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ASP: Language Extensions and Modelling

Lecture 11, 16th Jan 2023 // Foundations of Logic Programming, WS 2022/23

Previously . . .

- PROLOG-based logic programming focuses on **theorem proving**.
- LP based on stable model semantics focuses on **model generation**.
- The **stable model** of a positive program is its least (Herbrand) model.
- The **stable models** of a normal logic program P are those sets X for which X is the stable model of the positive program P^X (the reduct).
- The **well-supported** model semantics equals **stable** model semantics.

Example

Logic program $\{p \leftarrow {\sim}q,\ q \leftarrow {\sim}p\}$ has stable models $\{p\}$ and $\{q\}$.

Remember

A stable model is a supported model in which every true atom has well-founded support.

Overview

Language Extensions
- Integrity Constraints
- Choice Rules
- Cardinality Rules

Modelling
- Workflow
- A Case Study: Graph Colouring

Language Extensions

Basic Language Extensions

Fact

The expressiveness of a language can be enhanced by adding interesting language constructs.

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Questions

- What is the syntax of the new language construct?
- What is the semantics of the new language construct?
- How to implement the new language construct?

Basic Language Extensions

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Questions

- What is the syntax of the new language construct?
- What is the semantics of the new language construct?
- How to implement the new language construct?

Answers

- A way of providing semantics is to furnish a translation removing the new constructs.
- This translation might also be used for implementing the extension.

Integrity Constraint

Purpose: Eliminate unwanted solution candidates

Definition

An **integrity constraint** is of the form

$$\leftarrow a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n$$

where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$.

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Example: `:- edge(3,7), colour(3,red), colour(7,red).`

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Example Programs

$$\{\, a \leftarrow \sim b,\ b \leftarrow \sim a \,\}$$

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Example Programs

Program	Answer sets
$\{ a \leftarrow \sim b,\ b \leftarrow \sim a \}$	$\{a\}$ $\{b\}$
$\{ a \leftarrow \sim b,\ b \leftarrow \sim a \} \cup \{ \leftarrow a \}$	
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Embedding in Normal Rules

Translation

An integrity constraint of the form

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can be translated into the normal rule

$$x \leftarrow a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n, {\sim}x$$

where x is a new symbol.

Example Programs

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Choice Rule

Purpose: Provide choices over subsets of atoms

Definition

A **choice rule** is of the form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \sim a_{n+1}, \ldots, \sim a_o$$

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A choice rule of the form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \sim a_{n+1}, \ldots, \sim a_o$$

can be translated into $2m + 1$ normal rules

$$\begin{array}{rcl} x & \leftarrow & a_{m+1}, \ldots, a_n, \sim a_{n+1}, \ldots, \sim a_o \\ a_1 & \leftarrow & x, \sim x_1 \quad \ldots \quad a_m \leftarrow x, \sim x_m \\ x_1 & \leftarrow & \sim a_1 \quad\quad \ldots \quad x_m \leftarrow \sim a_m \end{array}$$

by introducing new atoms $x, x_1, \ldots, x_m$.

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Cardinality Rule

Purpose: Control (lower) cardinality of subsets of literals

Definition

A **cardinality rule** is the form

$$a_0 \leftarrow l \{ a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \}$$

where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
and l is a non-negative integer called **lower bound**.

Informal meaning: The head belongs to the stable model, if at least l positive/negative body literals are in/excluded in the stable model.

Example: `pass(c42) :- 2 { pass(a1); pass(a2); pass(a3) }.`

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$$\{ a \leftarrow 1 \{b, c\},\ b \leftarrow \}$$

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$$\{ \; a \leftarrow 1 \; \{b, c\}, \; b \leftarrow \; \} \qquad \{a, b\}$$

Embedding in Normal Rules

Translation

A cardinality rule of the form

$$a_0 \leftarrow l \, \{ \, a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \, \}$$

is translated into the normal rule $a_0 \leftarrow x(1, l)$

Idea: The atom $x(i, j)$ represents that at least j of the literals having an equal or greater index than i are in a stable model.

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is translated into the normal rule $a_0 \leftarrow x(1, l)$ and for $0 \leq k \leq l$ the rules

$$\begin{aligned}
x(i, k{+}1) &\leftarrow x(i+1, k), a_i \\
x(i, k) &\leftarrow x(i+1, k) && \text{for } 1 \leq i \leq m \\
x(j, k{+}1) &\leftarrow x(j+1, k), \sim a_j \\
x(j, k) &\leftarrow x(j+1, k) && \text{for } m+1 \leq j \leq n \\
x(n+1, 0) &\leftarrow
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An Example

- Program $\{\ a \leftarrow 1\ \{b, c\},\ b \leftarrow\ \}$ has the stable model $\{a, b\}$.

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- Program $\{\ a \leftarrow 1\ \{b, c\},\ b \leftarrow\ \}$ has the stable model $\{a, b\}$.
- Translating the cardinality rule yields the rules

$$
\begin{array}{rcl}
a & \leftarrow & x(1,1) \\
x(1,2) & \leftarrow & x(2,1), b \\
x(1,1) & \leftarrow & x(2,1) \\
x(2,2) & \leftarrow & x(3,1), c \\
x(2,1) & \leftarrow & x(3,1) \\
x(1,1) & \leftarrow & x(2,0), b \\
x(1,0) & \leftarrow & x(2,0) \\
x(2,1) & \leftarrow & x(3,0), c \\
x(2,0) & \leftarrow & x(3,0) \\
x(3,0) & \leftarrow &
\end{array}
\qquad\qquad
b \leftarrow
$$

having stable model $\{a, b, x(3,0), x(2,0), x(1,0), x(1,1)\}$.

Cardinality Rules with Upper Bounds

Translation

A rule of the form

$$a_0 \leftarrow l \{ a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \} u$$

where $0 \leq m \leq n$, each a_i is an atom for $1 \leq i \leq n$,
and l and u are non-negative integers

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where $0 \leq m \leq n$, each a_i is an atom for $1 \leq i \leq n$, and l and u are non-negative integers

is translated into

$$\begin{array}{rcl} a_0 & \leftarrow & x, {\sim}y \\ x & \leftarrow & l \; \{ \; a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n \; \} \\ y & \leftarrow & u{+}1 \; \{ \; a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n \; \} \end{array}$$

where x and y are new symbols.

The expression in the body of the cardinality rule is referred to as a cardinality constraint with lower and upper bound l and u.

Cardinality Rules with Upper Bounds

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A rule of the form

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \; \} \; u$$

where $0 \leq m \leq n$, each a_i is an atom for $1 \leq i \leq n$, and l and u are non-negative integers

is translated into

$$\begin{array}{rcl} a_0 & \leftarrow & x, \sim y \\ x & \leftarrow & l \; \{ \; a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \; \} \\ y & \leftarrow & u{+}1 \; \{ \; a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \; \} \end{array}$$

where x and y are new symbols.

The expression in the body of the cardinality rule is referred to as a cardinality constraint with lower and upper bound l and u.

Cardinality Constraints as Heads

Translation

A rule of the form

$$l\ \{a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n\}\ u \leftarrow a_{n+1}, \ldots, a_o, {\sim}a_{o+1}, \ldots, {\sim}a_p$$

where $0 \leq m \leq n \leq o \leq p$, each a_i is an atom for $1 \leq i \leq p$,
and l and u are non-negative integers

Cardinality Constraints as Heads

Translation

A rule of the form

$$l\ \{a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n\}\ u \leftarrow a_{n+1}, \ldots, a_o, {\sim}a_{o+1}, \ldots, {\sim}a_p$$

where $0 \leq m \leq n \leq o \leq p$, each a_i is an atom for $1 \leq i \leq p$,
and l and u are non-negative integers

Example: `1 {colour(2,red); colour(2,green); colour(2,blue)} 1.`

Cardinality Constraints as Heads

Translation

A rule of the form

$$l\ \{a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n\}\ u \leftarrow a_{n+1}, \ldots, a_o, {\sim}a_{o+1}, \ldots, {\sim}a_p$$

where $0 \leq m \leq n \leq o \leq p$, each a_i is an atom for $1 \leq i \leq p$, and l and u are non-negative integers

is translated into

$$\begin{aligned} x &\leftarrow a_{n+1}, \ldots, a_o, {\sim}a_{o+1}, \ldots, {\sim}a_p \\ \{a_1, \ldots, a_m\} &\leftarrow x \\ y &\leftarrow l\ \{a_1, \ldots, a_m, , {\sim}a_{m+1}, \ldots, {\sim}a_n\}\ u \\ &\leftarrow x, {\sim}y \end{aligned}$$

where x and y are new symbols.

Example: `1 {colour(2,red); colour(2,green); colour(2,blue)} 1.`

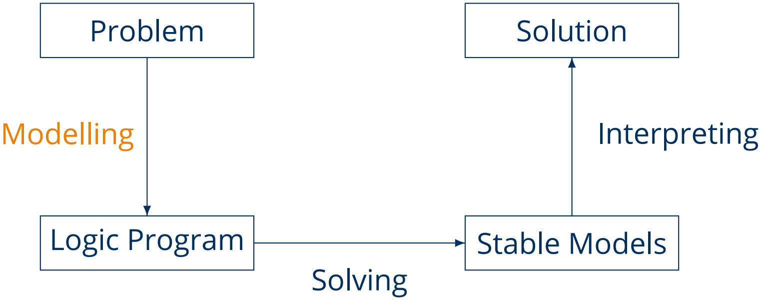
Quiz: Primes

Quiz

Consider the following answer set program P: …

Modelling

Modelling


Problem
Solution
Modelling
Interpreting
Logic Program
Stable Models
Solving

Guiding principle

Elaboration Tolerance (McCarthy, 1998)

"A formalism is elaboration tolerant [if] *it is convenient*
to modify a set of facts expressed in the formalism
to take into account new phenomena or changed circumstances."

Guiding principle

Elaboration Tolerance (McCarthy, 1998)

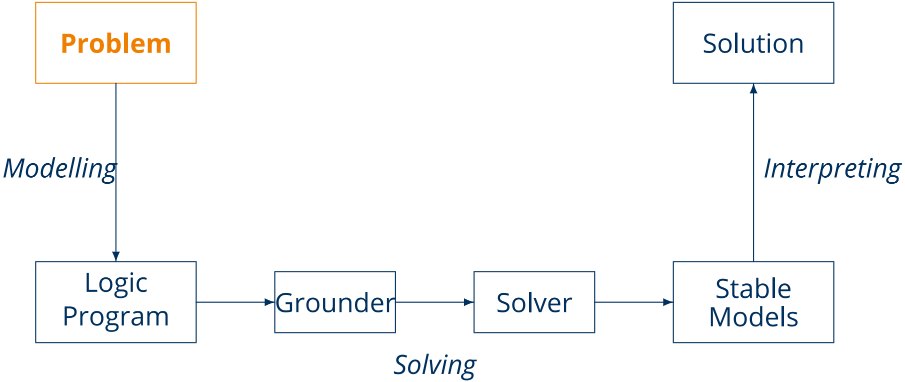
"A formalism is elaboration tolerant [if] *it is convenient to modify a set of facts expressed in the formalism to take into account new phenomena or changed circumstances."*

Uniform Problem Representation

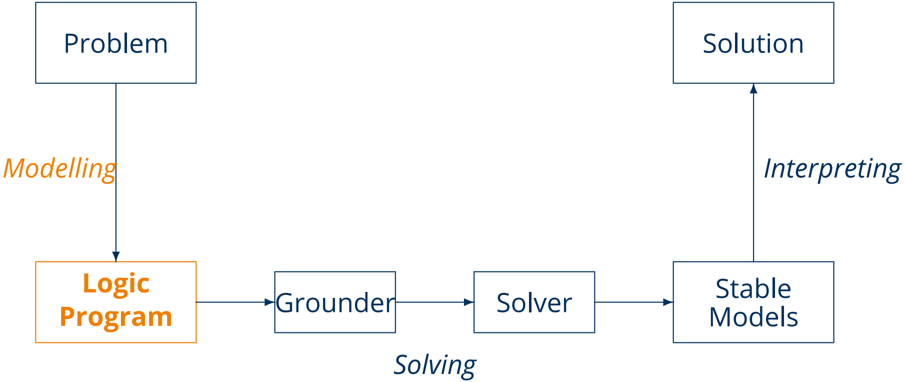
For solving a problem instance **I** of a problem class **C**,

- **I** is represented as a set of facts $P_{\mathbf{I}}$,
- **C** is represented as a set of rules $P_{\mathbf{C}}$, and
- $P_{\mathbf{C}}$ can be used to solve all problem instances in **C**

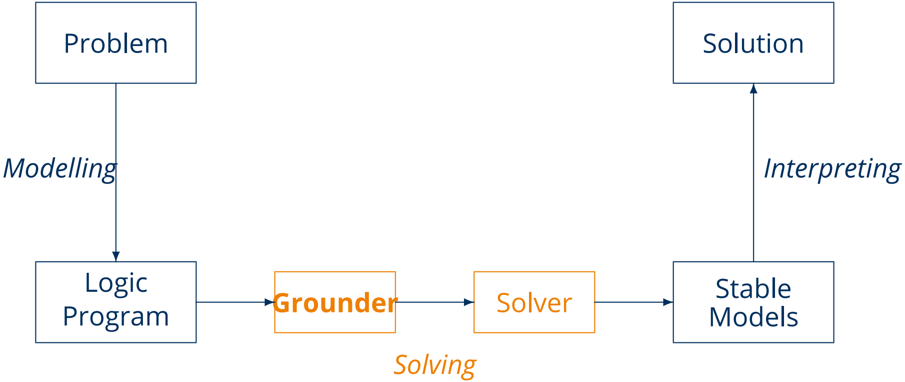
ASP workflow



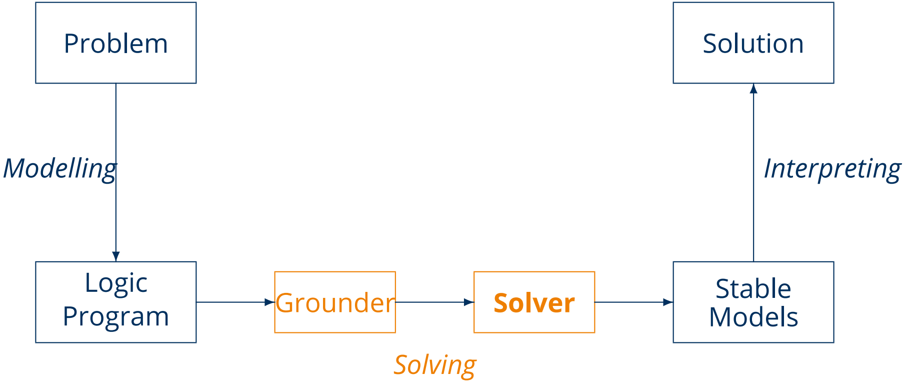
ASP workflow



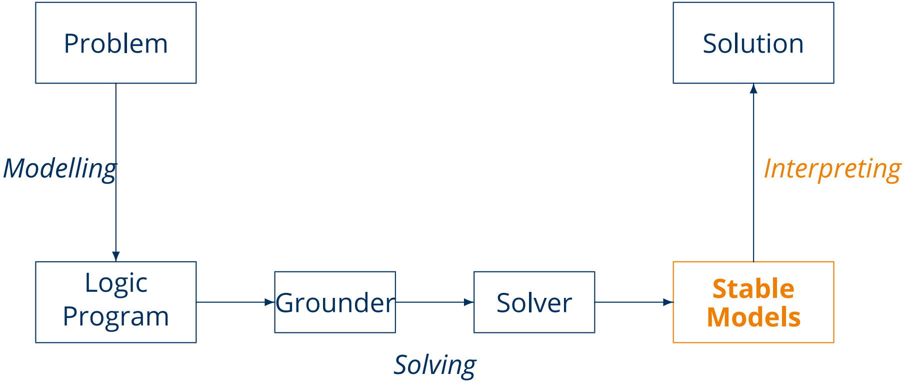
ASP workflow



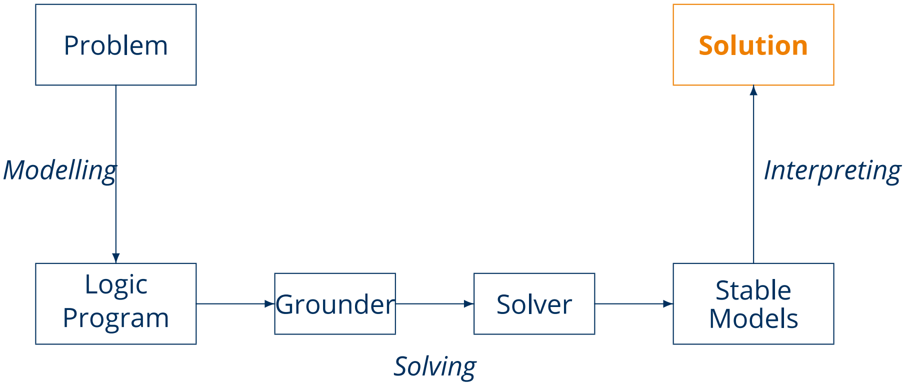
ASP workflow



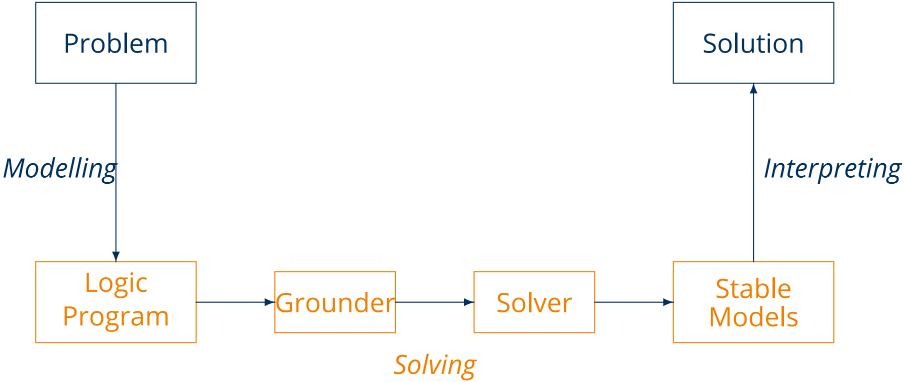
ASP workflow



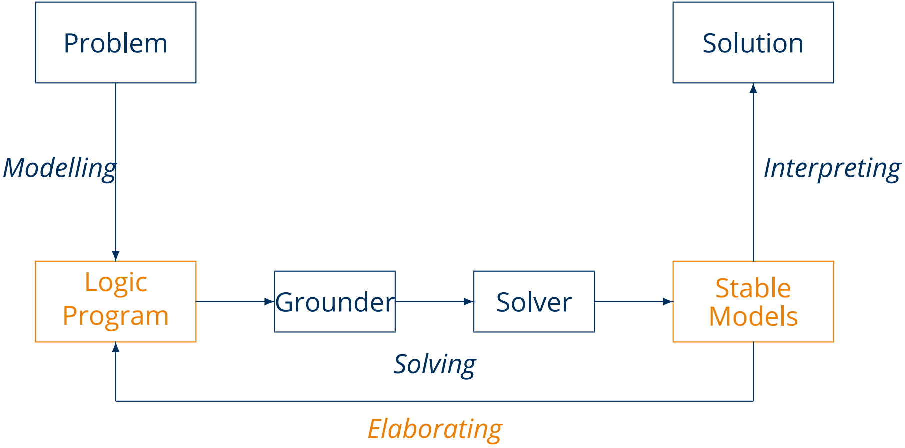
ASP workflow



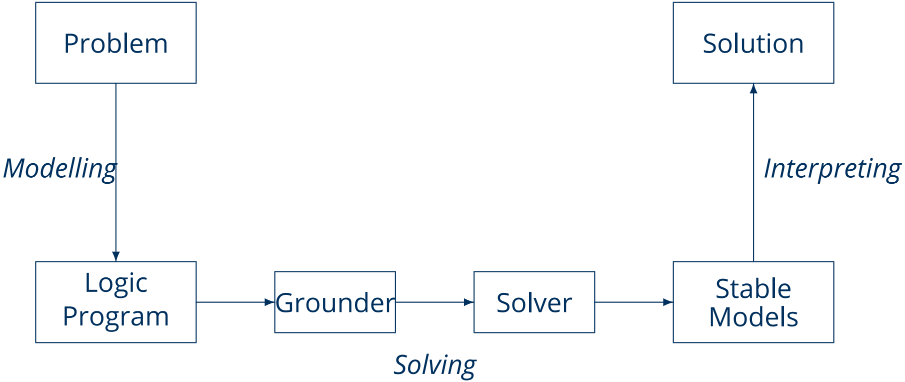
ASP workflow



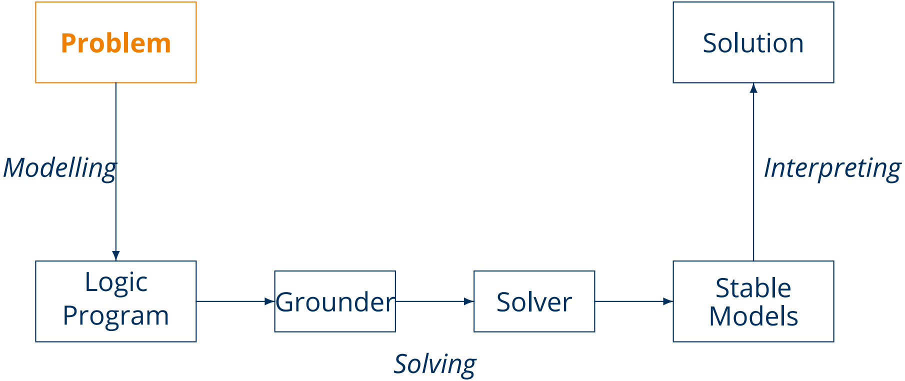
ASP workflow



ASP workflow



ASP workflow: Problem



A Case Study: Graph Colouring

A Case Study: Graph Colouring

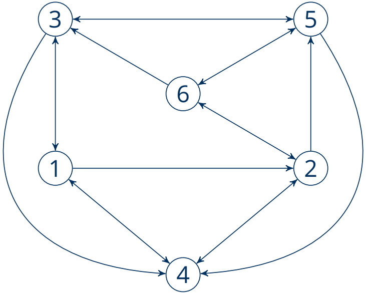
Problem instance:

A graph consisting of nodes and edges:

A Case Study: Graph Colouring

Problem instance:

A graph consisting of nodes and edges:

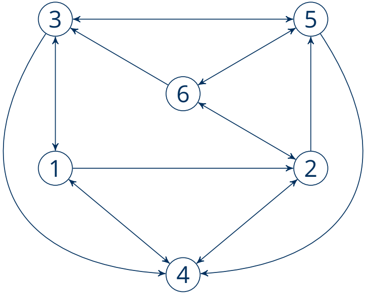


A Case Study: Graph Colouring

Problem instance:

A graph consisting of nodes and edges:

- facts using predicates `node/1` and `edge/2`



A Case Study: Graph Colouring

Problem instance:

A graph consisting of nodes and edges:

- facts using predicates `node/1` and `edge/2`
- facts using predicate `colour/1`

A Case Study: Graph Colouring

Problem instance:

A graph consisting of nodes and edges:

- facts using predicates `node/1` and `edge/2`
- facts using predicate `colour/1`

Problem class:

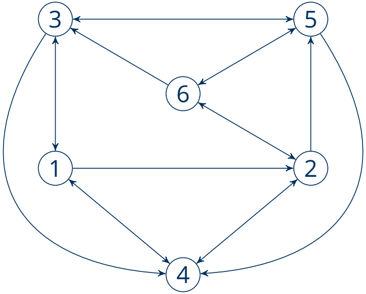
Assign each node one colour such that no two nodes connected by an edge have the same colour.

A Case Study: Graph Colouring

Problem instance:

A graph consisting of nodes and edges:

- facts using predicates `node/1` and `edge/2`
- facts using predicate `colour/1`



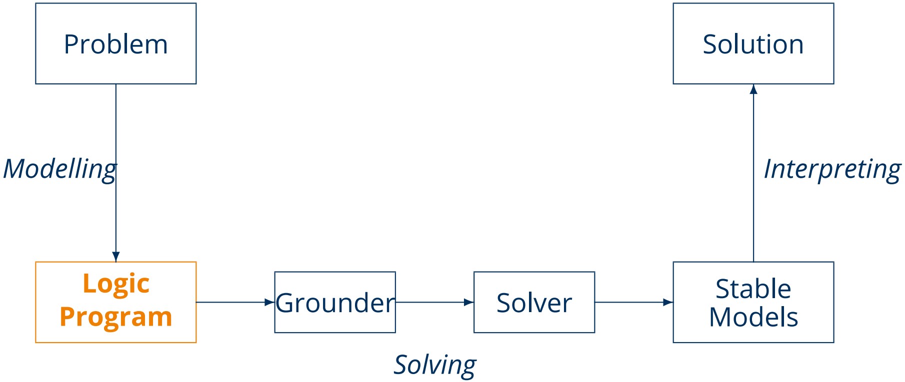
Problem class:

Assign each node one colour such that no two nodes connected by an edge have the same colour.

In other words:

1. Each node has one colour
2. Two connected nodes must not have the same colour

ASP Workflow: Problem Representation



Graph Colouring

Graph Colouring

```
node(1..6).
```

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).
```

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).
```

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).
```

} **Problem instance**

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).

1 { assign(N,C) : colour(C) } 1 :- node(N).
```

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).

1 { assign(N,r); assign(N,b); assign(N,g) } 1 :- node(N).
```

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
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colour(r). colour(b). colour(g).

1 { assign(N,C) : colour(C) } 1 :- node(N).
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Graph Colouring

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node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).

1 { assign(N,C) : colour(C) } 1 :- node(N).

:- edge(N,M), assign(N,C), assign(M,C).
```

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).

1 { assign(N,C) : colour(C) } 1 :- node(N).

:- edge(N,M), assign(N,C), assign(M,C).
```

Problem encoding

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).
```

} Problem instance

```
1 { assign(N,C) : colour(C) } 1 :- node(N).

:- edge(N,M), assign(N,C), assign(M,C).
```

} Problem encoding

Graph Colouring

graph.lp:

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

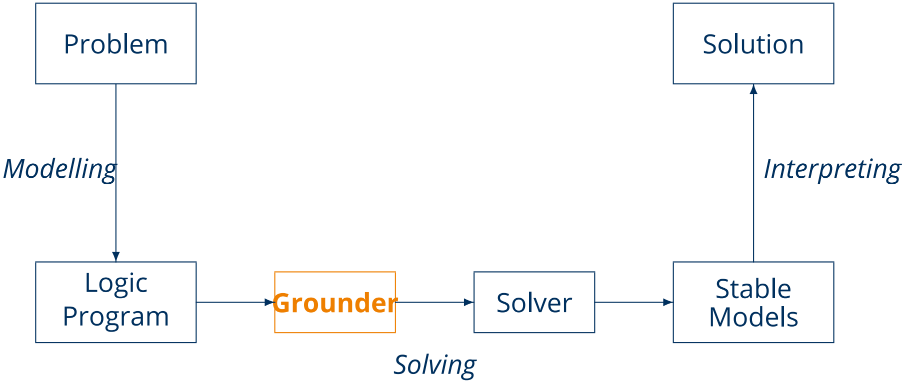
colour(r). colour(b). colour(g).
```

colour.lp:

```
1 { assign(N,C) : colour(C) } 1 :- node(N).

:- edge(N,M), assign(N,C), assign(M,C).
```

ASP Workflow: Grounding



Graph Colouring: Grounding

```
$ gringo -text graph.lp colour.lp
```

Graph Colouring: Grounding

$ gringo --text graph.lp colour.lp

```
node(1). node(2). node(3). node(4). node(5). node(6).

edge(1,2). edge(2,4). edge(3,1). edge(4,1). edge(5,3). edge(6,2).
edge(1,3). edge(2,5). edge(3,4). edge(4,2). edge(5,4). edge(6,3).
edge(1,4). edge(2,6). edge(3,5).            edge(5,6). edge(6,5).

colour(r). colour(b). colour(g).
```

Graph Colouring: Grounding

$ gringo −text graph.lp colour.lp

```
node(1). node(2). node(3). node(4). node(5). node(6).

edge(1,2). edge(2,4). edge(3,1). edge(4,1). edge(5,3). edge(6,2).
edge(1,3). edge(2,5). edge(3,4). edge(4,2). edge(5,4). edge(6,3).
edge(1,4). edge(2,6). edge(3,5).            edge(5,6). edge(6,5).

colour(r). colour(b). colour(g).

1{assign(1,r);assign(1,b);assign(1,g)}1.  1{assign(4,r);assign(4,b);assign(4,g)}1.
1{assign(2,r);assign(2,b);assign(2,g)}1.  1{assign(5,r);assign(5,b);assign(5,g)}1.
1{assign(3,r);assign(3,b);assign(3,g)}1.  1{assign(6,r);assign(6,b);assign(6,g)}1.
```

Graph Colouring: Grounding

```
$ gringo -text graph.lp colour.lp
```

```
node(1). node(2). node(3). node(4). node(5). node(6).

edge(1,2). edge(2,4). edge(3,1). edge(4,1). edge(5,3). edge(6,2).
edge(1,3). edge(2,5). edge(3,4). edge(4,2). edge(5,4). edge(6,3).
edge(1,4). edge(2,6). edge(3,5).            edge(5,6). edge(6,5).

colour(r). colour(b). colour(g).

1{assign(1,r);assign(1,b);assign(1,g)}1.  1{assign(4,r);assign(4,b);assign(4,g)}1.
1{assign(2,r);assign(2,b);assign(2,g)}1.  1{assign(5,r);assign(5,b);assign(5,g)}1.
1{assign(3,r);assign(3,b);assign(3,g)}1.  1{assign(6,r);assign(6,b);assign(6,g)}1.

:- assign(1,r),assign(2,r).  :- assign(2,r),assign(4,r). [...] :- assign(6,r),assign(2,r).
:- assign(1,b),assign(2,b).  :- assign(2,b),assign(4,b).       :- assign(6,b),assign(2,b).
:- assign(1,g),assign(2,g).  :- assign(2,g),assign(4,g).       :- assign(6,g),assign(2,g).
:- assign(1,r),assign(3,r).  :- assign(2,r),assign(5,r).       :- assign(6,r),assign(3,r).
:- assign(1,b),assign(3,b).  :- assign(2,b),assign(5,b).       :- assign(6,b),assign(3,b).
:- assign(1,g),assign(3,g).  :- assign(2,g),assign(5,g).       :- assign(6,g),assign(3,g).
:- assign(1,r),assign(4,r).  :- assign(2,r),assign(6,r).       :- assign(6,r),assign(5,r).
:- assign(1,b),assign(4,b).  :- assign(2,b),assign(6,b).       :- assign(6,b),assign(5,b).
:- assign(1,g),assign(4,g).  :- assign(2,g),assign(6,g).       :- assign(6,g),assign(5,g).
```

Graph Colouring: Grounding

$ clingo −text graph.lp colour.lp

```
node(1). node(2). node(3). node(4). node(5). node(6).

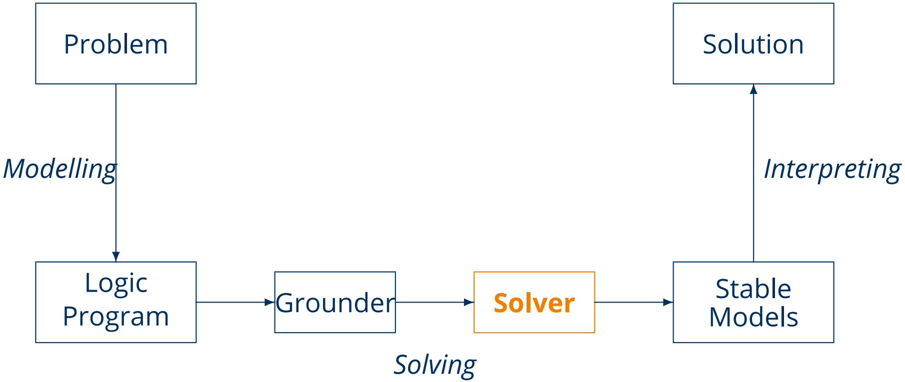
edge(1,2). edge(2,4). edge(3,1). edge(4,1). edge(5,3). edge(6,2).
edge(1,3). edge(2,5). edge(3,4). edge(4,2). edge(5,4). edge(6,3).
edge(1,4). edge(2,6). edge(3,5).            edge(5,6). edge(6,5).

colour(r). colour(b). colour(g).

1{assign(1,r);assign(1,b);assign(1,g)}1.  1{assign(4,r);assign(4,b);assign(4,g)}1.
1{assign(2,r);assign(2,b);assign(2,g)}1.  1{assign(5,r);assign(5,b);assign(5,g)}1.
1{assign(3,r);assign(3,b);assign(3,g)}1.  1{assign(6,r);assign(6,b);assign(6,g)}1.

:- assign(1,r),assign(2,r).  :- assign(2,r),assign(4,r). [...] :- assign(6,r),assign(2,r).
:- assign(1,b),assign(2,b).  :- assign(2,b),assign(4,b).       :- assign(6,b),assign(2,b).
:- assign(1,g),assign(2,g).  :- assign(2,g),assign(4,g).       :- assign(6,g),assign(2,g).
:- assign(1,r),assign(3,r).  :- assign(2,r),assign(5,r).       :- assign(6,r),assign(3,r).
:- assign(1,b),assign(3,b).  :- assign(2,b),assign(5,b).       :- assign(6,b),assign(3,b).
:- assign(1,g),assign(3,g).  :- assign(2,g),assign(5,g).       :- assign(6,g),assign(3,g).
:- assign(1,r),assign(4,r).  :- assign(2,r),assign(6,r).       :- assign(6,r),assign(5,r).
:- assign(1,b),assign(4,b).  :- assign(2,b),assign(6,b).       :- assign(6,b),assign(5,b).
:- assign(1,g),assign(4,g).  :- assign(2,g),assign(6,g).       :- assign(6,g),assign(5,g).
```

ASP Qorkflow: Solving



Graph Colouring: Solving

```
$ gringo graph.lp colour.lp | clasp 0
```

Graph Colouring: Solving

```
$ gringo graph.lp colour.lp | clasp 0
```

```
clasp version 2.1.0
Reading from stdin
Solving...
Answer: 1
node(1) [...] assign(6,b) assign(5,g) assign(4,b) assign(3,r) assign(2,r) assign(1,g)
Answer: 2
node(1) [...] assign(6,r) assign(5,g) assign(4,r) assign(3,b) assign(2,b) assign(1,g)
Answer: 3
node(1) [...] assign(6,g) assign(5,b) assign(4,g) assign(3,r) assign(2,r) assign(1,b)
Answer: 4
node(1) [...] assign(6,r) assign(5,b) assign(4,r) assign(3,g) assign(2,g) assign(1,b)
Answer: 5
node(1) [...] assign(6,g) assign(5,r) assign(4,g) assign(3,b) assign(2,b) assign(1,r)
Answer: 6
node(1) [...] assign(6,b) assign(5,r) assign(4,b) assign(3,g) assign(2,g) assign(1,r)
SATISFIABLE

Models      : 6
Time        : 0.002s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
```

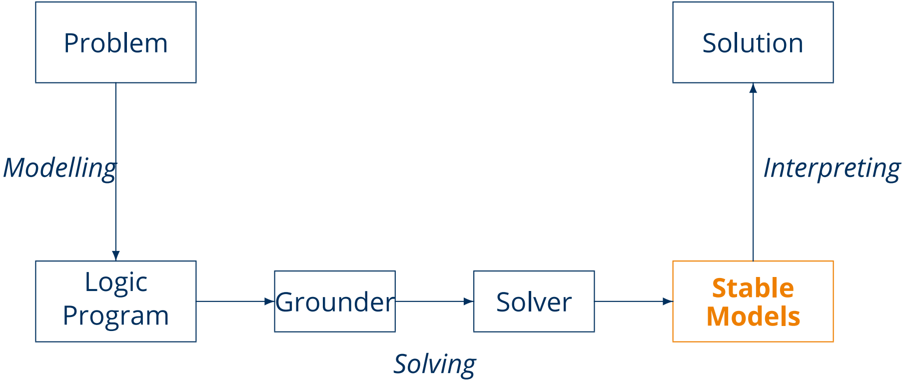
Graph Colouring: Solving

$ clingo graph.lp colour.lp 0

```
   clasp version 2.1.0
   Reading from stdin
   Solving...
   Answer: 1
node(1) [...] assign(6,b) assign(5,g) assign(4,b) assign(3,r) assign(2,r) assign(1,g)
   Answer: 2
node(1) [...] assign(6,r) assign(5,g) assign(4,r) assign(3,b) assign(2,b) assign(1,g)
   Answer: 3
node(1) [...] assign(6,g) assign(5,b) assign(4,g) assign(3,r) assign(2,r) assign(1,b)
   Answer: 4
node(1) [...] assign(6,r) assign(5,b) assign(4,r) assign(3,g) assign(2,g) assign(1,b)
   Answer: 5
node(1) [...] assign(6,g) assign(5,r) assign(4,g) assign(3,b) assign(2,b) assign(1,r)
   Answer: 6
node(1) [...] assign(6,b) assign(5,r) assign(4,b) assign(3,g) assign(2,g) assign(1,r)
   SATISFIABLE

   Models      : 6
Time      : 0.002s (Solving: 0.00s 1st Model: 0.00s Unsat: 0.00s)
```

ASP Workflow: Stable models

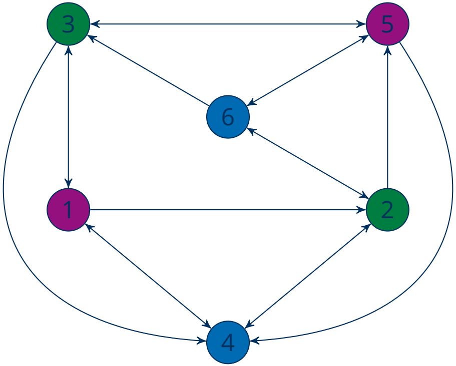


A Colouring

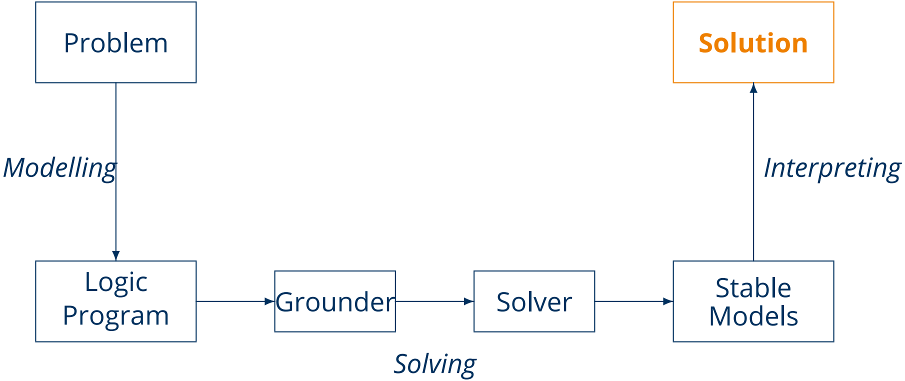
```
Answer: 6
node(1)   [...]    \
assign(6,b) assign(5,r) assign(4,b) assign(3,g) assign(2,g) assign(1,r)
```

A Colouring

```
Answer: 6
node(1)   [...]    \
assign(6,b) assign(5,r) assign(4,b) assign(3,g) assign(2,g) assign(1,r)
```



ASP Workflow: Solutions



Basic Methodology

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Generate and Test (or: Guess and Check)

Generator Generate potential stable model candidates
(typically through non-deterministic constructs)

Tester Eliminate invalid candidates
(typically through integrity constraints)

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Nutshell

Logic program = Data + Generator + Tester (+ Optimizer)

Graph Colouring

```
node(1..6).

edge(1,2). edge(1,3). edge(1,4).
edge(2,4). edge(2,5). edge(2,6).
edge(3,1). edge(3,4). edge(3,5).
edge(4,1). edge(4,2).
edge(5,3). edge(5,4). edge(5,6).
edge(6,2). edge(6,3). edge(6,5).

colour(r). colour(b). colour(g).
```

Problem instance

```
1 {assign(N,C) : colour(C) } 1 :- node(N).

:- edge(N,M), assign(N,C), assign(M,C).
```

Problem encoding

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Data

```
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Generator

```
:- edge(N,M), assign(N,C), assign(M,C).
```

Tester

Conclusion

Summary

- The language of normal logic programs can be extended by constructs:
  - **Integrity constraints** for eliminating unwanted solution candidates
  - **Choice rules** for choosing subsets of atoms
  - **Cardinality rules** for counting certain present/absent atoms
- All of them can be translated back into normal logic program rules.
- The modelling methodology of ASP is **generate and test**:
- Generate solution candidates, eliminate infeasible ones.

Suggested action points:

- Model solving Sudoku puzzles using a ternary predicate $\mathtt{num}(i, j, k)$ expressing that the field in row i and column j of the Sudoku grid contains the number k ($i, j, k \in \{1, \ldots, 9\}$). Initial hints are given by `num/3` facts.