TECHNISCHE UNIVERSITÄT DRESDEN

DATABASE THEORY

Lecture 8: Tree-Like Conjunctive Queries (2)

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Knowledge-Based Systems

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Review: Treewidth

Graphs of bounded treewidth as a generalisation of (undirected) trees:

- Trees have treewidth 1
- Graphs of higher treewidth resemble trees with "thicker branches"
- It is (in theory) not hard to check if a graph has treewidth $\leq k$ for some k
- It is (in theory) not hard to answer BCQs whose primal graph has a bounded treewidth

Practically feasible only for lower treewidths

However, bounded treewidth does not generalise the notion of hypergraph acyclicity (acyclic families of hypergraphs may have unbounded treewidth)

Is there a better notion of tree-likeness for hypergraphs?

Query Width

Idea of Chekuri and Rajamaran [1997]:

- Create tree structure similar to tree decomposition
- But consider bags of query atoms instead of bags of variables
- Two connectedness conditions:
  (1) Bags that refer to a certain variable must be connected
  (2) Bags that refer to a certain query atom must be connected

Query width: least number of atoms needed in bags of a query decomposition

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Theorem 8.1: Given a query decomposition for a BCQ, the query answering problem can be decided in time polynomial in the query width.

Problems with Query Width

Theorem 8.2 (Gottlob et al. 1999): Deciding if a query has query width at most k is NP-complete.

In particular, it is also hard to find a query decomposition

$\leadsto$ Query answering complexity drops from NP to P ...
...but we need to solve another NP-hard problem first!

Generalised Hypertree Width

Gottlob, Leone, and Scarcello had another idea on defining tree-like hypergraphs:

Intuition:

- Combine key ideas of tree decomposition and query decomposition
- Start by looking at a tree decomposition
- But define the width based on query atoms:
  How many atoms do we need to cover all variables in a bag?

$\leadsto$ Generalised hypertree width

$\leadsto$ A technical condition is needed to get a simpler-to-check notion

Hypertree Width

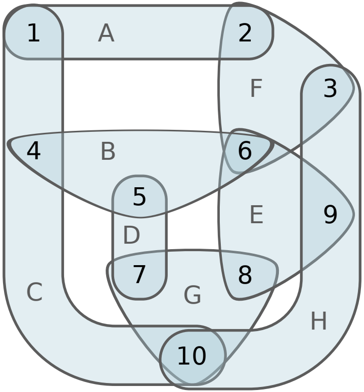
Definition 8.3: Consider a hypergraph $G = \langle V, E \rangle$. A hypertree decomposition of G is a tree structure T where each node n of T is associated with a bag of variables $B_n \subseteq V$ and with a set of edges $G_n \subseteq E$, such that:

- T with B_n yields a tree decomposition of the primal graph of G.
- For each node n of T:
  (1) the vertices used in the edges G_n are a superset of B_n,
  (2) if a vertex v occurs in an edge of G_n and this vertex also occurs in B_m for some node m below n in T, then $v \in B_n$.

The width to T is the largest number of edges in a set G_n.
The hypertree width of G, hw(G), is the least width of its hypertree decompositions.

((2) is the "special condition": without it we get the generalised hypertree width)

Hypertree Width: Example



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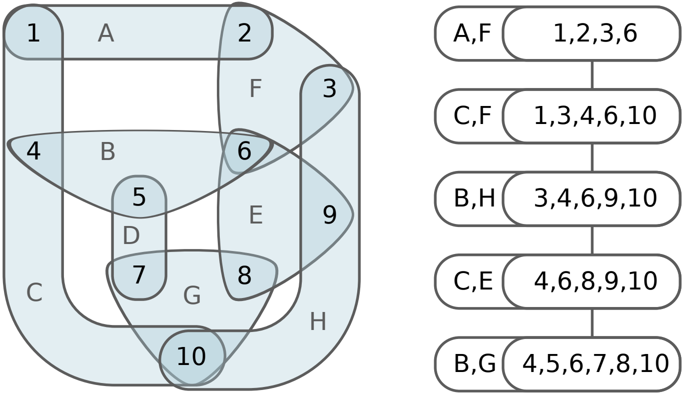

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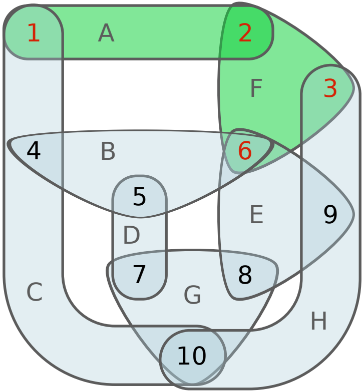

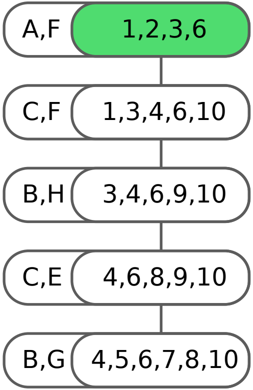

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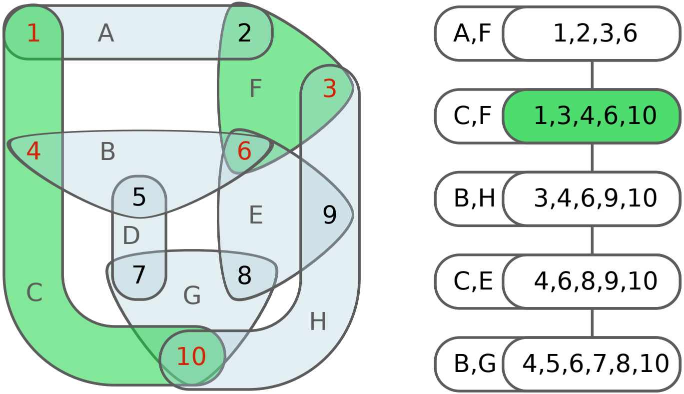

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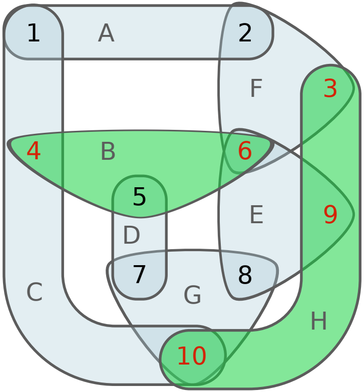

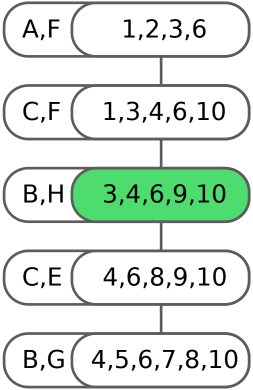

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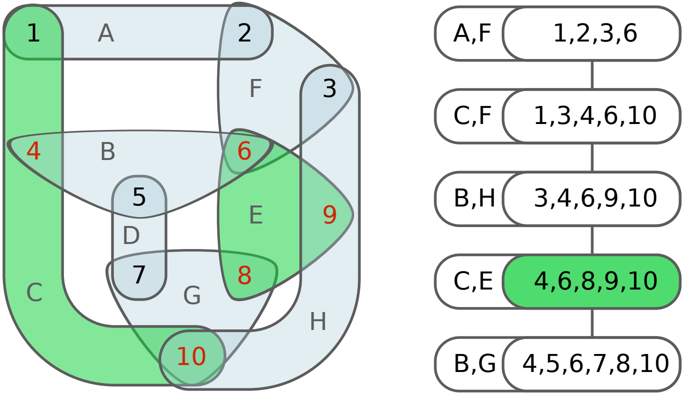

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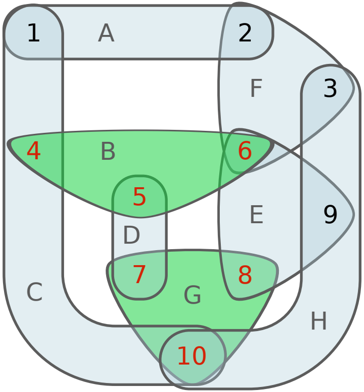

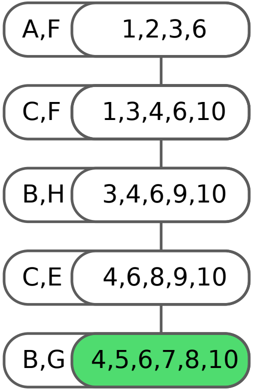

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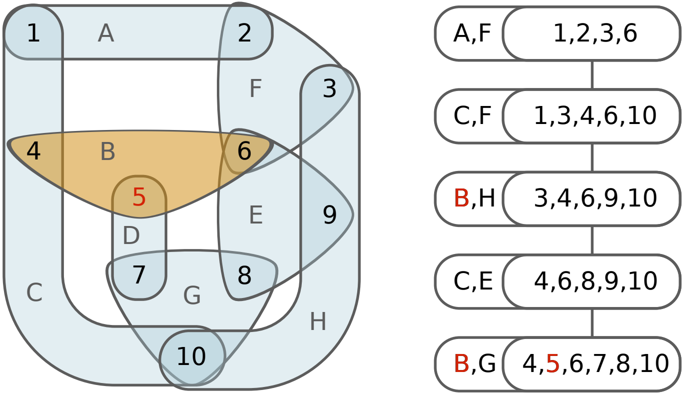

Special condition violated $\rightsquigarrow$ no hypertree decomposition

$\rightsquigarrow$ But generalised hypertree decomposition of width 2

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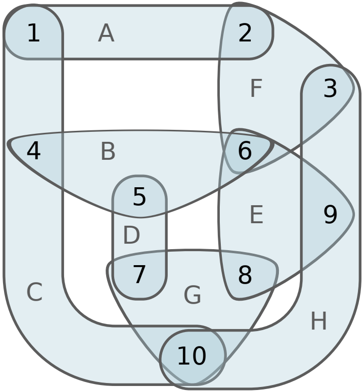

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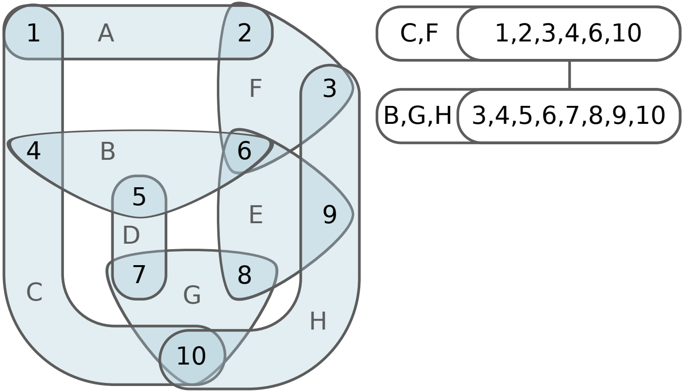

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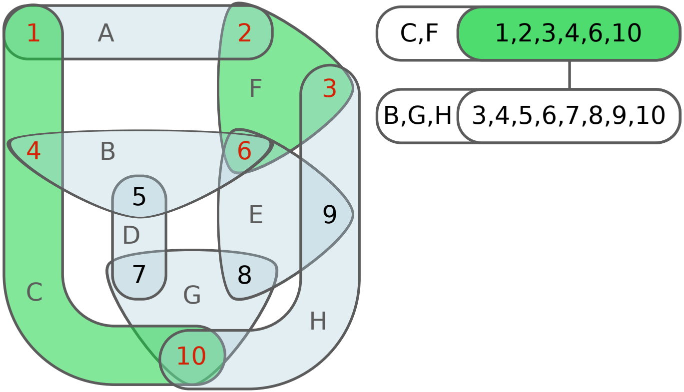

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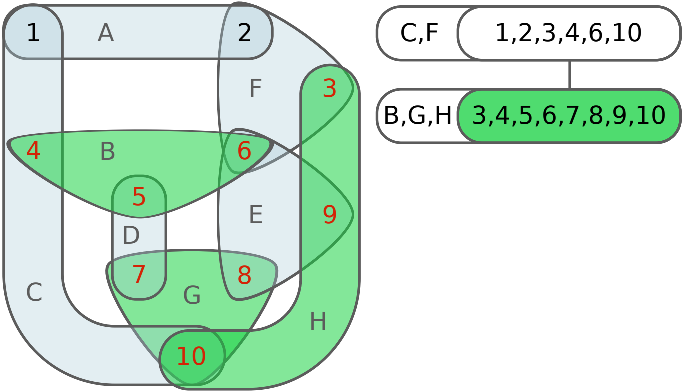

Special condition satisfied $\rightsquigarrow$ hypertree decomposition of width 3

Hypertree Width: Observations

Observation 8.4: If $\langle T, (B_n), (G_n)\rangle$ is a hypertree decomposition for a hypergraph $\langle V, E\rangle$, then the union of all sets G_n might be a proper subset of E.

Proof: Indeed, we only require that every bag B_n is "covered" by the edges in G_n, not that every edge in E is actually used for this purpose. $\square$

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Proof: Since $T, (B_n)$ is a tree decomposition of the primal graph, and every edge $e \in E$ gives rise to a $|e|$-clique in this graph, the variables of e must occur together in one bag of the tree decomposition. □

Complete Hypertree Decompositions

We can make sure that all atoms are in fact used in some set G_n of the decomposition:

Theorem 8.6: If $\langle T, (B_n), (G_n)\rangle$ is a (generalised) hypertree decomposition for a hypergraph $\langle V, E\rangle$, then there is a (generalised) hypertree decomposition $\langle T', (B'_n), (G'_n)\rangle$ of the same width and of size $O(|T| + |E|)$ such that, for all $e \in E$, there is a node n in T' with $e \in G'_n$.

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Proof: For every edge $e \in E$ that does not appear in (G_n) yet:

- extend T with a new node m that is a child of an existing node n with $e \subseteq B_n$ (this must exist as just observed)
- define $B_m = e$ and $G_m = \{e\}$

This establishes the claim for e and preserves all conditions in the definition of (generalised) hypertree decomposition. □

Such hypertree decompositions are called complete.

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This easily corresponds to a hypertree decomposition (using the same tree structure, singleton edge sets $G_n = \{e\}$ and vertex bags $B_n = e$ if n is associated with e)

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Proof: ($\Leftarrow$) For a hypergraph $\langle V, E\rangle$, consider a hypertree decomposition $\langle T, (B_n), (G_n)\rangle$ of width 1 that is complete (w.l.o.g.).

We modify the decomposition so that, for every edge $e \in E$, there is exactly one node n_e in T such that $G_{n_e} = \{e\}$ and $B_{n_e} = e$.

Modification procedure:

- Choose an arbitrary total order $\prec$ on the nodes of T such that nodes are before their child nodes (i.e., $\prec$ is a topological order wrt. T)
- For each $e \in E$:
  1. Find the $\prec$-least node n_e of T with $G_{n_e} = \{e\}$ and $B_{n_e} = e$ (exists since we have a complete decomposition of width 1)
  2. For every node $n \neq n_e$ with $G_n = \{e\}$: re-attach all children of n to n_e and delete n

Note: Since we have hypertree width 1, the set G_{n_e} in step (1) must be singleton.

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The modified hypertree decomposition corresponds to a join tree:

- each node is associated with a single edge
- no edge is associated with more than one node
- the vertices satisfy the connectedness condition for join trees (since T is a tree decomposition of the primal graph)

Hence the hypergraph has a join tree and is therefore acyclic. □

Efficient Query Answering

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We first construct a modified BCQ q', hypertree decomposition $\langle T, (B_n), (G'_n)\rangle$ of q', and a database instance $\mathcal{I}'$, such that $\mathcal{I} \models q$ iff $\mathcal{I}' \models q'$ and $\bigcup G'_n = B_n$ for all nodes n of T

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- For each node n and atom $r(\vec{x}) \in G_n$
- create a new relation r' and let $\vec{y}$ be a list of all variables in $\vec{x} \cap B_n$
- replace $r(\vec{x}) \in G_n$ by $r'(\vec{y}) \in G'_n$
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BCQ q', hypertree decomposition $\langle T, (B_n), (G'_n)\rangle$, and database instance $\mathcal{I}'$ are of size polynomial in the input.

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($\Leftarrow$) Every match of q' in $\mathcal{I}'$ is also a match of q in $\mathcal{I}$ since

- For every atom $r(\vec{x})$ of q, there is a node n of T with $\vec{x} \subseteq B_n$ (observed before)
- so $r(\vec{x})$ is an atom of q' as well

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Proof: We now construct an acyclic BCQ $\bar{q}$, database $\bar{\mathcal{I}}$, and join tree J of $\bar{q}$, such that $\mathcal{I}' \models q'$ iff $\bar{\mathcal{I}} \models \bar{q}$.

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  The overall claim now follows by applying Yannakakis' Algorithm to answer the query. $\square$

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Theorem 8.9: For a BCQ of (generalised) hypertree width k, query answering can be decided in polynomial time, and is complete for LOGCFL.

…but the degree of the polynomial time bound is greater than k

Hypertree Width via Games

There is also a game characterisation of (generalised) hypertree width.

The Marshals-and-Robber Game

- The game is played on a hypergraph
- There are k marshals, each controlling one hyperedge, and one robber located at a vertex
- Otherwise similar to cops-and-robber game
- Special condition: Marshals must shrink the space that is left for the robber in every turn!

Hypertree width $\leq k$ if and only if k marshals have a winning strategy
$\leadsto$ hypergraph is acyclic iff 1 marshal has a winning strategy

Hypertree Width via Logic

There is also a logical characterisation of hypertree width.

Loosely k-Guarded Logic

- Fragment of FO with $\exists$ and $\wedge$
- Special form for all $\exists$ subexpressions:

$$\exists x_1, \ldots, x_n.(G_1 \wedge \ldots \wedge G_k \wedge \varphi)$$

  where G_i are atoms ("guards") and every variable x_j from $x_1, \ldots, x_n$ co-occurs with any free variable of φ in one G_i.

A query has hypertree width $\leq k$ if and only if it can be expressed as a loosely k-guarded formula

$\leadsto$ tree queries correspond to loosely 1-guarded formulae

("loosely 1-guarded" logic is better known as guarded logic and widely studied)

Summary and Outlook

Besides tree queries, there are other important classes of CQs that can be answered in polynomial time:

- Bounded treewidth queries
- Bounded hypertree width queries

General idea: decompose the query in a tree structure

Other possible characterisations via games and logic

Open questions:

- What else is there besides query answering? $\leadsto$ optimisation
- Measure expressivity rather than just complexity