On Classical Decidable Logics extended with Percentage Quantifiers and Arithmetics

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Abstract

During the last decades, a lot of effort was put into identifying decidable fragments of first-order logic. Such efforts gave birth, among the others, to the two-variable fragment and the guarded fragment, depending on the type of restriction imposed on formulae from the language. Despite the success of the mentioned logics in areas like formal verification and knowledge representation, such first-order fragments are too weak to express even the simplest statistical constraints, required for modelling of influence networks or in statistical reasoning.

In this work we investigate the extensions of these classical decidable logics with percentage quantifiers, specifying how frequently a formula is satisfied in the indented model. We show, surprisingly, that all the mentioned decidable fragments become undecidable under such extension, sharpening the existing results in the literature. Our negative results are supplemented by decidability of the two-variable guarded fragment with even more expressive counting, namely Presburger constraints. Our results can be applied to infer decidability of various modal and description logics, *e.g.* Presburger Modal Logics with Converse or $\mathcal{ALCI}$, with expressive cardinality constraints.

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1 Introduction

Since the works of Church, Turing and Trakhtenbrot, it is well-known that the (finite) satisfiability and validity problems for the First-Order Logic (FO) are undecidable [32]. Such results motivated researchers to study restricted classes of FO that come with decidable satisfiability problem, such as the prefix classes [9], fragments with fixed number of variables [30], restricted forms of quantification [1, 33] and the restricted use of negation [6]. These fragments have found many applications in the areas of knowledge representation, automated reasoning and program verification, just to name a few. To the best of our knowledge, none

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of the known decidable logics incorporate a feature that allows for stating even a very modest statistical property. For example, one may want to state that "to qualify to be a major, one must have at least 51% of the total votes", which may be useful to formalise, *e.g.* the voting systems.

Our results. In this paper, we revisit the satisfiability problem for some of the most prominent fragments of FO, namely the two-variable fragment FO^2 and the guarded fragment GF. We extend them with the so-called *percentage* quantifiers, in two versions: *local* and *global*. Global percentage quantifiers are quantifiers of the form $\exists^{=q\%}x\ \varphi(x)$, which states that the formula $\varphi(x)$ holds on exactly $q\%$ of the domain elements. Their local counterparts are quantifiers of the form $\exists_R^{=q\%}y\ \varphi(x,y)$, which intuitively means that exactly $q\%$ of the R-successors of an element x satisfy φ.

In this paper, we show that both FO^2 and GF become undecidable when extended with percentage quantifiers of any type. In fact, the undecidability of GF already holds for its three variable fragment GF^3. Our results strengthen the existing undecidability proofs of $\mathcal{ALCISCC}^{++}$ from [4] and of FO^2 with equicardinality statements (implemented via the Härtig quantifier) from [19] and contrast with the decidability of FO^2 with counting quantifiers (C^2) [18, 25, 28] and modulo and ultimately-periodic counting quantifiers [8].

Additionally, we show that the decidability status of GF can be regained if we consider GF^2, *i.e.* the intersection of GF and FO^2, which is still a relevant fragment of FO that captures standard description logics up to $\mathcal{ALCIH}b^{\mathsf{self}}$ [5, 15]. We in fact show a stronger result here: GF^2 remains decidable when extended with *local Presburger quantifiers*, which are essentially Presburger constraints on the neighbouring elements, *e.g.* we can say that the number of red outgoing edges plus twice the number of blue outgoing edges is at least three times as many as the number of green incoming edges.

We stress here that the semantics of global percentage quantifiers makes sense only over finite domains and hence, we study the satisfiability problem over finite models only. Similarly, the semantics of local percentage quantifiers only makes sense if the models are finitely-branching. While we stick again to the finite structures, our results on local percentage quantifiers also can be transferred to the case of (possibly infinite) finitely-branching structures.

Related works. Some restricted fragments of GF^2 extended with arithmetics, namely the (multi) modal logics, were already studied in the literature [11, 21, 2, 4], where the decidability results for their finite and unrestricted satisfiability were obtained. However, the logics considered there do not allow the use of the inverse of relations. Since GF^2 captures the extensions of all the aforementioned logics with the inverse relations, our decidability results subsume those in [11, 21, 2, 4]. We note that prior to our paper, it was an open question whether any of these decidability results still hold when inverse relations are allowed [4]. In our approach, despite the obvious difference in expressive power, we show that GF^2 with Presburger quantifiers can be encoded directly into the two-variable logic with counting quantifiers [18, 25, 28], which we believe is relatively simple and avoids cumbersome reductions of the satisfiability problem into integer programming.

2 Preliminaries

We employ the standard terminology from finite model theory [22]. We refer to structures/-models with calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{M}$ and to their universes with the corresponding capital letters A, B, M. We work only on structures with *finite* universes over purely relational (*i.e.* constant- and function-free) signatures of arity ≤ 2 containing the equality predicate $=$. We usually use $a, b, \ldots$ to denote elements of structures, $\bar{a}, \bar{b}, \ldots$ for tuples of elements, $x, y, \ldots$ for variables and $\bar{x}, \bar{y}, \ldots$ for tuples of variables (all of these possibly with some decorations). We write $\varphi(\bar{x})$ to indicate that all free variables of φ are in $\bar{x}$. We write $\mathcal{M}, x/a \models \varphi(x)$ to denote that $\varphi(x)$ holds in the structure $\mathcal{M}$ when the free variable x is assigned with element a. Its generalization to arbitrary number of free variables is defined similarly. The (finite) satisfiability problem is to decide whether an input formula has a (finite) model.

2.1 Percentage quantifiers

For a formula $\varphi(x)$ with a single free-variable x, we write $|\varphi(x)|_{\mathcal{M}}$ to denote the total number of elements of $\mathcal{M}$ satisfying $\varphi(x)$. Likewise, for an element $a \in M$ and a formula $\varphi(x, y)$ with free variables x and y, we write $|\varphi(x, y)|_{\mathcal{M}}^{x/a}$ to denote the total number of elements $b \in M$ such that (a, b) satisfies $\varphi(x, y)$.

The *percentage quantifiers* are quantifiers of the form $\exists^{=q\%} x\, \varphi(x, y)$, where q is a rational number between 0 and 100, stating that exactly $q\%$ of domain elements satisfy $\varphi(x, y)$ with y known upfront. Formally:

$$\mathcal{M}, y/a \models \exists^{=q\%} x\, \varphi(x, y) \qquad \text{iff} \qquad |\varphi(x, y)|_{\mathcal{M}}^{y/a} = \frac{q}{100} \cdot |M|.$$

Percentage quantifiers for other thresholds (*e.g.* for $<$) are defined analogously. We stress here that the above quantifiers count *globally*, *i.e.* they take the whole universe of $\mathcal{M}$ into account. This motivates us to define their local counterpart, as follows: for a binary[1] relation R and a rational q between 0 and 100, we define the quantifier $\exists_R^{=q\%} y\, \varphi(x, y)$, which evaluates to true whenever exactly $q\%$ of R-successors y of x satisfy $\varphi(x, y)$. Formally,

$$\mathcal{M}, x/a \ \models\ \exists_R^{=q\%} y\, \varphi(x, y) \qquad \text{iff} \qquad |R(x, y) \wedge \varphi(x, y)|_{\mathcal{M}}^{x/a} = \frac{q}{100} \cdot |R(x, y)|_{\mathcal{M}}^{x/a}.$$

We define the percentage quantifiers w.r.t. R^- (*i.e.* the inverse of R) and for other thresholds analogously.

2.2 Local Presburger quantifiers

The *local Presburger quantifiers* are expressions of the following form:

$$\sum_{i=1}^{n} \lambda_i \cdot \#_y^{r_i}[\varphi_i(x, y)] \quad \circledast \quad \delta$$

where λ_i, δ are integers; r_i is either R or R^- for some binary relation R; $\varphi_i(x, y)$ is a formula with free variables x and y; and $\circledast$ is one of $=, \neq, \leq, \geq, <, >, \equiv_d$ or $\not\equiv_d$, where $d \in \mathbb{N}_+$. Here $\equiv_d$ denotes the congruence modulo d. Note that the above formula has one free variable x.

Intuitively, the expression $\#_y^{r_i}[\varphi_i(x, y)]$ denotes the number of y's that satisfy $r_i(x, y) \wedge \varphi_i(x, y)$ and evaluates to true on x, if the (in)equality $\circledast$ holds. Formally,

$$\mathcal{M}, x/a \quad \models \quad \sum_{i=1}^{n} \lambda_i \cdot \#_y^{r_i}[\varphi_i(x, y)] \ \circledast\ \delta \quad \text{iff} \quad \sum_{i=1}^{n} \lambda_i \cdot |r_i(x, y) \wedge \varphi_i(x, y)|_{\mathcal{M}}^{x/a} \quad \circledast \quad \delta$$

Note that local percentage quantifiers can be expressed with Presburger quantifiers, *e.g.* $\exists_R^{50\%} y \varphi(x, y)$ can be expressed as local Presburger quantifier: $\#_y^R[\varphi(x, y)] - \frac{1}{2}\#_y^R[\top] = 0$.

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[1] Local percentage quantifiers for predicates of arity higher than two can also be defined but we will never use them. Hence, for simplicity, we define such quantifiers only for binary relations.

2.3 Logics

In this paper we mostly consider two fragments of first-order logic, namely *the two-variable fragment* FO^2 and *the guarded fragment* GF. The former logic is a fragment of FO in which we can only use the variables x and y. By allowing local and global percentage quantifiers in addition to the standard universal and existential quantifiers, we obtain the logics $\mathrm{FO}^2_{loc\%}$ and $\mathrm{FO}^2_{gl\%}$. The latter logic is defined by relativising quantifiers with relations. More formally, GF is the smallest set of first-order formulae such that the following holds.

- GF contains all atomic formulae $R(\bar{x})$ and equalities between variables.
- GF is closed under boolean connectives.
- If $\psi(\bar{x}, \bar{y})$ is in GF and $\gamma(\bar{x}, \bar{y})$ is a relational atom containing all free variables of ψ, then both $\forall \bar{y}\ \gamma(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y})$ and $\exists \bar{y}\ \gamma(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{y})$ are in GF.

By allowing global percentage quantifiers additionally in place of existential ones, we obtain the logic $\mathrm{GF}_{gl\%}$. We obtain the logic $\mathrm{GF}_{loc\%}$ by extending GF's definition with the rule:[2]

- $\exists_R^{=q\%} y\ \varphi(x, y)$ is in $\mathrm{GF}_{loc\%}$ iff $\varphi(x, y)$ in GF with free variables x, y.

Similarly, we obtain GF_{pres} by extending GF's definition with the rule:

- $\sum_{i=1}^{n} \lambda_i \cdot \#_y^{r_i}[\varphi_i(x, y)] \circledast \delta$ is in GF_{pres} iff $\varphi_i(x, y)$ are in GF with free variables x, y.

Finally, we use $\mathrm{GF}^k_{gl\%}$, $\mathrm{GF}^k_{loc\%}$ and GF^k_{pres} to denote the k-variable fragments of the mentioned logics. Specifically, we use $\mathrm{GF}^2_{gl\%}$, $\mathrm{GF}^2_{loc\%}$ and GF^2_{pres} for the two-variable fragments.

2.4 Semi-linear sets

Since we will exploit the semi-linear characterization of Presburger constraints, we introduce some terminology. The term *vector* always means *row vectors*. For vectors $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_k \in \mathbb{N}^\ell$, we write $L(\bar{v}_0; \bar{v}_1, \ldots, \bar{v}_k)$ to denote the set:

$$L(\bar{v}_0; \bar{v}_1, \ldots, \bar{v}_k) \quad := \quad \left\{ \ \bar{u} \in \mathbb{N}^\ell \ \middle| \ \bar{u} = \bar{v}_0 + \textstyle\sum_{i=1}^{k} n_i \bar{v}_i \text{ for some } n_1, \ldots, n_k \in \mathbb{N} \ \right\}$$

A set $S \subseteq \mathbb{N}^\ell$ is a *linear* set, if $S = L(\bar{v}_0; \bar{v}_1, \ldots, \bar{v}_k)$, for some $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_k \in \mathbb{N}^\ell$. In this case, the vector $\bar{v}_0$ is called the *offset* vector of S, and $\bar{v}_1, \ldots, \bar{v}_k$ are called the *period* vectors of S. We denote by offset(S) the offset vector of S, *i.e.* $\bar{v}_0$ and prd(S) the set of period vectors of S, *i.e.* $\{\bar{v}_1, \ldots, \bar{v}_k\}$. A *semilinear* set is a finite union of linear sets.

The following theorem is a well-known result by Ginsburg and Spanier [13] which states that every set $S \subseteq \mathbb{N}^\ell$ definable by Presburger formula is a semilinear set. See [13] for the formal definition of Presburger formula.

▶ **Theorem 1** ([13]). *For every Presburger formula $\varphi(x_1, \ldots, x_\ell)$ with free variables $x_1, \ldots, x_\ell$, the set $\{\bar{u} \in \mathbb{N}^\ell \mid \varphi(\bar{u}) \text{ holds in } \mathbb{N}\}$ is semilinear. Moreover, given the formula $\varphi(x_1, \ldots, x_\ell)$, one can effectively compute a set of tuples of vectors $\{(\bar{v}_{1,0}, \ldots, \bar{v}_{1,k_1}), \ldots, (\bar{v}_{p,0}, \bar{v}_{p,1}, \ldots, \bar{v}_{p,k_p})\}$ such that $\{\bar{u} \in \mathbb{N}^\ell \mid \varphi(\bar{u}) \text{ holds in } \mathbb{N}\}$ is equal to $\bigcup_{i=1}^{p} L(\bar{v}_{i,0}; \bar{v}_{i,1}, \ldots, \bar{v}_{i,k_i})$.*

2.5 Types and neighbourhoods

A 1-*type* over a signature Σ is a maximally consistent set of unary predicates from Σ or their negations, where each atom uses only one variable x. Similarly, a 2-*type* over Σ is a

[2] Note that R in the subscript of a quantifier serves the role of a "guard".

maximally consistent set of binary predicates from Σ or their negations containing the atom $x \neq y$, where each atom or its negation uses two variables x and y.[3]

Note that 1-types and 2-types can be viewed as quantifier-free formulae that are the conjunction of their elements. We will use the symbols π and η (possibly indexed) to denote 1-type and 2-type, respectively. When viewed as formula, we write $\pi(x)$ and $\eta(x, y)$, respectively. We write $\pi(y)$ to denote formula $\pi(x)$ with x being substituted with y. The 2-type that contains only the negations of atomic predicates is called the *null* type, denoted by η_{null}. Otherwise, it is called a *non-null* type.

For a Σ-structure $\mathcal{M}$, the *type of an element* $a \in M$ is the unique 1-type π that a satisfies in $\mathcal{M}$. Similarly, the type of a pair $(a, b) \in M \times M$, where $a \neq b$, is the unique 2-type that (a, b) satisfies in $\mathcal{M}$. For an element $a \in M$, the *η-neighbourhood* of a, denoted by $\mathcal{N}_{\mathcal{M},\eta}(a)$, is the set of elements b such that η is the 2-type of (a, b). Formally,

$$\mathcal{N}_{\mathcal{M},\eta}(a) \ := \ \left\{ \ b \in M \ \middle| \ \mathcal{M}, x/a, y/b \models \eta(x, y) \ \right\}.$$

The *η-degree* of a, denoted by $\deg_{\mathcal{M},\eta}(a)$, is the cardinality of $\mathcal{N}_{\mathcal{M},\eta}(a)$.

Let $\eta_1, \ldots, \eta_\ell$ be an enumeration of all the non-null types. The *degree of a in $\mathcal{M}$* is defined as the vector $\deg_{\mathcal{M}}(a) \ := \ (\deg_{\mathcal{M},\eta_1}(a), \cdots, \deg_{\mathcal{M},\eta_\ell}(a))$. Intuitively, $\deg_{\mathcal{M}}(a)$ counts the number of elements adjacent to a with non-null type. We note that our logic can be easily extended with atomic predicates of the form of a linear constraint C over the variables $\deg_\eta(x)$'s or $\deg(x) \in S$, where S is a semilinear set. Semantically, $\mathcal{M}, x/a \models C$ iff the linear constraint C evaluates to true when each $\deg_\eta(x)$ is substituted with $\deg_{\mathcal{M},\eta}(a)$ and $\mathcal{M}, x/a \models \deg(x) \in S$ iff $\deg_{\mathcal{M}}(a) \in S$. We stress that these atomic predicates will only be used to facilitate the proof of our decidability result.

3 Negative results

In this section we turn our attention to the negative results announced in the introduction.

3.1 Two-Variable Fragment

We start by proving that the two-variable fragment of FO extended with percentage quantification has undecidable finite satisfiability problem. Actually, in our proof, we will only use the $\exists^{=50\%}$ quantifier. Our results strengthen the existing undecidability proofs of $\mathcal{ALCISCC}^{++}$ from [4] and of FO^2 with equicardinality statements (implemented via the Härtig quantifier) from [19]. Roughly speaking, our counting mechanism is weaker: we cannot write arbitrary Presburger constraints (as it is done in [4]) nor compare sizes of any two sets (as it is done in [19]). Nevertheless, we will see that in our framework we can express "functionality" of a binary relation and "compare" cardinalities of sets, but under some technical assumptions of dividing the intended models into halves. Due to such technicality, we cannot simply encode the undecidability proofs of [4, 19] and we need to prepare our proof "from scratch".

Our proof relies on encoding of Hilbert's tenth problem, whose simplified version is introduced below. In the classical version of *Hilbert's tenth problem* we ask whether a *diophantine equation*, *i.e.* a polynomial equation with integer coefficients, has a solution over $\mathbb{N}$. It is

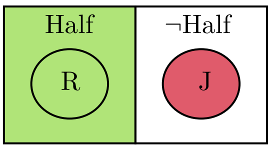
[3] We should remark here that the standard definition of 2-type, such as in [17, 28], a 2-type also contains unary predicates or its negation involving variable x or y. However, for our purpose, it is more convenient to define a 2-type as consisting of only binary predicates that strictly use both variables x and y. Note also that we view a binary predicate such as $R(x, x)$ as a unary predicate.

well-known that such problem is undecidable [23]. By employing some routine transformations (*e.g.* by rearranging terms with negative coefficients, by replacing exponentiation by multiplication and by introducing fresh variable for partial results of multiplications or addition), one can reduce any diophantine equation to an equi-solvable system of equations, where the only allowed operations are addition or multiplication of two variables or assigning the value one to some of them. We refer to the problem of checking solvability (over $\mathbb{N}$) of such systems of equations as SHTP (*simpler Hilbert's tenth problem*) and present its precise definition next. Note that, by the described reduction, SHTP is undecidable.

▶ **Definition 2** (SHTP). *An input of* SHTP *is a system of equations ε, where each of its entries ε_i is in one of the following forms: (i) $u_i = 1$, (ii) $u_i = v_i + w_i$, (iii) $u_i = v_i \cdot w_i$, where u_i, v_i, w_i are pairwise distinct* variables *from some countably infinite set* Var*. In* SHTP *we ask whether an input system of equations ε, as described before, has a solution over $\mathbb{N}$.*

3.1.1 Playing with percentage quantifiers

Before reducing SHTP to $\mathrm{FO}^2_{gl\%}$, let us gain more intuitions of $\mathrm{FO}^2_{gl\%}$ and introduce a useful trick employing percentage quantifiers to express equi-cardinality statements. Let $\mathcal{M}$ be a finite structure and let $\mathrm{Half}, \mathrm{R}, \mathrm{J}$ be unary predicates. We say that $\mathcal{M}$ is (*Half, R, J*)*-separated* whenever it satisfies the following conditions: (a) exactly half of the domain elements from $\mathcal{M}$ satisfy Half (b) the satisfaction of R implies the satisfaction of Half (c) the satisfaction of J implies the non-satisfaction of Half. Roughly speaking, the above conditions entail that the elements satisfying R and those satisfying J are in different halves of the model. We show that under these assumptions one can enforce the equality $|\mathrm{R}(x)|_{\mathcal{M}} = |\mathrm{J}(x)|_{\mathcal{M}}$. Indeed, such a property can be expressed in $\mathrm{FO}^2_{gl\%}$ with the following formula $\varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J})$:

$\mathcal{A} :=$ 

$\models \varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J}) := \exists^{=50\%} x\ (\mathrm{Half}(x) \wedge \neg \mathrm{R}(x)) \vee \mathrm{J}(x)$

For intuitions on $\varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J})$, consult the above picture. We basically take all the elements satisfying Half (so exactly half of the domain elements, indicated by the green area). Next, we discard the elements labelled with R (so we get the green area without the circle inside) and replace them with the elements satisfying J (the red circle, note that $\mathrm{J}^{\mathcal{A}}$ and $\mathrm{R}^{\mathcal{A}}$ are disjoint!). The total number of selected elements is equal to half of the domain, thus $|\mathrm{J}^{\mathcal{M}}| = |\mathrm{R}^{\mathcal{M}}|$. The following fact is a direct consequence of the semantics of $\mathrm{FO}^2_{gl\%}$.

▶ **Fact 1.** *For* (*Half, R, J*)*-separated $\mathcal{M}$ we have $\mathcal{M} \models \varphi_{eq}(Half, R, J)$ iff $|R(x)|_{\mathcal{M}} = |J(x)|_{\mathcal{M}}$.*

3.1.2 Undecidability proof

Until the end of this section, let us fix ε, a valid input of SHTP. By $\mathrm{Var}(\varepsilon) = \{u, v, w, \ldots\}$ we denote the set of all variables appearing in ε, and with $|\varepsilon|$ we denote the total number of entries in ε. Let $\mathcal{M}$ be a finite structure.

The main idea of the encoding is fairly simple: in the intended model $\mathcal{M}$ some elements will be labelled with A_u predicates, ranging over variables $u \in \mathrm{Var}(\varepsilon)$, and the number of such elements will indicate the value of u in an example solution to ε. The only tricky part here is to encode multiplication of variables. Once ε contains an entry $w = u \cdot v$, we need to ensure that $|A_w(x)|_{\mathcal{M}} = |A_u(x)|_{\mathcal{M}} \cdot |A_v(x)|_{\mathcal{M}}$ holds. It is achieved by linking, via a binary relation $\mathrm{Mult}^{\mathcal{M}}$, each element from $A_u^{\mathcal{M}}$ with exactly $|A_v(x)|_{\mathcal{M}}$ elements satisfying A_w, which relies on imposing equicardinality statements. To ensure that the performed multiplication is correct, each element labelled with $A_w^{\mathcal{M}}$ has exactly one predecessor from $A_u^{\mathcal{M}}$ and hence the relation $\mathrm{Mult}^{\mathcal{M}}$ is backward-functional.

We start with a formula inducing a labelling of elements with variable predicates and ensuring that all elements of $\mathcal{M}$ satisfy at most one variable predicate. Note that it can happen that there will be auxiliary elements that are not labelled with any of the variable predicates.

($\varphi^{\varepsilon}_{\mathsf{var}}$) $\forall x \bigwedge_{u \neq v \in \mathrm{Var}(\varepsilon)} \neg(A_u(x) \wedge A_v(x))$.

We now focus on encoding the entries of ε. For an entry ε_i of the form $u_i = 1$ we write:

($\varphi_{u_i=1}$) $\exists x\ A_{u_i}(x) \wedge \forall x \forall y\ (A_{u_i}(x) \wedge A_{u_i}(y)) \rightarrow x = y$

▶ **Fact 2.** *$\mathcal{M} \models \varphi_{u_i=1}$ holds iff there is exactly one element in $\mathcal{M}$ satisfying $A_u(x)$.*

To deal with entries ε_i of the form $w_i = u_i + v_i$ or $w_i = u_i \cdot v_i$ we need to "prepare an area" for the encoding, similarly to Section 3.1.1. First, we cover domain elements of $\mathcal{M}$ by *layers*. The i-th layer is divided into halves with $\mathrm{FHalf}^{[i]}$ and $\mathrm{SHalf}^{[i]}$ predicates with:

($\varphi^{i}_{\mathsf{halves}}$) $\forall x \left(\mathrm{FHalf}^{[i]}(x) \leftrightarrow \neg \mathrm{SHalf}^{[i]}(x)\right) \wedge \exists^{=50\%} x\ \mathrm{FHalf}^{[i]}(x)$

▶ **Fact 3.** *$\mathcal{M} \models \varphi^{i}_{halves}$ holds iff exactly half of the domain elements from $\mathcal{M}$ are labelled with $FHalf^{[i]}$ and the other half of elements are labelled with $SHalf^{[i]}$.*

Second, we need to ensure that in the i-th layer of $\mathcal{M}$, the elements satisfying A_{u_i} or A_{v_i} are in the first half, whereas elements satisfying A_{w_i} are in the second half. We do it with:

($\varphi^{i}_{\mathsf{parti}}(u_i, v_i, w_i)$) $\forall x \left([(A_{u_i}(x) \vee A_{v_i}(x)) \rightarrow \mathrm{FHalf}^{[i]}(x)] \wedge [A_{w_i}(x) \rightarrow \mathrm{SHalf}^{[i]}(x)]\right)$

▶ **Fact 4.** *$\mathcal{M} \models \varphi^{i}_{parti}(u_i, v_i, w_i)$ holds iff for all elements $a \in M$, if a satisfies $A_{u_i}(x) \vee A_{v_i}(x)$ then a also satisfies $FHalf^{[i]}(x)$ and if a satisfies $A_{w_i}(x)$ then a also satisfies $SHalf^{[i]}(x)$.*

Gathering the presented formulae, we call a structure $\mathcal{M}$ *well-prepared*, if it satisfies the conjunction of all previous formulae over $1 \leq i \leq |\varepsilon|$ and over all entries ε_i from the system ε. The forthcoming encodings will be given under the assumption of *well-preparedness*.

Now, for the encoding of addition, assume that ε_i is of the form $u_i + v_i = w_i$. Thus in our encoding, we would like to express that $|A_{u_i}(x)|_{\mathcal{M}} + |A_{v_i}(x)|_{\mathcal{M}} = |A_{w_i}(x)|_{\mathcal{M}}$, which is clearly equivalent to $|A_{w_i}(x)|_{\mathcal{M}} - |A_{u_i}(x)|_{\mathcal{M}} - |A_{v_i}(x)|_{\mathcal{M}} = 0$ and also to $|A_{w_i}(x)|_{\mathcal{M}} + |\mathrm{FHalf}^{[i]}(x)|_{\mathcal{M}} - |A_{u_i}(x)|_{\mathcal{M}} - |A_{v_i}(x)|_{\mathcal{M}} = |\mathrm{FHalf}^{[i]}(x)|_{\mathcal{M}}$. Knowing that exactly 50% of domain elements of an intended model satisfy $\mathrm{FHalf}^{[i]}$ and that A_{u_i}, A_{v_i} and A_{w_i} label disjoint parts of the model, we can write the obtained equation as an $\mathrm{FO}^2_{gl\%}$ formula:

($\varphi^{i}_{\mathsf{add}}(u_i, v_i, w_i)$) $\exists^{=50\%} x \left(A_{w_i}(x) \vee (\mathrm{FHalf}^{[i]}(x) \wedge \neg A_{u_i}(x) \wedge \neg A_{v_i}(x))\right)$

Note that the above formula is exactly the $\varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J})$ formula from Section 3.1.1, with $\mathrm{Half} = \mathrm{FHalf}^{[i]}(x)$, $\mathrm{J} = A_{w_i}$ and R defined as a union of A_{u_i} and A_{v_i}. Hence, we conclude:

▶ **Lemma 3.** *A well-prepared $\mathcal{M}$ satisfies $\varphi^{i}_{add}(u_i, v_i, w_i)$ iff $|A_{u_i}(x)|_{\mathcal{M}} + |A_{v_i}(x)|_{\mathcal{M}} = |A_{w_i}(x)|_{\mathcal{M}}$.*

The only missing part is the encoding of multiplication. Take ε_i of the form $u_i \cdot v_i = w_i$. As already described in the overview, our definition of multiplication requires three steps:

(link) A binary relation $\mathrm{Mult}_i^{\mathcal{M}}$ links each element from $A_{w_i}^{\mathcal{M}}$ to some element from $A_{u_i}^{\mathcal{M}}$.

(count) Each element from M satisfying $A_{u_i}(x)$ has exactly $|A_{v_i}(x)|_{\mathcal{M}}$ $\mathrm{Mult}_i{}^{\mathcal{M}}$-successors.
(bfunc) The binary relation $\mathrm{Mult}_i{}^{\mathcal{M}}$ is backward-functional.

Such properties can be expressed with the help of $\exists^{=50\%}$ quantifier, as presented below:

$(\varphi^i_{\mathsf{link}}(u_i, w_i))$ $\forall y\ A_{w_i}(y) \to \exists x\ \mathrm{Mult}_i(x, y) \wedge \forall x \forall y\ \mathrm{Mult}_i(x, y) \to (A_{u_i}(x) \wedge A_{w_i}(y))$
$(\varphi^i_{\mathsf{count}}(u_i, v_i, w_i))$ $\forall x\ A_{u_i}(x) \to \exists^{=50\%} y \left([\mathrm{SHalf}^{[i]}(y) \wedge \neg \mathrm{Mult}_i(x, y)] \vee A_{v_i}(y) \right)$
$(\varphi^i_{\mathsf{bfunc}}(u_i, v_i, w_i))$ $\forall x\ A_{w_i}(x) \to \exists^{=50\%} y \left([\mathrm{SHalf}^{[i]}(y) \wedge x \neq y] \vee \mathrm{Mult}_i(y, x) \right)$

While the first formula, namely $\varphi^i_{\mathrm{link}}(u_i, w_i)$, is immediate to write, the next two are more involved. A careful reader can notice that they are actually instances of $\varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J})$ formula from Section 3.1.1. In the case of $\varphi^i_{\mathrm{count}}(u_i, v_i, w_i)$ we have $\mathrm{Half} = \mathrm{SHalf}^{[i]}$, $\mathrm{J} = A_{v_i}$ and the Mult_i-successors of x play the role of elements labelled by R. For the last formula one can see that we remove exactly one element from $\mathrm{SHalf}^{[i]}$ (y that is equal to x) and we replace it with the Mult_i-predecessors of x, which implies that there is the unique such predecessor. We summarise the mentioned facts as follows:

▶ **Lemma 4.** *Let $\mathcal{M}$ be a well-prepared structure satisfying $\varphi^i_{link}(u_i, w_i)$. We have that (i) $\mathcal{M}$ satisfies $\varphi^i_{count}(u_i, v_i, w_i)$ iff every $a \in M$ satisfying A_{u_i} is connected via $Mult_i$ to exactly $|A_{v_i}|$ elements satisfying A_{w_i} and (ii) $\mathcal{M}$ satisfies $\varphi^i_{bfunc}(u_i, v_i, w_i)$ iff the binary relation $Mult_i{}^{\mathcal{M}}$ linking elements satisfying $A_{u_i}(x)$ with those satisfying $A_{w_i}(x)$ is backward-functional.*

Putting the last three properties together, we encode multiplication as their conjunction:

$(\varphi^i_{\mathsf{mult}}(u_i, v_i, w_i))$ $\quad \varphi^i_{\mathrm{link}}(u_i, v_i, w_i) \wedge \varphi^i_{\mathrm{count}}(u_i, v_i, w_i) \wedge \varphi^i_{\mathrm{bfunc}}(u_i, v_i, w_i)$

▶ **Lemma 5.** *If a well-prepared $\mathcal{M}$ satisfies $\varphi^i_{mult}(u_i, v_i, w_i)$, then $|A_{u_i}(x)|_{\mathcal{M}} \cdot |A_{v_i}(x)|_{\mathcal{M}} = |A_{w_i}(x)|_{\mathcal{M}}$.*

Let $\varphi^{\varepsilon}_{\mathrm{red}}$ be $\varphi^{\varepsilon}_{\mathrm{var}}$ supplemented with a conjunction of formulae $\varphi^{\varepsilon_i}_{\mathrm{entry}}$, where $\varphi^{\varepsilon_i}_{\mathrm{entry}}$ is respectively: (i) $\varphi_{u_i=1}$ if ε_i is equal to $u_i{=}1$, (ii) $\varphi^i_{\mathrm{halves}} \wedge \varphi^i_{\mathrm{parti}}(u_i, v_i, w_i) \wedge \varphi^i_{\mathrm{add}}(u_i, v_i, w_i)$ for ε_i of the form $u_i + v_i = w_i$ and (iii) $\varphi^i_{\mathrm{halves}} \wedge \varphi^i_{\mathrm{parti}}(u_i, v_i, w_i) \wedge \varphi^i_{\mathrm{mult}}(u_i, v_i, w_i)$ for ε_i of the form $u_i \cdot v_i = w_i$. As the last piece in the proof we show that each solution of the system ε corresponds to some model of $\varphi^{\varepsilon}_{\mathrm{red}}$. Its proof is routine and relies on the correctness of all previously announced facts (consult [7, Appendix B] for more details). Hence, by the undecidability of SHTP, we immediately conclude:

▶ **Theorem 6.** *The finite satisfiability problem for $\mathrm{FO}^2_{gl\%}$ is undecidable, even when the only percentage quantifier allowed is $\exists^{=50\%}$.*

Note that in our proof above, all the presented formulas can be easily transformed to formulae under the local semantics of percentage quantifiers as follows. First, we introduce a fresh binary symbol U and enforce it to be interpreted as the universal relation with $\forall x \forall y\ U(x, y)$. Then, we replace every occurrence of $\exists^{=50\%} x\ \varphi$ by $\exists^{=50\%}_{U} x\ \varphi$. Obviously, the resulting formula is FO^2 formula with local percentage quantifiers. Thus we conclude:

▶ **Corollary 7.** *The finite satisfiability problem for $\mathrm{FO}^2_{loc\%}$ is undecidable.*

3.2 Guarded Fragment

We now focus on the second seminal fragment of FO considered in this paper, namely on the guarded-fragment GF. We start from the global semantics of percentage quantifiers. Consider a unary predicate H, whose interpretation is constrained to label exactly half of the domain with $\exists^{=50\%} x\ \mathrm{H}(x)$. We then employ the formula

$$\forall x\ x = x \to \exists^{=50\%} y\, [\mathrm{U}(x, y) \wedge \mathrm{H}(y)] \wedge \exists^{=50\%} y\, [\mathrm{U}(x, y) \wedge \neg \mathrm{H}(y)]\,,$$

whose satisfaction by $\mathcal{M}$ entails that $U^{\mathcal{M}}$ is the universal relation. Hence, by putting U as a dummy guard in every formula in the undecidability proof of $\mathrm{FO}^2_{loc\%}$, we conclude:

▶ **Corollary 8.** *The finite satisfiability problem for* $\mathrm{GF}_{gl\%}$ *is undecidable, even when restricted to its two-variable fragment* $\mathrm{GF}^2_{gl\%}$.

It turns out that the undecidability still holds for GF once we switch from the global to the local semantics of percentage counting. In order to show it, we present a reduction from $\mathrm{GF}^3[F]$ (*i.e.* the three-variable fragment of GF with a distinguished binary F interpreted as a functional relation), whose finite satisfiability was shown to be undecidable in [16].

▶ **Theorem 9.** *The finite satisfiability problem for* $\mathrm{GF}_{loc\%}$ *(and even* $\mathrm{GF}^3_{loc\%}$*) is undecidable.*

Proof sketch. By reduction from $\mathrm{GF}^3[F]$ it suffices to express that F is functional. Let H, R be fresh binary relational symbols. We use a similar trick to the one from Section 3.1.1, where $H(x, \cdot)$ plays the role of Half (note that H may induce different partitions for different x), $R(\cdot, y)$ plays the role of R and y in $x = y$ plays the role of J.

The functionality of F can be expressed with:

$$\varphi_{func} := \forall x\; x = x \rightarrow [(\forall y\; F(x,y) \rightarrow R(x,y)) \wedge (\exists^{=50\%}_{R} y\; H(x,y)) \wedge$$

$$(\forall y\; F(x,y) \rightarrow (\neg H(x,y) \vee x = y)) \wedge (\exists^{=50\%}_{R} y\; ((H(x,y) \wedge x \neq y) \vee F(x,y)))]$$

In the appendix we will show that if $\mathcal{M} \models \varphi_{func}$ then F is indeed functional and every structure $\mathcal{M}$ with functional F can be extended by H and R, such that φ_{func} holds. ◀

The similar proof techniques do not work for GF^2, since GF^2 with counting is decidable [29]. Thus, in the forthcoming section we show that decidability status transfers not only to GF^2 with percentage counting, but also with Presburger arithmetics. This can be then applied to infer decidability of several modal and description logics, see [7, Appendix A].

4 Positive results

We next show that the finite satisfiability problem for GF^2_{pres} is decidable, as stated below.

▶ **Theorem 10.** *The finite satisfiability problem for* GF^2_{pres} *is decidable.*

It is also worth pointing out that Theorem 10 together with a minor modification of existing techniques [3] yields decidability of conjunctive query entailment problem for GF^2_{pres}, *i.e.* a problem of checking if an existentially quantified conjunction of atoms is entailed by GF^2_{pres} formula. This is a fundamental object of study in the area of logic-based knowledge representation. All the proofs and appropriate definitions are moved to [7, Appendix D].

▶ **Theorem 11.** *Finite conjunctive query entailment for* GF^2_{pres} *is decidable.*

The rest of this section will be devoted to the proof of Theorem 10, which goes by reduction to the two-variable fragment of FO with counting quantifiers $\exists^{=k}, \exists^{\leq k}$ for $k \in \mathbb{N}$ with their obvious semantics. Since the finite satisfiability of C^2 is decidable [28], Theorem 10 follows.[4]

[4] Note that we propose a reduction into C^2, not into the *guarded* C^2, which might seem to be more appropriate. As we will see soon, a bit of non-guarded quantification is required in our proof.

4.1 Transforming GF^2_{pres} formulae into C^2

It is convenient to work with formulae in the appropriate normal form. Following a routine renaming technique (see *e.g.* [20]) we can convert in linear time a GF^2_{pres} formula into the following equisatisfiable normal form (over an extended signature):

$$\Psi_0 \quad := \quad \forall x\; \gamma(x) \wedge \bigwedge_{i=1}^{n} \Big(\forall x \forall y\; e_i(x,y) \to \alpha_i(x,y)\Big) \wedge \bigwedge_{i=1}^{m} \forall x \Big(\sum_{j=1}^{n_i} \lambda_{i,j} \cdot \#_y^{r_{i,j}}[x \neq y] \circledast \delta_i\Big),$$

where $\gamma(x)$ and each $\alpha_i(x,y)$ are quantifier-free formulae, each $e_i(x,y)$ is atomic predicate and all $\lambda_{i,j}$'s and δ_i's are integers, and $\circledast$ is as in Section 2.2.

Then, for every non-null type η, we replace each of the expressions $\#_y^{r_{i,j}}[x \neq y]$ with the sum of all the degrees $\deg_\eta(x)$ with η containing $r_{i,j}(x,y)$, *i.e.* the sum $\sum_{r_{i,j}(x,y)\in\eta} \deg_\eta(x)$. Moreover, since $\bigwedge\forall$ commutes, we obtain the following formula:

$$\Psi' \;:=\; \forall x\; \gamma(x) \wedge \bigwedge_{i=1}^{n} \Big(\forall x \forall y\; e_i(x,y) \to \alpha_i(x,y)\Big) \wedge \forall x \bigwedge_{i=1}^{m} \Big(\sum_{j=1}^{n_i} \lambda_{i,j} \cdot \sum_{r_{i,j}(x,y)\in\eta} \deg_\eta(x) \;\circledast\; \delta_i\Big)$$

Note that the conjunction $\bigwedge_{i=1}^{m} \Big(\sum_{j=1}^{n_i} \lambda_{i,j} \cdot \sum_{r_{i,j}(x,y)\in\eta} \deg_\eta(x) \;\circledast\; \delta_i\Big)$ is a Presburger formula with free variables $\deg_\eta(x)$'s, for every non-null type η.[5] Thus, by Theorem 1, we can compute a set of tuples of vectors $\{(\bar{v}_{1,0}, \bar{c}_{1,1}, \ldots, \bar{v}_{1,k_1}), \ldots, (\bar{v}_{p,0}, \bar{v}_{p,1}, \ldots, \bar{v}_{p,k_p})\}$ and further rewrite Ψ' into the following formula:

$$\Psi = \forall x\; \gamma(x) \;\wedge\; \textstyle\bigwedge_{i=1}^{n} \Big(\forall x \forall y\; e_i(x,y) \to \alpha_i(x,y)\Big) \;\wedge\; \forall x\;\; \deg(x) \in S$$

where $S = \bigcup_{i=1}^{p} \; L(\bar{c}_{i,0}; \bar{c}_{i,1}, \ldots, \bar{c}_{i,k_i})$. We stress that technically Ψ is no longer in GF^2_{pres}.

In the following we will show how to transform Ψ into a C^2 formula Ψ^* such that they are (finitely) equi-satisfiable. For every $i = 1, \ldots, p$, let $S_i = L(\bar{v}_{i,0}; \bar{v}_{i,1}, \ldots, \bar{v}_{i,k_i})$. Recall that $\mathrm{offset}(S_i)$ is the offset vector $\bar{v}_{i,0}$ and $\mathrm{prd}(S_i)$ is the set of periodic vectors of S_i, *i.e.* $\{\bar{v}_{i,1}, \ldots, \bar{v}_{i,k_i}\}$. Consider the following formulae ξ and ϕ.

$$\xi \;:=\; \forall x \bigvee_{i=1}^{p} \underline{\deg(x){=}\mathrm{offset}(S_i)} \vee \underline{\deg(x){\in}\mathrm{prd}(S_i)}, \quad \phi \;:=\; \forall x \bigwedge_{i=1}^{p} \underline{\deg(x){\neq}\mathrm{offset}(S_i)} \to \exists y\; \varphi(x,y)$$

where $\varphi(x,y)$ is the conjunction expressing the following properties:

- The 1-types of x and y equal. It can be expressed with the formula $\bigwedge_U U(x) \leftrightarrow U(y)$, where U ranges over unary predicates appearing in Ψ.
- $\deg(x) \in \mathrm{prd}(S_j)$ and $\deg(y) = \mathrm{offset}(S_j)$ for some $1 \leq j \leq p$.

Note that $\underline{\deg(x) = \mathrm{offset}(S_i)}$ can be written as a C^2 formula. For example, if $\bar{v}_{i,0} = (d_1, \ldots, d_\ell)$, it is written as $\bigwedge_{j=1}^{\ell} \exists^{=d_j} y\;\; \eta_j(x,y)$. We can proceed with $\underline{\deg(x) \in \mathrm{prd}(S_i)}$ similarly, since $\mathrm{prd}(S_i)$ contains only finitely many vectors. Finally, we put Ψ^* to be

$$\Psi^* \quad := \quad \forall x\; \gamma(x) \;\wedge\; \bigwedge_{i=1}^{n} \forall x \forall y\; e_i(x,y) \to \alpha_i(x,y) \quad \wedge \quad \xi \quad \wedge \quad \phi.$$

We will show that Ψ and Ψ^* are finitely equi-satisfiable, as stated formally below.

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[5] Technically speaking, in the standard definition of Presburger formula, the equality $f \equiv_d g$ is not allowed. However, it can be rewritten as $\exists x_1 \exists x_2 (f + x_1 d = g + x_2 d)$.

▶ **Lemma 12.** *Ψ is finitely satisfiable if and only if Ψ^* is.*

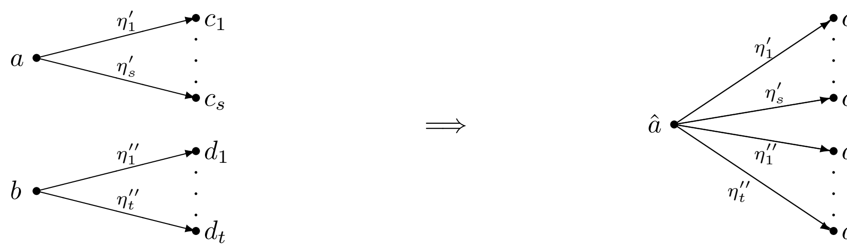
We delegate the proof of Lemma 12 to the next section. We conclude by stating that the complexity of our decision procedure is 3NExpTime. For more details of our analysis, see Section 4.3. Note that if we follow the decision procedure described in [13] for converting a system of linear equations to its semilinear set representation we will obtain a non-elementary complexity. This is because we need to perform $k-1$ intersections, where k is the number of linear constraints in the formula Ψ', and the procedure in [13] for handling each intersection yields an exponential blow-up. Instead, we use the results in [12, 27, 10] and obtain the complexity 3NExpTime, which though still high, falls within the elementary class.

4.2 Correctness of the translation

Before we proceed with the proof, we need to define some terminology. Let $\mathcal{M}$ be a finite model. Let $a, b \in A$ be such that the 2-type of (a, b) is η_{null}, *i.e.* the null-type and that a and b have the same 1-type. Suppose $c_1, \ldots, c_s$ are all elements such that the 2-type of each (a, c_j), denoted by η'_j, is non-null. Likewise, $d_1, \ldots, d_t$ are all the elements such that the 2-type of (b, d_j), denoted by η''_j, is non-null. Moreover, $c_1, \ldots, c_s, d_1, \ldots, d_t$ are pair-wise different.

"Merging" a and b into one new element $\hat{a}$ is defined similarly to the one in the graph-theoretic sense where a and b are merged into $\hat{a}$ such that the following holds.

- The 2-types of each $(\hat{a}, c_j)$ are equal to the original 2-types of (a, c_j), for all $j = 1, \ldots, s$.
- The 2-types of each $(\hat{a}, d_j)$ are equal to the original 2-types of (a, d_j), for all $j = 1, \ldots, t$.
- The 2-types of $(\hat{a}, a')$ is the null type, for every $a' \notin \{c_1, \ldots, c_2, d_1, \ldots, d_t\}$.
- The 1-type of $\hat{a}$ is the original 1-type of a (which is the same as the 1-type of b).



Note that we require that the original 2-type of (a, b) is the null type. Thus, after the merging, the degree of $\hat{a}$ is the sum of the original degrees of a and b. Moreover, the 1-type of $\hat{a}$ is the same as the original 1-type of a and b. Thus, if $\forall x \forall y\ e_i(x, y) \to \alpha_i(x, y)$ holds in $\mathcal{M}$, after the merging, it will still hold. Likewise, if $\forall x\ \gamma(x)$ holds in $\mathcal{M}$, it will still hold after the merging.

For the inverse, we define the "splitting" of an element $\hat{a}$ into two elements a and b as illustrated above, where the 1-type of a and b is the same as the 1-type of $\hat{a}$ and the 2-type of (a, b) is set to be η_{null}. After the splitting, the sum of the degrees of a and b is the same as the original degree of $\hat{a}$. Moreover, since the 2-type of (a, b) is η_{null}, $\mathcal{M}, x/a, y/b \not\models e_i(x, y)$. Thus, if $\forall x \forall y\ e_i(x, y) \to \alpha_i(x, y)$ holds in the original $\mathcal{M}$, it will still hold after the splitting.

▶ **Lemma 13.** *If Ψ is finitely satisfiable then Ψ^* is.*

Proof. Let $\mathcal{M}$ be a finite model of Ψ. We will construct a finite model $\mathcal{M}^* \models \Psi^*$ by splitting every element in $\mathcal{M}$ into several elements so that their degrees are either one of the offset vectors of S or one of the period vectors.

Let $a \in A$ and $\deg_{\mathcal{M}}(a) \in S_i$, for some $1 \le i \le p$. Suppose $\deg_{\mathcal{M}}(a) = \bar{v}_{i,0} + \sum_{j=1}^{k_i} n_j \bar{v}_{i,j}$, for some $n_1, \ldots, n_{k_i} \ge 0$. Let $N = 1 + \sum_{j=1}^{k_i} n_j$. We split a into N elements $b_1, \ldots, b_N$. Let

$\mathcal{M}^*$ denote the resulting model after such splitting. Note that it should be finite since the degree of a is finite. It is straightforward to show that $\mathcal{M}^* \models \Psi^*$. ◀

▶ **Lemma 14.** *If Ψ^* is finitely satisfiable then Ψ is.*

Proof. Let $\mathcal{M}^*$ be a finite model of Ψ^*. Note that the degree of every element in $\mathcal{M}^*$ is either the offset vector or one of the period vectors of S_i, for some $1 \leq i \leq p$. To construct a finite model $\mathcal{M} \models \Psi$, we can appropriately "merge" elements so that the degree of every element is a vector in S_i, for some $1 \leq i \leq p$.

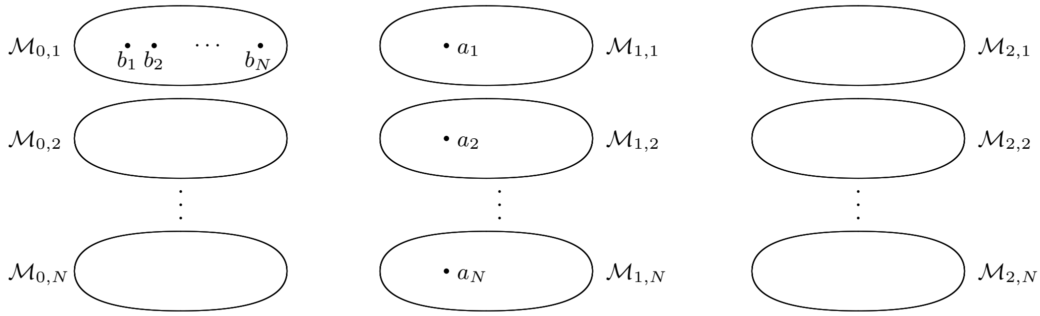
To this end, we call an element a in $\mathcal{M}^*$ a *periodic* element, if its degree is not an offset vector of some S_i. Let N be the number of periodic elements in $\mathcal{M}^*$. We make $3N$ copies of $\mathcal{M}^*$, which we denote by $\mathcal{M}_{i,j}$, where $0 \leq i \leq 2$ and $1 \leq j \leq N$. Let $\mathcal{M}$ be a model obtained by the disjoint union of all of $\mathcal{M}_{i,j}$'s, where for every b, b' that do not come from the same $\mathcal{M}_{i,j}$, the 2-type of (b, b') is the null-type.

We will show how to eliminate periodic elements in $\mathcal{M}$ by appropriately "merging" its elements. We need the following terminology. Recall that $S = S_1 \cup \cdots \cup S_p$, where each S_i is a linear set. For two vectors $\bar{u}$ and $\bar{v}$, we say that *$\bar{u}$ and $\bar{v}$ are compatible* (w.r.t. the semilinear set S), if there is S_i such that $\bar{u}$ is the offset vector of S_i and $\bar{v}$ is one of the period vectors of S_i. We say that two elements a and b in $\mathcal{M}$ are *merge-able*, if their 1-types are the same and their degrees are compatible.

We show how to merge periodic elements in $\mathcal{M}_{0,j}$, for every $j = 1, \ldots, N$.

- Let $b_1, \ldots, b_N$ be the periodic elements in $\mathcal{M}_{0,j}$.
- For each $l = 1, \ldots, N$, let a_l be an offset element in $\mathcal{M}_{1,l}$ such that every b_l and a_l are merge-able. (Such b_l exists, since $\mathcal{M}^*$ satisfies Ψ^* and each $\mathcal{M}_{i,j}$ is isomorphic to $\mathcal{M}^*$.)
- Then, merge a_l and b_l into one element, for every $l = 1, \ldots, k$.

See below, for an illustration for the case when $j = 1$.



Obviously, after this merging, there is no more periodic element in $\mathcal{M}_{0,j}$, for every $j = 1, \ldots, N$. We can perform similar merging between the periodic elements in $\mathcal{M}_{1,1} \cup \cdots \cup \mathcal{M}_{1,N}$ and the offset elements in $\mathcal{M}_{2,1} \cup \cdots \cup \mathcal{M}_{2,N}$, and between the periodic elements in $\mathcal{M}_{2,1} \cup \cdots \cup \mathcal{M}_{2,N}$ and the offset elements in $\mathcal{M}_{0,1} \cup \cdots \cup \mathcal{M}_{0,N}$.

After such merging, there is no more periodic element in $\mathcal{M}$ and the degree of every element is now a vector in S_i, for some $1 \leq i \leq p$. Moreover, since the merging preserves the satisfiability of $\forall x\ \gamma(x)$ and each $\forall x \forall y\ e_i(x, y) \to \alpha_i(x, y)$, the formula Ψ holds in $\mathcal{M}$. That is, Ψ is finitely satisfiable. ◀

4.3 Complexity analysis of the decision procedure

We need to introduce more terminology. For a vector/matrix X, we write $\|X\|$ to denote its L_∞-norm, *i.e.* the maximal absolute value of its entries. For a set of vector/matrices B, we write $\|B\|$ to denote $\max_{X \in B} \|X\|$.

Let $P = \{\bar{v}_1, \ldots, \bar{v}_k\} \subseteq \mathbb{N}^\ell$ be a finite set of (row) vectors of natural number components. To avoid clutter, we write $L(\bar{u}; P)$ to denote the linear set $L(\bar{u}; \bar{v}_1, \ldots, \bar{v}_k)$. For a finite set $B \subseteq \mathbb{N}^\ell$, we write $L(B; P)$ to denote the set $\bigcup_{\bar{u} \in B} L(\bar{u}; P)$.

We will use the following fact from [12, 27]. See also Proposition 2 in [10].

▶ **Proposition 15.** *Let $A \in \mathbb{Z}^{\ell \times m}$ and $\bar{c} \in \mathbb{Z}^m$. Let Γ be the space of the solutions of the system $\bar{x}A = \bar{c}$ (over the set of natural numbers $\mathbb{N}$).*[6] *Then, there are finite sets $B, P \subseteq \mathbb{N}^\ell$ such that the following holds.*

- $L(B; P) = \Gamma$.
- $\|B\| \leq ((m+1)\|A\| + \|\bar{c}\| + 1)^\ell$.
- $\|P\| \leq (m\|A\| + 1)^\ell$.
- $|B| \leq (m+1)^\ell$.
- $|P| \leq m^\ell$.

By repeating some of the vectors, if necessary, we can assume that Proposition 15 states that $|B| = |P| = (m+1)^\ell$.

Proposition 15 immediately implies the following naïve construction of the sets B and P in deterministic double-exponential time (in the size of input A and $\bar{c}$).

- Enumerate all possible sets $B, P \subseteq \mathbb{N}^\ell$ of cardinality $(m+1)^\ell$ whose entries are all bounded above by $((m+1)\|A\| + \|\bar{c}\| + 1)^\ell$.
- For each pair B, P, where $P = \{\bar{v}_1, \ldots, \bar{v}_k\}$, check whether for every $i_1, \ldots, i_k \in \mathbb{N}$ and every $\bar{u} \in B$, the following equation holds.

$$(\bar{u} + \sum_{j=1}^{k} i_j \bar{v}_j)A \quad = \quad \bar{c}. \tag{1}$$

The number of bits needed to represent the sets B and P is $O(\ell^2(m+1)^\ell \log K)$, where $K = (m+1)\|A\| + \|\bar{c}\| + 1$. Since Eq. 1 can be checked in deterministic exponential time (more precisely, it takes non-deterministic polynomial time to check if there is $i_1, \ldots, i_k$ such that Eq. 1 does not hold) in the length of the bit representation of the vectors in B, P, A and the vector $\bar{c}$, see, *e.g.* [26], constructing the sets B and P takes double-exponential time.

For completeness, we repeat the complexity analysis in Section 4. First, the formula Ψ_0 takes linear time in the size of the input formula. Constructing the formula Ψ' requires exponential time (in the number of binary predicates), *i.e.* $\ell = 2^k - 1$, where k is the number of binary predicates. Thus, constructing the sets B and P takes deterministic triple exponential time in the size of Ψ_0. However, the size of B and P is $O(2^{2k}(m+1)^{2^k} \log K)$, *i.e.* double exponential in the size of Ψ_0. The C^2 formulas ξ and ϕ are constructed in polynomial time in the size of B and P. Since both the satisfiability and finite satisfiability of C^2 formulas is decidable in nondeterministic exponential time, we have another exponential blow-up. Altogether, our decision procedure runs in 3NExpTime.

5 Concluding remarks

In the paper we studied the finite satisfiability problem for classical decidable fragments of FO extended with percentage quantifiers (as well as arithmetics in the full generality), namely the two-variable fragment FO^2 and the guarded fragment GF. We have shown that even in the presence of percentage quantifiers they quickly become undecidable.

[6] Recall that vectors in this paper are row vectors. So, $\bar{x}$ and $\bar{c}$ are row vectors of ℓ variables and m constants, respectively.

The notable exception is the intersection of GF and FO^2, *i.e.* the two-variable guarded fragment, for which we have shown that it is decidable with elementary complexity, even when extended with local Presburger arithmetics. The proof is quite simple and goes via an encoding into the two-variable logic with counting (C^2). One of the bottlenecks in our decision procedure is the conversion of systems of linear equations into the semilinear set representations, which incurs a double-exponential blow-up. We leave it for future work whether a decision procedure with lower complexity is possible and/or whether the conversion to semilinear sets is necessary.

We stress that our results are also applicable to the unrestricted satisfiability problem (whenever the semantics of percentage quantifiers make sense), see [7, Appendix C].

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A From Presburger modal logic with converse to GF^2 with Presburger

In this section we give a rather standard translation from the (multi)modal logic with converse, extended by Presburger arithmetics [11] into GF^2_{pres}. A similar translation can be defined also for $\mathcal{ALCSCC}$ with inverse from [2], whose semantics is a bit more complicated but essentially means the same.

We employ a countable set of propositional variables $\mathsf{Prop} = \{p_1, p_2, \ldots\}$ and a countable set of relation symbols $\mathsf{Rel} = \{R_1, R_2, \ldots\}$. We define the formulae of Presburger Modal Logic with converse, as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid t \sim b \mid t \equiv_k c$$
$$t ::= a \cdot \#^{R}\varphi \mid a \cdot \#^{R^-}\varphi \mid t + t$$

where $p \in \mathsf{Prop}$, $R \in \mathsf{Rel}$, $b, c \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{0,1\}$, $a \in \mathbb{Z} \setminus \{0\}$, $\sim \in \{\leq, <, >, =, \geq\}$.

A Kripke structure is a tuple $\mathcal{M} = (W, (R^{\mathcal{M}})_{R \in \mathsf{Rel}}, \ell)$, where W is a set of *worlds*, each $R^{\mathcal{M}}$ is a binary relation and $\ell : W \to 2^{\mathsf{Prop}}$ label worlds with atomic propositions. Now, for the semantics, we define $\mathcal{M}, w \models \varphi$ as follows:

- $\mathcal{M}, w \models p$ iff $p \in \ell(w)$
- $\mathcal{M}, w \models \neg\varphi$ iff not $\mathcal{M}, w \models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \varphi'$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \varphi'$
- $\mathcal{M}, w \models \Sigma_i a_i \cdot \#^{S_i}\varphi_i \sim b$ iff $\left(\Sigma_i a_i \cdot |\{w' \mid (w, w') \in S_i^{\mathcal{M}}\}|\right) \sim b$ (where in the case of $S_i = R^-$ we think about the inverse relation of $R^{\mathcal{M}}$). And similarly for $\equiv_k c$ (meaning the congruence modulo k).

We say that φ is globally satisfiable if there is a structure structure $\mathcal{M}$ such that for all its worlds w we have $\mathcal{M}, w \models \varphi$.

We next define a translation of Presburger modal logic into GF^2_{pres}. We let v denote either x or y and $\bar{v}$ the other variable.

- $\mathfrak{tr}_v(p) = p(v)$
- $\mathfrak{tr}_v(\neg\varphi) = \neg\mathfrak{tr}_v(\varphi)$
- $\mathfrak{tr}_v(\varphi \wedge \varphi') = \mathfrak{tr}_v(\varphi) \wedge \mathfrak{tr}_v(\varphi')$
- $\mathfrak{tr}_v(\Sigma_i a_i \cdot \#^{S_i}\varphi_i \sim b) = \sum_{i=1}^{n} a_i \cdot \#^{S_i}_{\bar{v}}[\mathfrak{tr}_{\bar{v}}(\varphi)] \sim b$
- $\mathfrak{tr}_v(\Sigma_i a_i \cdot \#^{S_i}\varphi_i \equiv_k c) = \sum_{i=1}^{n} a_i \cdot \#^{S_i}_{\bar{v}}[\mathfrak{tr}_{\bar{v}}(\varphi)] \equiv_k c$

Hence, for a given φ let $\mathfrak{tr}(\varphi) = \forall x\; x = x \to \mathfrak{tr}_x(\varphi)$. It is easy to see that $\mathfrak{tr}(\varphi)$ has a (finite) model iff φ is globally (finitely) satisfiable. Hence, we obtain a logspace reduction from the global satisfiability for Presburger modal logic with converse into GF^2_{pres}.

B Appendix for Section 3

B.1 Proof of Fact 1

Proof. The fact that a finite $(\mathrm{Half}, \mathrm{R}, \mathrm{J})$-separated $\mathcal{A}$ such that $|\mathrm{R}(x)|_{\mathcal{A}} = |\mathrm{J}(x)|_{\mathcal{A}}$ satisfies $\varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J})$ is obvious from the semantics of $\mathrm{FO}^2_{gl\%}$ and the definition of $(\mathrm{Half}, \mathrm{R}, \mathrm{J})$-separability. For the opposite direction let $\mathcal{A}$ be a finite, $(\mathrm{Half}, \mathrm{R}, \mathrm{J})$-separated model of $\varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J})$. Then we can see that, by the satisfaction of $\exists^{=50\%}$, $\mathcal{A}$ has an even number of elements (call it $2n$). Moreover, we know that the sets $\mathrm{R}^{\mathcal{A}}$ and $\mathrm{J}^{\mathcal{A}}$ are disjoint, hence by the satisfaction of $\exists^{=50\%}x\; (\mathrm{Half}(x) \wedge \neg\mathrm{R}(x)) \vee \mathrm{J}(x)$ we conclude that $n = (n - |\mathrm{R}(x)|_{\mathcal{A}}) + |\mathrm{J}(x)|_{\mathcal{A}}$, which implies that $|\mathrm{R}(x)|_{\mathcal{A}} = |\mathrm{J}(x)|_{\mathcal{A}}$. ◀

B.2 Proof of Fact 2, Fact 3 and Fact 4

Proof. An immediate consequence of the semantics of $\mathrm{FO}^2_{gl\%}$. ◀

B.3 Proof of Lemma 3

Proof. Identical to the proof Fact 1, by taking $\mathrm{Half} = \mathrm{FHalf}^{[i]}(x)$, $\mathrm{J} = A_{w_i}$ and R defined as the union of A_{u_i} and A_{v_i}. ◀

B.4 Proof of Lemma 4

Proof. Observe $\varphi^i_{\mathrm{count}}(u_i, v_i, w_i)$ that is actually an instance of $\varphi_{\mathrm{eq}}(\mathrm{Half}, \mathrm{R}, \mathrm{J})$ formula from Section 3.1.1 with $\mathrm{Half} = \mathrm{SHalf}^{[i]}$, $\mathrm{J} = A_{v_i}$ and the Mult_i-successors of x play the role of elements labelled by R. For $\varphi^i_{\mathrm{bfunc}}(u_i, v_i, w_i)$ we proceed similarly. ◀

B.5 Proof of Lemma 5

Indeed, assume that a well prepared $\mathcal{M}$ satisfies $\varphi^i_{\mathrm{mult}}(u_i, v_i, w_i)$. Then from Lemma 4(i) it follows that every domain element labelled by A_{u_i} is connected via $\mathrm{Mult}_i{}^{\mathcal{M}}$ relation with exactly $|A^{\mathcal{M}}_{v_i}|$ that satisfies A_{w_i}. By backward-functionality of $\mathrm{Mult}_i{}^{\mathcal{M}}$ (guaranteed by Lemma 4(ii)) the sets of elements labelled with A_{w_i} and connected to elements satisfying A_{u_i} are disjoint. Hence, by combining such facts, we infer that $|A^{\mathcal{M}}_{w_i}| \geq |A^{\mathcal{M}}_{u_i}| \cdot |A^{\mathcal{M}}_{v_i}|$. But by Lemma 4(ii) we also know that every element from $A^{\mathcal{M}}_{w_i}$ is connected via $\mathrm{Mult}_i{}^{\mathcal{M}}$ to some element from $A^{\mathcal{M}}_{u_i}$. Hence the mentioned inequality becomes the equality.

B.6 Proof of Theorem 6

We need to show that ε has a solution iff $\varphi^{\varepsilon}_{\mathrm{red}}$ is finitely satisfiable. For the "if" direction take any finite model $\mathcal{M}$ of $\varphi^{\varepsilon}_{\mathrm{red}}$ and let $S : \mathrm{Var}(\varepsilon)$ be a function that maps a variable v to $|A^{\mathcal{M}}_v|$. From Lemma 5, Lemma 3, Fact 2 it is easy to conclude that S is indeed the solution of ε.

For the second direction assume that ε has a solution and call it S. Let $\mathcal{M}$ be any finite structure satisfying all the following conditions:

- **a)** $|M|$ is even and $\frac{|M|}{2} > max\{S(v) + S(u) : v, u \in \mathrm{Var}(\varepsilon)\}$ and $|M| > \sum_{i \in \mathrm{Var}(\varepsilon)} S(i)$
- **b)** All $a \in M$ satisfies at most one $A_u(x)$ for $u \in \mathrm{Var}(\varepsilon)$
- **c)** For all $v \in \mathrm{Var}(\varepsilon)$ we have $|A_v(x)|_x = S(v)$.
- **d)** For all entry ε_i of the form $u_i + v_i = w_i$ or $u_i \cdot v_i = w_i$ all domain elements satisfying $A_{u_i}(x) \vee A_{v_i}(x)$ are labelled with $\mathrm{FHalf}^{[i]}$ and all elements satisfying $A_{w_i}(x)$ are labelled with $\mathrm{SHalf}^{[i]}$.
- **e)** For al $i \leq |\varepsilon|$ exactly half of domain elements are labelled with $\mathrm{FHalf}^{[i]}$ while the other half is labelled with $\mathrm{SHalf}^{[i]}$.

Additionally, for all ε_i of the form $u_i \cdot v_i = w_i$ the structure $\mathcal{M}$ should satisfy:

- **A)** The relation $\mathrm{Mult}_i{}^{\mathcal{M}}$ is a subset of $A^{\mathcal{M}}_{u_i} \times A^{\mathcal{M}}_{w_i}$.
- **B)** For each $a \in A^{\mathcal{M}}_{u_i}$, that the total number of $\mathrm{Mult}_i{}^{\mathcal{M}}$-successors of a is equal to $|A^{\mathcal{M}}_{v_i}|$.
- **C)** Every element from $A^{\mathcal{M}}_{w_i}$ has exactly one $\mathrm{Mult}_i{}^{\mathcal{M}}$-predecessor.

One can readily check, by routine case enumeration, that $\mathcal{M} \models \varphi^{\varepsilon}_{\mathrm{red}}$. What remains to be done is to show that such a structure $\mathcal{M}$ actually exists. Hence, let $\mathcal{M}$ be any finite structure satisfying (a). It is obvious that such a structure exists. Due to the fact that $|M| > \sum_{i \in \mathrm{Var}(\varepsilon)} S(i)$ we can extend $\mathcal{M}$ by interpret predicates A_u (where $u \in \mathrm{Var}(\varepsilon)$) in $\mathcal{M}$, such that (b) and (c) are satisfied.

We next interpret the relational symbols $\mathrm{FHalf}^{[i]}$ in $\mathcal{M}$ such that the extended structure satisfy (d) and (e). It can be done due to the fact that $|A_{u_i}(x) \vee A_{v_i}(x)|_x = S(u_i) + S(v_i)$ and from the fact that $\frac{M}{2} > S(u_i) + S(v_i)$. Next we put $(\mathrm{SHalf}^{[i]})^{\mathcal{M}} = M \setminus (\mathrm{FHalf}^{[i]})^{\mathcal{M}}$. Next, we interpret Mult_i in such a way that it satisfies all the remaining conditions. It is possible due to (a) and (b) and the fact that $|A_{w_i}|^{\mathcal{M}} = |A_{u_i}^{\mathcal{M}}| \cdot |A_{v_i}^{\mathcal{M}}|$. Since each of the steps is correct, the structure $\mathcal{M}$ actually exists, which finishes the proof.

B.7 More details on the proof of Theorem 9

Let us recall that φ_{func} is defined as follows:

$$\forall x \quad x = x \rightarrow \tag{2}$$
$$(\forall y \quad F(x,y) \rightarrow R(x,y)) \quad \wedge \tag{3}$$
$$\exists_R^{=50\%} y R(x,y) \wedge H(x,y) \quad \wedge \tag{4}$$
$$(\forall y \quad F(x,y) \rightarrow (\neg H(x,y) \vee x = y)) \quad \wedge \tag{5}$$
$$(\exists_R^{=50\%} y (R(x,y) \wedge ((H(x,y) \wedge x \neq y) \vee F(x,y))) \tag{6}$$

Call the above lines (a), (b), (c), (d), (e). We prove two lemmas.

▶ **Lemma 16.** *For all models $\mathcal{M}$ of φ_{func} we have that $F^{\mathcal{M}}$ is functional.*

Proof. In the proof we write $|\varphi(x,y)|_{R,y}$ means the total number of y satisfying both $R(x,y)$ and $\varphi(x,y)$. From (c) and (e) we have $|((H(x,y) \wedge x \neq y) \vee F(x,y)|_{R,y} = |H(x,y)|_{R,y}$. By applying De Morgan's law in (d) we can see that for all x the set of elements y witnessing $F(x,y)$ and the set of elements y satisfying $(H(x,y) \wedge x \neq y)$ are disjoint. Thus we know that the equation $|((H(x,y) \wedge x \neq y) \vee F(x,y)|_{R,y} = |H(x,y) \wedge x \neq y|_{R,y} + |F(x,y)|_{R,y}$ holds. By applying simple transformations we conclude $|F(x,y)|_{R,y} = |H(x,y)|_{R,y} - |H(x,y) \wedge x \neq y|_{R,y}$. We can see that $|H(x,y)|_{R,y} - |H(x,y) \wedge x \neq y|_{R,y}$ is equal 1 if both $H(x,x)$ and $R(x,x)$ and 0 otherwise. Hence, we conclude that for all x the number of R-successors y of x satisfying $F(x,y)$ is either zero or one. By (b) we know that $F^{\mathcal{M}}$ is a subset of $R^{\mathcal{M}}$ thus F is indeed functional. ◀

▶ **Lemma 17.** *Every finite $\mathcal{M}$ with a functional $F^{\mathcal{M}}$ can be extended by interpretation of R, H in a way that $\mathcal{M} \models \varphi_{func}$.*

Proof. Take any such $\mathcal{M}$ and for all element $a \in M$ we define R_a and H_a as follows.

- If there is $b \neq a$ such that $(a,b) \in F^{\mathcal{M}}$ we put $R_a = \{(a,b),(a,a)\}$ and $H_a = \{(a,a)\}$
- If $(a,a) \in F^{\mathcal{M}}$ holds then we take any $b \neq a$ and put define R_a, H_a as above.
- If there is no b such that $(a,b) \in F^{\mathcal{M}}$ we keep R_a and H_a empty.

Now put $R^{\mathcal{M}} = \bigcup_{a \in M} R_a$ and $H^{\mathcal{M}} = \bigcup_{a \in M} H_a$. It is easy to verify that such an extended $\mathcal{M}$ satisfies φ_{func}. ◀

C Results that are transferable to general satisfiability

Undecidability results. Since the semantics of global percentage quantifiers only make sense over finite domains, all our undecidability results involving global percentage quantifiers hold also for general satisfiability (*i.e.* general = finite). For our results concerning their local counterparts, namely Corollary 7, note that the relation $U^{\mathcal{M}}$ is forced to be universal. Thus, by the fact that we consider models where the total number of $U^{\mathcal{M}}$-successors of any node is finite, this implies that the whole domain is also finite. Hence, also in this case we have that general satisfiability and the finite one are the same. Finally, the undecidability of GF with local percentage quantifiers over arbitrary finite-branching structures can be concluded by routinely checking that in the undecidability proof of Graedel [19] the constructed models are finite branching.

Decidability results. It is not difficult to see that the decision procedure described in Section 4 also holds for the general satisfiability problem for GF^2_{pres}. Indeed, note that the general satisfiability problem for C^2 is decidable [28]. The only part that is different is the correctness proof when the model $\mathcal{M}^* \models \Psi^*$ is infinite. In this case, to construct the model $\mathcal{M} \models \Psi$, we make infinitely many copies: $\mathcal{M}_{i,j}$, where $i \in \{0,1,2\}$ and $j \in \mathbb{N}$. The merging process to eliminate the periodic elements is the same.

We believe that a rather standard technique à la tableaux of constructing a tree model of GF^2_{pres} level-by-level can be employed here and yields ExpTime upper bound for the general satisfiability problem. We stress that this approach exploit the "infiniteness" of models in a rather heavy way: at any level of the infinite tree model we can always pick fresh witnesses in order to satisfy formulae and we do not need to worry that at some point the models must be rolled-up to form a finite structure. Since we are not ready with all the details at the time of submission, we delegate this result to the journal version of the paper.

D Appendix for conjunctive query entailment

In this section we formally prove Theorem 11. Before we start, let us revisit the basics on conjunctive query entailment problem over ontologies.

D.1 Preliminaries on queries, homomorphisms and Gaifman graphs

Conjunctive queries (CQs) are conjunctions of *atoms* of the form $r(x,y)$ or $\mathrm{A}(z)$, where r is a binary relational symbol, A is a unary relational symbol and x, y, z are variables from some countably infinite set of variable names. We denote with $|q|$ the number of its atoms and with $\mathrm{Var}(q)$ the set of all variables that appear in q.

Let $\mathcal{A}$ be a structure, q a CQ and $\eta : \mathrm{Var}(q) \to A$ be a variable assignment. We write $\mathcal{A} \models_\eta r(x,y)$ if $(\eta(x), \eta(y)) \in r^{\mathcal{A}}$ and $\mathcal{A} \models_\eta \mathrm{A}(z)$ if $\eta(z) \in \mathrm{A}^{\mathcal{A}}$. We say that η is a *match* for $\mathcal{A}$ and q if $\mathcal{A} \models_\eta \alpha$ holds for every atom $\alpha \in q$ and that $\mathcal{A}$ *satisfies* q (denoted with: $\mathcal{A} \models q$) whenever $\mathcal{A} \models_\eta q$ for some match η. The definitions are lifted to formulae: q is *(finitely) entailed* by a GF^2_{pres} formula φ, written: $\varphi \models q$ ($\varphi \models_{\mathrm{fin}} q$) if every (finite) model of φ satisfies q. When $\mathcal{A} \models q$ but $\mathcal{A} \not\models q$, we call $\mathcal{A}$ a *countermodel* for φ and q. Moreover, if such an $\mathcal{A}$ is finite, we call it a *finite countermodel*. The *(finite) query entailment* problem for GF^2_{pres} is defined as follows: given a formula φ and a CQ q verify if φ (finitely) entails q.

Note that CQs may be seen as structure: for a query q we define a structure $\mathcal{Q}_q$ satisfying $(x,y) \in r^{\mathcal{Q}_q}$ iff $r(x,y) \in q$ and $x \in \mathrm{A}^{\mathcal{Q}_q}$ iff $\mathrm{A}(x) \in q$.

A *homomorphism* $\mathfrak{h} : \mathcal{A} \to \mathcal{B}$ is a function that maps every element of A to some element from B and it preserves unary and binary relations, *i.e.* we have that $a \in \mathrm{A}^{\mathcal{A}}$ implies that $\mathfrak{h}(a) \in \mathrm{A}^{\mathcal{B}}$ and $(a,b) \in r^{\mathcal{A}}$ implies $(\mathfrak{h}(a), \mathfrak{h}(b)) \in r^{\mathcal{B}}$ for all binary relational symbols r, unary relational symbols A and elements $a, b \in A$. Since queries can be seen as structures, their matches can be seen as homomorphisms.

We employ the notion of Gaifman graphs. Intuitively, the *Gaifman graphs* $G_{\mathcal{A}}$ of a structure $\mathcal{A}$ is the underlying undirected graph structure of $\mathcal{A}$. More precisely the graph $G_{\mathcal{A}} = (V_{\mathcal{A}}, E_{\mathcal{A}})$ is composed of nodes $V_{\mathcal{A}} := A$ and edges $E_{\mathcal{A}}$, for which $(a,b) \in E_{\mathcal{A}}$ if $a \neq b$ and there is a binary relation name r such that $(a,b) \in r^{\mathcal{A}}$ or $(b,a) \in r^{\mathcal{A}}$. The *girth* of $G_{\mathcal{A}}$ is the length of its shortest cycle or ∞ if $G_{\mathcal{A}}$ does not have any cycles. The *girth* of $\mathcal{A}$ is the girth of its Gaifman Graph.[7] A structure $\mathcal{A}$ is tree-shaped if $G_{\mathcal{A}}$ is a tree.

The notion of Gaifman graphs is adjusted to queries q, *i.e.* the *query graph* G_q is simply the Gaifman graph of its corresponding structure. Similarly, we speak about tree-shaped queries as well as tree-shaped matches, *i.e.* query matches whose induced substructures are tree-shaped.

D.2 Reducing conjunctive query entailment to satisfiability

Our goal is to reduce (finite) conjunctive query entailment problem to (finite) satisfiability (in exponential time, which is optimal). We rely on previous results by the first author that appeared in the workshop paper [3]. Our proof methods rely on two facts:

1. Tree-shaped query matches (*i.e.* those whose underlying graph forms a tree, more precisely a graph of treewidth 1) can be efficiently blocked.
2. If there is a countermodel $\mathcal{A}$ for a formula φ and a query q then there is also a countermodel $\mathcal{B}$ of high *girth*, meaning that it is sufficiently tree-like for a query q to match only in a tree-shaped way. In other words, from a given structure we "eliminate" all non-tree-shaped query matches of q. We achieve this by an appropriate model transformation, called *pumping*, similar to the one recently introduced by the first author and his coauthors in the workshop paper [3].

We discuss them in separate sections.

D.3 Eliminating tree-shaped query matches

We start from the first point. A query q' is a *treeification* of q if it is tree-shaped and can be obtained from q by selecting, possibly multiple times, two of its variables and identifying them. The set $\mathsf{Tree}(q)$, *i.e.* the set of all its treeifications of q, is clearly of at most exponential size in $|q|$.

To detect tree-shaped query matches we employ the well-known rolling-up technique [31, 14] that for a given tree-shaped q produces a GF^2 defining an unary relation C_q, whose interpretation is non-empty in $\mathcal{A}$ iff there is a match of q in $\mathcal{A}$. Hence, by imposing that such unary predicates are empty in $\mathcal{A}$ for all treeifications of q, we can conclude that $\mathcal{A}$ does not have any tree-shaped query matches. Let $\preceq$ be a descendant ordering on variables of a tree-shaped q. Then for any leaf v of $G_q = (V_q, E_q)$ (*i.e.* the $\preceq$-maximal element of $\preceq$) we define a unary relation C_q^v as

$$\mathrm{C}_q^v(x) := \bigwedge_{\mathrm{A}(v)\in q} \mathrm{A}(x) \wedge \bigwedge_{r(v,v)\in q} r(x,x),$$

while for non-leaf node v the unary relation C_q^v is defined with as above but we additionally append:

$$\bigwedge_{(v,u)\in E_q, u\prec v} \exists y \left(\bigwedge_{r(u,v)\in q} r(x,y) \ \wedge \bigwedge_{r(v,u)\in q} r(y,x) \right) \wedge \mathrm{C}_q^u(y)$$

We put $\mathrm{C}_q := \mathrm{C}_q^{v_r}$, where v_r is the root variable of q (*i.e.* the $\preceq$-minimal variable). Note that the size of C_q is linear in $|q|$.

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[7] Self-loops in $\mathcal{A}$ are not counted as cycles as $G_{\mathcal{A}}$ is simple.

▶ **Fact 5.** *For a tree-shaped query q we have $\mathcal{A} \models q$ if and only if $(\mathrm{C}_q)^{\mathcal{A}} \neq \emptyset$.*

Proof. We proceed by induction over $\preceq$, where the inductive assumption is that for all variables $u \prec v$ we have that $a \in (\mathrm{C}_q^u)^{\mathcal{A}}$ iff there is a homomorphism $\mathfrak{h}$ from the subtree rooted at u to $\mathcal{A}$ with $\mathfrak{h}(u) = a$.

- Base case. We have that the following unary predicate is non-empty and contains a

$$\mathrm{C}_q^v(x) := \bigwedge_{\mathrm{A}(v) \in q} \mathrm{A}(x) \wedge \bigwedge_{r(v,v) \in q} r(x,x),$$

  and it basically states that a satisfies all atoms involving a as a sole variable. Hence, $\mathfrak{h}(v) := a$ is indeed a homomorphism. For the opposite way, the existence of a homomorphism ensures us that all the conjuncts of C_q^v are satisfied at a.
- Take any variable v and assume that there is a homomorphism $\mathfrak{h}$ from the subtree rooted at v to $\mathcal{A}$ with $\mathfrak{h}(v) = a$. We want to show that a satisfies

$$\bigwedge_{(v,u) \in E_q, u \prec v} \exists y \left(\bigwedge_{r(u,v) \in q} r(x,y) \wedge \bigwedge_{r(v,u) \in q} r(y,x) \right) \wedge \mathrm{C}_q^u(y)$$

  , since the rest is in the base case. By the existence of a homomorphism we know that there are domain elements $\mathfrak{h}(u) = a_u$. Since $\mathfrak{h}$ is a homomorphism we know that if $r(v,u) \in q$ then also $(a, a_u) \in r^{\mathcal{A}}$ holds. Similarly for inverse roles. By the inductive assumption we also know that $a_u \in (\mathrm{C}_q^v)^{\mathcal{A}}$. Hence, $a \in (\mathrm{C}_q^v)^{\mathcal{A}}$. For the opposite direction the homomorphism is constructed by taking $\mathfrak{h}(v) = a$ and then by selecting a witness a_u for a formula

$$\bigwedge_{(v,u) \in E_q, u \prec v} \exists y \left(\bigwedge_{r(u,v) \in q} r(x,y) \wedge \bigwedge_{r(v,u) \in q} r(y,x) \right) \wedge \mathrm{C}_q^u(y)$$

  and putting $\mathfrak{h}(u) := a_u$.

◀

The above fact can be justified by induction over $\preceq$, where the inductive assumption is that for all variables $u \prec v$ we have that $a \in (\mathrm{C}_q^u)^{\mathcal{A}}$ iff there is a homomorphism $\mathfrak{h}$ from the subtree rooted at u to $\mathcal{A}$ with $\mathfrak{h}(u) = a$.

D.4 Obtaining counter-models of high girth

We start from auxiliary definitions. For a given finite set S we denote with 2^S the set of all boolean functions with the domain S. The unique function of the form $S \to \{0\}$ is denoted with zero_S. We drop the subscript S, whenever it is known from the context. Once two boolean functions $f, g \in 2^S$ and an element $s \in S$ are given, we say that f is *nearly-s-equal* g, symbolized by $f \approx_s g$, if f and g are equal on all arguments from S except for s, for which $g(s) = 1 - f(s)$ holds.

In what follows, we will construct from $\mathcal{A}$ a structure $\mathsf{pump}(\mathcal{A})$ having the girth doubled. The construction of $\mathsf{pump}(\mathcal{A})$ is quite technical, so we provide some informal intuitions behind it. We construct $\mathsf{pump}(\mathcal{A})$ from $\mathcal{A}$ by equipping each domain element from A with a set of coins (implemented as a boolean function), one coin per each edge from the initial structure. Then, we define connections between elements in the domain of $\mathsf{pump}(\mathcal{A})$ in such a way that two elements u and v are connected only if they carry nearly the same sets of coins except that they differ on the side of unique coin responsible for the edge (u, v). Thus, in every round-trip, visited edges will appear even number of times (due to the fact that crossing an edge requires tossing the appropriate coin). Hence the girth of the obtained structure will be strictly greater than the girth of $\mathcal{A}$.

▶ **Definition 18.** *Let $\mathcal{A}$ be a structure and let $\mathsf{G}_{\mathcal{A}} = (\mathsf{V}_{\mathcal{A}}, \mathsf{E}_{\mathcal{A}})$ be its Gaifman graph. We define the structure $\mathsf{pump}(\mathcal{A})$ with the domain $A \times 2^{\mathsf{E}_{\mathcal{A}}}$ as follows:*

- *For all elements $a \in A$ and all unary relational names C we put $(a, f) \in \mathrm{C}^{\mathsf{pump}(\mathcal{A})}$ for all boolean function $f \in 2^{\mathsf{E}_{\mathcal{A}}}$ whenever $a \in \mathrm{C}^{\mathcal{A}}$ holds.*
- *For any binary relational symbols r and any pair $p = (v_1, v_2)$ of domain elements $v_1{=}(a_1, f_1)$ and $v_2{=}(a_2, f_2)$ with $a_1{\neq}a_2$ we set $p \in r^{\mathsf{pump}(\mathcal{A})}$ iff $(a_1, a_2) \in r^{\mathcal{A}}$ and $f_1 \approx_{(a_1,a_2)} f_2$.*
- *For all domain elements a and binary relational symbols r s.t. $(a, a) \in r^{\mathcal{A}}$ we put $((a, f), (a, f)) \in r^{\mathsf{pump}(\mathcal{A})}$ for all $f \in 2^{\mathsf{E}_{\mathcal{A}}}$.*

We next aim to prove that the pumping method works as desired, *i.e.* that it increases the girth and it is model-preserving. We show the former goal first. The proof hinges upon the fact that in the projection of a cycle in $\mathsf{pump}(\mathcal{A})$ some edge from $\mathcal{A}$ is repeated twice.

▶ **Lemma 19.** *For any structure $\mathcal{A}$ with a finite girth, the girth of $\mathsf{pump}(\mathcal{A})$ is strictly greater than the girth of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is finite then $\mathsf{pump}(\mathcal{A})$ is also finite.*

Proof. Let $\mathsf{G}_{\mathcal{A}}$ [resp. $\mathsf{G}_{\mathsf{pump}(\mathcal{A})}$] be the Gaifman graph of $\mathcal{A}$ [resp. $\mathsf{pump}(\mathcal{A})$]. Let $\rho_{\mathsf{pump}(\mathcal{A})} :=$ $(v_1, f_1) \to^{e_1} \ldots \to^{e_n} (v_{n+1}, f_{n+1}){=}(v_1, f_1)$ be an arbitrary shortest cycle in $\mathsf{G}_{\mathsf{pump}(\mathcal{A})}$. From the construction of $\mathsf{pump}(\mathcal{A})$ we know that $((v_i, f_i), (v_{i+1}, f_{i+1})) \in \mathsf{E}_{\mathsf{pump}(\mathcal{A})}$ holds iff $(v_i, v_{i+1}) \in \mathsf{E}_{\mathcal{A}}$ holds. Thus the projection $\rho_{\mathcal{A}}$ of $\rho_{\mathsf{pump}(\mathcal{A})}$ onto $\mathcal{A}$ (*i.e.* $\rho_{\mathcal{A}} = v_1 \to^{e'_1} \ldots \to^{e'_n} v_n$) is a cycle in $\mathcal{A}$. To conclude that $|\rho_{\mathcal{A}}| > \mathsf{girth}(\mathcal{A})$ it is sufficient to prove that some pair of elements (v_i, v_{i+1}) appear at least twice in $\rho_{\mathcal{A}}$. Fix any edge $e = (u, v)$ from $\rho_{\mathcal{A}}$. W.l.o.g. assume that $f_1(e) = 0$ and let i be the minimal index, such that $((u, f_i), (v, f_{i+1})) \in \rho_{\mathsf{pump}(\mathcal{A})}$. Hence $f_i(e) = 0$, but $f_{i+1}(e) = 1$ (from the construction of $\mathsf{pump}(\mathcal{A})$). If there would be no $j > i$, s.t. $((u, f_j), (v, f_{j+1})) \in \rho_{\mathsf{pump}(\mathcal{A})}$ then $f_{n+1}(e) = 1$ (since crossing the edge e is the only way to flip the value of $f(e)$), which leads to a contradiction with $f_1 = f_{n+1}$. Thus e appears at least twice on the cycle $\rho_{\mathcal{A}}$. ◀

We next prove the latter, *i.e.* that the pumping method is model-preserving.

▶ **Lemma 20.** *For all GF^2_{pres} φ we have that if $\mathcal{A} \models \varphi$ then $\mathsf{pump}(\mathcal{A}) \models \varphi$.*

Proof. We can focus on φ in normal forms. First, note that by the construction a satisfies the same unary relations as (a, f) and has the same self-loops. Thus all atomic concepts, their boolean combinations and "self-loop" concepts from φ are satisfied in (a, f) in $\mathsf{pump}(\mathcal{A})$ iff there are satisfied by a in $\mathcal{A}$. Second, observe that the "neighbourhood" of $a \in A$ is preserved: an element (a, f) is connected to the elements from $\{(b, f_b)\}$ with a and b originally connected in $\mathcal{A}$ and $f \approx_{(a,b)} f_b$. Moreover, by the construction, $((a, f), (b, f_b)) \in r^{\mathsf{pump}(\mathcal{A})}$ iff $(a, b) \in r^{\mathcal{A}}$. Hence, it follows that all two-types of all pairs of elements are not only identical but the degree of every node form $\mathcal{A}$ is the same as the degree of its corresponding elements in $\mathsf{pump}(\mathcal{A})$. Thus the satisfaction of Presburger constraints is preserved, finishing the proof. ◀

D.5 Exponential reduction from querying to satisfiability

We conclude the section by applying the pumping method to the (finite) conjunctive query entailment problem. Take a GF^2_{pres} formula φ and a CQ q. Our reduction simply checks the (finite) satisfiability of $\varphi' = \varphi \wedge \bigwedge_{q' \in \mathsf{Tree}(q)} \forall x \neg \mathrm{C}_{q'}(x)$. The following lemma claims the correctness:

▶ **Lemma 21.** *For any* GF^2_{pres} *we have that* $\varphi \models q$ *(in the finite) iff* φ' *is (finitely) unsatisfiable.*

Proof. We show the proof for both finite and unrestricted setting at the same. For the first direction assume that $\varphi \models q$ (in the finite) and ad absurdum assume that φ' has a (finite) model $\mathcal{A}$. By applying the pumping method to $\mathcal{A}$ at least $|q|$ times, by Lemma 19 we obtain a (finite) $\mathcal{B}$ of girth greater than $|q|$. By Lemma 20 we conclude that $\mathcal{B} \models \varphi'$. Since $\varphi' \models \varphi$ we also know that $\mathcal{B} \models \varphi$. Since $\varphi \models q$ we infer that there is a tree-shaped match of q on $\mathcal{B}$ due to the fact that the girth of $\mathcal{B}$ is greater than the number of atoms of q. Hence, there is a treeification q' of q that matches $\mathcal{B}$ implying (by Fact 5) that $\mathrm{C}^{\mathcal{B}}_{q'} \neq \emptyset$. It contradicts the satisfaction of an extra conjuncts in φ' enforcing that $\mathrm{C}^{\mathcal{B}}_{q'} = \emptyset$. For the second direction, by contraposition, it suffices to take a (finite) countermodel for φ and q, which by Fact 5 is also a model of φ'. ◀

Hence, by the decidability of finite and unrestricted satisfiability problem for GF^2_{pres} we conclude Theorem 11 and its unrestricted version.