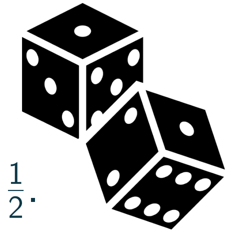

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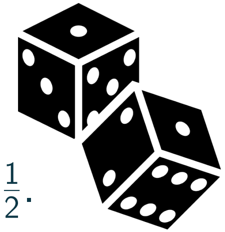

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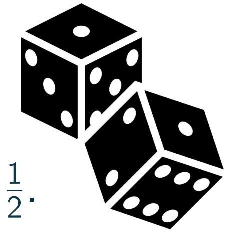

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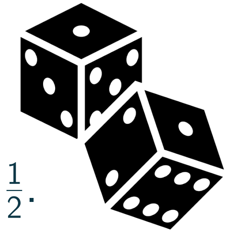

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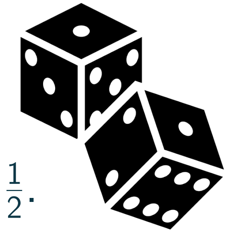

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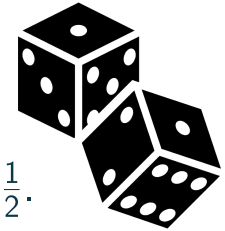

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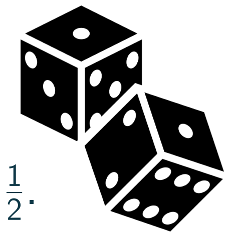

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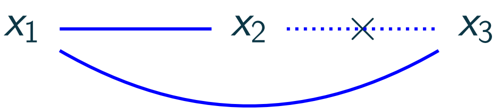

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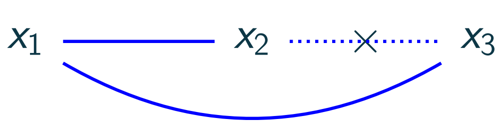

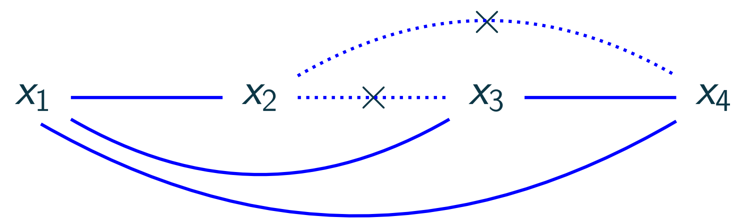

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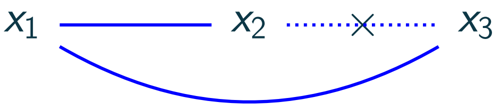

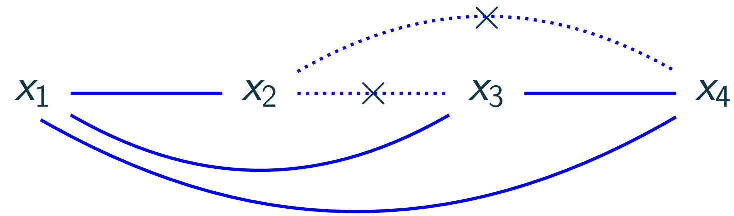

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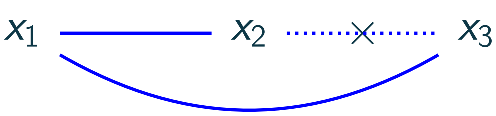

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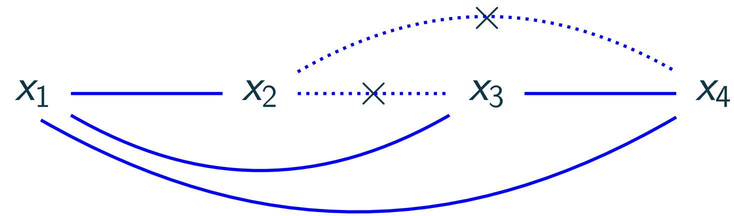

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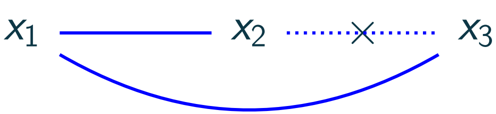

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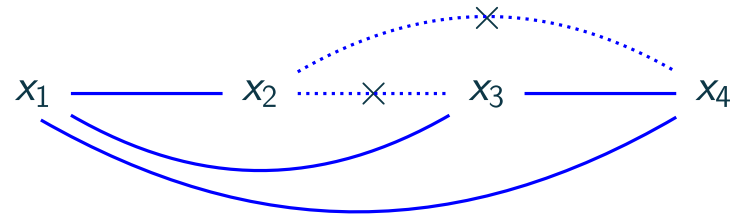

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A $(k+1)$-type t extends a k-type s if conjuncts$(s) \subseteq$ conjuncts(t) (c.f. the above picture).

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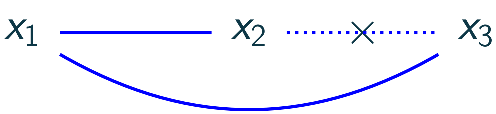
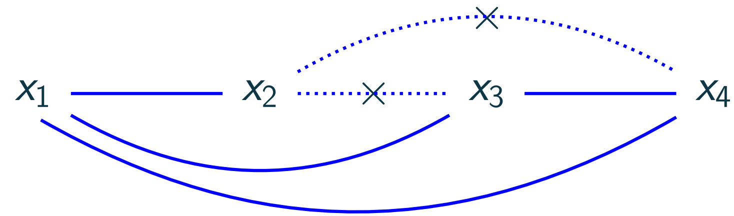


$$\subseteq$$



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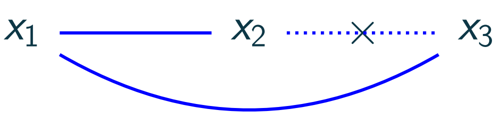

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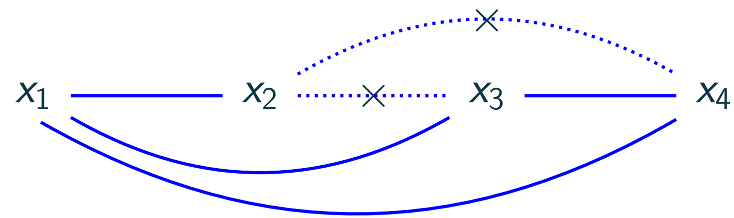

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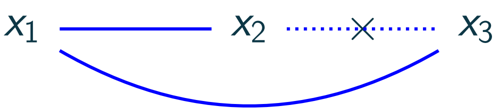

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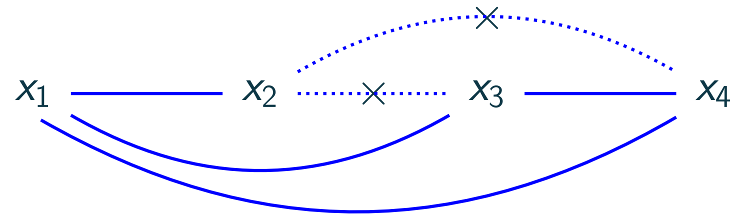

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The sum $\sum_{\sigma \in \mathbb{EA}_0} \mu_n(\neg\sigma)$ converges to 0 for $n \to \infty$, concluding $\mu_\infty(\varphi) = 1$. ∎

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Thus $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \neg\varphi$ (since $\mathfrak{B} \models \neg\varphi$). A contradiction!

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induction — back and forth — first not yet covered — exploit types realized by $\bar{a}$ — ind. ass. — $\mathfrak{B} \models \mathbb{EA}$ — Choose a witness

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Let $\mathfrak{G} = (V, \mathrm{E})$ be a graph such that $V = \mathbb{N}_+$ and $(i, j) \in \mathrm{E}^{\mathfrak{G}}$ iff $p_i \mid j$ or $p_j \mid i$ (p_i is the i-th prime number)

Lemma

$$\mathfrak{G} \models \sigma_{s,t} := \forall x_1 \ldots \forall x_n \; s(x_1, \ldots, x_n) \to \exists x_{n+1} \; t(x_1, \ldots, x_n, x_{n+1})$$

Proof

Take any $a_1, \ldots, a_k$ such that $\mathfrak{G} \models s(a_1, \ldots, a_k)$. Goal: Find a_{k+1} such that $\mathfrak{G} \models t(a_1, \ldots, a_k, a_{k+1})$.

We divide indices $1, 2, \ldots, k$ into Con $:= \{i \mid \mathrm{E}(x_i, x_{k+1}) \in t\}$ and DisC $:= \{i \mid \neg\mathrm{E}(x_i, x_{k+1}) \in t\}$.

Thus, our a_{k+1} must be connected to all a_i with $i \in$ Con and disconnected from all a_i with $i \in$ DisC.

Divide $x_1, x_2, \ldots, x_k$ biased on type connections with $k+1$

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(Dis)connected with $x \approx$ (non)dividable by the x-th prime number

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$a_{k+1} := \prod_{i \in \mathrm{Con}} p_{a_i} \cdot q$, where q is any prime number bigger than $\prod_{i=1}^{k} p_{a_i}$

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$a_{k+1} := \Pi_{i \in \mathrm{Con}} \; p_{a_i} \cdot q$, where q is any prime number bigger than $\Pi_{i=1}^{k} \; p_{a_i}$

And now it is easy to check our choice of a_{k+1} is correct. ■

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