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Description Logics of Context with Rigid Roles Revisited

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Description Logics of Context with Rigid Roles Revisited

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To represent and reason about contextualized knowledge often two-dimensional Description Logics (DLs) are employed, where one DL is used to describe contexts (or possible worlds) and the other DL is used to describe the objects, i.e. the relational structure of the specific contexts. Previous approaches for DLs of context that combined pairs of DLs resulted in undecidability in those cases where so-called rigid roles are admitted, i.e. if parts of the relational structure are the same in all contexts. In this paper, we present a novel combination of pairs of DLs and show that reasoning stays decidable even in the presence of rigid roles. We give complexity results for various combinations of DLs involving $\mathcal{ALC}$, $\mathcal{SHOQ}$, and $\mathcal{EL}$.

1 Introduction

Description logics (DLs) of context can be employed to represent and reason about contextualized knowledge [BS03; BVS+09; KG10; KG11b; KG11a]. Such contextualized knowledge naturally occurs in practice. Consider, for instance, the rôles played by a person in different contexts. The person Bob, who works for the company Siemens, plays the rôle of an employee of Siemens while at work, i.e. in the work context, whereas he might play the rôle of a customer of Siemens in the context of private life. In this example, access restrictions to the data of Siemens might critically depend on the rôle played by Bob. Moreover, DLs capable of representing contexts are vital to integrate distributed knowledge as argued in [BS03; BVS+09].

In DLs, we use *concept names* (unary predicates) and *complex concepts* (using certain constructors) to describe subsets of an interpretation domain and *roles* (binary predicates) that are interpreted as binary relations over the interpretation domain. Thus, DLs are well-suited to describe contexts as formal objects with formal properties that are organized in relational structures, which are fundamental requirements for modeling contexts [McC87; McC93].

However, classical DLs lack expressive power to formalize furthermore that some individuals satisfy certain concepts and relate to other individuals depending on a specific context. Therefore, often two-dimensional DLs are employed [KG10; KG11b; KG11a]. The approach is to have one DL $\mathcal{L}_M$ as the *meta* or *outer logic* to represent the contexts and their relationships to each other. This logic is combined with the *object* or *inner logic* $\mathcal{L}_O$ that captures the relational structure within each of the contexts. Moreover, while some pieces of information depend on the context, other pieces of information are shared throughout all contexts. For instance, the name of a person typically stays the same independent of the actual context. To be able to express that, some concepts and roles a designated to be *rigid*, i.e. they are required to be interpreted the same in all contexts. Unfortunately, if rigid roles are admitted, reasoning in the above mentioned two-dimensional DLs of context turns out to be undecidable; see [KG10].

We propose and investigate a family of two-dimensional context DLs $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ that meet the above requirements, but are a restricted form of the ones defined in [KG10] in the sense that we limit the interaction of $\mathcal{L}_M$ and $\mathcal{L}_O$. More precisely, in our family of context DLs the outer logic can refer to the internal structure of each context, but not vice versa. That means that information is viewed in a top-down manner, i.e. information of different contexts is strictly capsuled and can only be accessed from the meta level. This means that we cannot express, for instance, that everybody who is employed by Siemens has a certain property in the context of private life. Interestingly, reasoning in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ stays decidable with such a restriction, even in the presence of rigid roles. In some sense this restriction is similar to what was done in [BGL08; BGL12; Lip14] to obtain a decidable temporalized DL with rigid roles. Even though, our techniques to show complexity results are very similar to the ones employed for those temporalized DLs, we cannot simply reuse these results to reason in our context DLs, and more effort is needed to obtain tight complexity bounds.

For providing better intuition on how our formalism works, we examine the above mentioned example a bit further. Consider the following axioms:

$$
\begin{aligned}
\top &\sqsubseteq [\![\exists\, \mathit{worksFor}.\{\mathit{Siemens}\} \sqsubseteq \exists\, \mathit{hasAccessRights}.\{\mathit{Siemens}\}]\!] && (1)\\
\textsc{Work} &\sqsubseteq [\![\mathit{worksFor}(\mathit{Bob}, \mathit{Siemens})]\!] && (2)\\
[\![(\exists\, \mathit{worksFor}.\top)(\mathit{Bob})]\!] &\sqsubseteq \exists\, \mathit{related}.(\textsc{Private} \sqcap [\![\mathit{HasMoney}(\mathit{Bob})]\!]) && (3)\\
\top &\sqsubseteq [\![\exists\, \mathit{isCustomerOf}.\top \sqsubseteq \mathit{HasMoney}]\!] && (4)\\
\textsc{Private} &\sqsubseteq [\![\mathit{isCustomerOf}(\mathit{Bob}, \mathit{Siemens})]\!] && (5)\\
\textsc{Private} \sqcap \textsc{Work} &\sqsubseteq \bot && (6)\\
\neg\textsc{Work} &\sqsubseteq [\![\exists\, \mathit{worksFor}.\top \sqsubseteq \bot]\!] && (7)
\end{aligned}
$$

The first axiom states that it holds true in all contexts that somebody who works for Siemens also has access rights to certain data. The second axiom states that Bob is an employee of Siemens in any work context. Furthermore, Axioms 3 and 4 say intuitively that if Bob has a job, he will earn money, which he can spend as a customer. Axiom 5 formalises that Bob is a customer of Siemens in any private context. Moreover, Axiom 6 ensures that the private contexts are disjoint from the work contexts. Finally, Axiom 7 states that the *worksFor* relation only exists in work contexts.

A fundamental reasoning task is to decide whether the above mentioned axioms are *consistent* altogether, i.e. whether there is a common model. In our example, this is the case; Figure 1 depicts a model. In this model, we also have Bob's social security number linked to him using a rigid role *hasSSN*. We require this role to be rigid since Bob's social security number does not change over the contexts. Furthermore the axioms entail more knowledge such as for example that in any private context nobody will have access rights to work data of Siemens, i.e.

$$\textsc{Private} \sqsubseteq [\![\exists\, \mathit{hasAccessRights}.\{\mathit{Siemens}\} \sqsubseteq \bot]\!]$$

The remainder of the technical report is structured as follows. In the next section, we introduce the syntax and semantics of our family of context DLs $\mathcal{L}_M[\![\mathcal{L}_O]\!]$. For this, we repeat some basic notions of DLs. In Section 3, we show decidability of the consistency problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ for $\mathcal{L}_M$ and $\mathcal{L}_O$ being DLs between $\mathcal{ALC}$ and $\mathcal{SHOQ}$. There we consider the cases without rigid names, with rigid concepts and roles, and with rigid concepts only, and analyze the computational complexity of the consistency problem in these cases. Thereafter, in Section 4 we investigate the complexity of deciding consistency in DLs of context $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ where $\mathcal{L}_M$ or $\mathcal{L}_O$ is the sub-Boolean DL $\mathcal{EL}$. Again

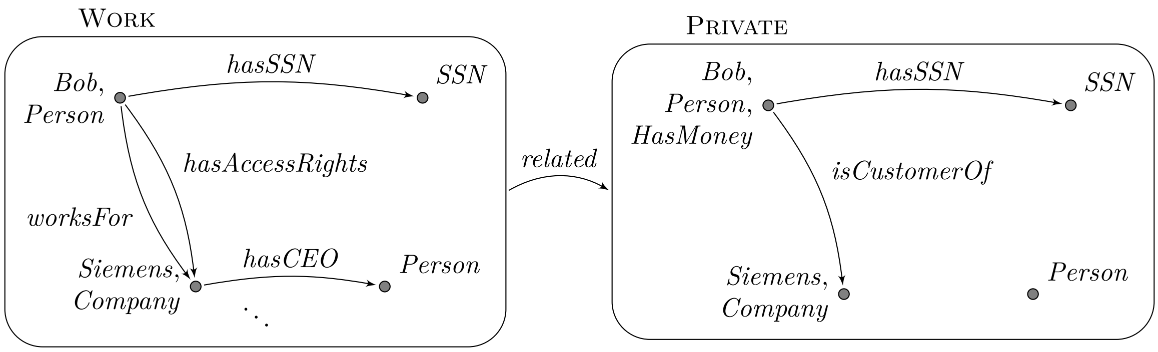

Figure 1: Model of Axioms 1–7

we consider the cases with rigid names, with rigid concepts and roles, and with rigid concepts only. Section 5 concludes the report and lists some possible future work.

2 Basic Notions

As argued in the introduction, our family of two-dimensional context DLs $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ consists of combinations of two DLs: $\mathcal{L}_M$ and $\mathcal{L}_O$. We focus on the case where $\mathcal{L}_M$ and $\mathcal{L}_O$ are the lightweight DL $\mathcal{EL}$ or DLs between $\mathcal{ALC}$ and $\mathcal{SHOQ}$. First, we recall the basic definitions of those DLs; for a thorough introduction to DLs, we refer the reader to [BCM+07].

Definition 1 [*Syntax of DLs*]
Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be non-empty, pairwise disjoint sets of *concept names*, *role names*, and *individual names*, respectively. Furthermore, let $\mathsf{N} := (\mathsf{N_C}, \mathsf{N_R}, \mathsf{N_I})$. The set of *concepts over* N is inductively defined starting from concept names $A \in \mathsf{N_C}$ using the constructors in the upper part of Table 1, where $r, s \in \mathsf{N_R}$, $a, b \in \mathsf{N_I}$, $n \in \mathbb{N}$, and C, D are concepts over N. The lower part of Table 1 shows how *axioms over* N are defined.

Moreover, an *RBox* $\mathcal{R}$ *over* N is a finite set of role inclusions over N and transitivity axioms over N. A *Boolean axiom formula over* N is defined inductively as follows:

- every GCI over N is a Boolean axiom formula over N,
- every concept and role assertion over N is a Boolean axiom formula over N,
- if $\mathcal{B}_1, \mathcal{B}_2$ are Boolean axiom formulas over N, then so are $\neg\mathcal{B}_1$ and $\mathcal{B}_1 \wedge \mathcal{B}_2$, and
- nothing else is a Boolean axiom formula over N.

Finally, a *Boolean knowledge base (BKB) over* N is a pair $\mathfrak{B} = (\mathcal{B}, \mathcal{R})$, where $\mathcal{B}$ is a Boolean axiom formula over N and $\mathcal{R}$ is an RBox over N. ◇

Note that in this definition we refer to the triple N explicitly although it is usually left implicit in standard definitions. This turns out to be useful as we need to distinguish between the symbols used in $\mathcal{L}_M$ and $\mathcal{L}_O$. Sometimes we omit N, however, if it is clear from the context. As usual, we use the following abbreviations:

- $C \sqcup D$ (disjunction) for $\neg(\neg C \sqcap \neg D)$,
- $\top$ (top concept) for $A \sqcup \neg A$, where $A \in \mathsf{N_C}$ is arbitrary but fixed,
- $\bot$ (bottom concept) for $\neg\top$,
- $\forall r.C$ (value restriction) for $\neg\exists r.\neg C$,

Table 1: Syntax and Semantics of DLs

	syntax	semantics
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists\, r.C$	$\{d \in \Delta^{\mathcal{I}} \mid \text{there is an } e \in C^{\mathcal{I}} \text{ with } (d,e) \in r^{\mathcal{I}}\}$
nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
at-most restriction	$\leqslant_n r.C$	$\{d \in \Delta^{\mathcal{I}} \mid \sharp\{e \in C^{\mathcal{I}} \mid (d,e) \in r^{\mathcal{I}}\} \leq n\}$
general concept inclusion (GCI)	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$r(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$
role inclusion	$r \sqsubseteq s$	$r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$
transitivity axiom	$\mathsf{trans}(r)$	$r^{\mathcal{I}} = (r^{\mathcal{I}})^+$

- $\geqslant_n s.C$ (at-least restriction) for $\neg(\leqslant_{n-1} s.C)$, and
- $\mathcal{B}_1 \vee \mathcal{B}_2$ (disjunction) for $\neg(\neg\mathcal{B}_1 \wedge \neg\mathcal{B}_2)$.

Which concept constructors and types of axioms are available depends on the specific DL used. In the smallest propositionally closed DL $\mathcal{ALC}$, the only allowed concept constructors are negation, conjunction, and existential restriction. Thus disjunction, top and bottom concept, and value restriction can be used as abbreviations. Moreover, no role inclusions and transitivity axioms are allowed in $\mathcal{ALC}$. If additional concept constructors or types of axioms are allowed, this is denoted by concatenating a corresponding letter: $\mathcal{O}$ means nominals, $\mathcal{Q}$ means at-most restrictions (qualified number restrictions), and $\mathcal{H}$ means role inclusions (role hierarchies). For instance, the DL $\mathcal{ALCHO}$ is the extension of $\mathcal{ALC}$ that also allows for nominals and role inclusions. The extension of $\mathcal{ALC}$ with transitivity axioms is denoted by $\mathcal{S}$. Hence, the DL allowing for all the concept constructors and types of axioms introduced here is called $\mathcal{SHOQ}$. The sub-Boolean DL $\mathcal{EL}$ is the fragment of $\mathcal{ALC}$ where only conjunction, existential restriction, and the top concept (which cannot be expressed as an abbreviation anymore due to the lack of negation) are admitted as concept constructors. We sometimes write $\mathcal{L}$-concept over N ($\mathcal{L}$-BKB over $\mathsf{N}, \dots$) for some DL $\mathcal{L}$ to make clear which DL is used.

The semantics of DLs are defined in a model-theoretic way through the notion of *interpretations*.

Definition 2 [*Semantics of DLs*]
Let $\mathsf{N} = (\mathsf{N_C}, \mathsf{N_R}, \mathsf{N_I})$. An N-*interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set (called *domain*), and $\cdot^{\mathcal{I}}$ is a mapping assigning a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ to every $A \in \mathsf{N_C}$, a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to every $r \in \mathsf{N_R}$, and a domain element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ to every $a \in \mathsf{N_I}$. The function $\cdot^{\mathcal{I}}$ is extended to concepts over N inductively as shown in the upper part of Table 1, where $\sharp S$ denotes the cardinality of the set S.

Moreover, $\mathcal{I}$ is a model of the axiom α over N if the condition in the lower part of Table 1 is satisfied, where $\cdot^+$ denotes the transitive closure of a binary relation. This is extended to Boolean axiom formulas over N inductively as follows:

- $\mathcal{I}$ is a model of $\neg\mathcal{B}_1$ if it is not a model of $\mathcal{B}_1$, and
- $\mathcal{I}$ is a model of $\mathcal{B}_1 \wedge \mathcal{B}_2$ if it is a model of both $\mathcal{B}_1$ and $\mathcal{B}_2$.

We write $\mathcal{I} \models \mathcal{B}$ if $\mathcal{I}$ is a model of the Boolean axiom formula $\mathcal{B}$ over N. Futhermore, $\mathcal{I}$ is a model of an RBox $\mathcal{R}$ over N (written $\mathcal{I} \models \mathcal{R}$) if it is a model of each axiom in $\mathcal{R}$.

Finally, $\mathcal{I}$ is a model of the BKB $\mathfrak{B} = (\mathcal{B}, \mathcal{R})$ over N (written $\mathcal{I} \models \mathfrak{B}$) if it is a model of both $\mathcal{B}$ and $\mathcal{R}$. We call $\mathfrak{B}$ *consistent* if it has a model. $\diamond$

We call a role name $r \in \mathsf{N_R}$ *transitive* (w.r.t. $\mathcal{R}$) if every model of $\mathcal{R}$ is a model of $\mathsf{trans}(r)$. Moreover, r is a *subrole* of a role name $s \in \mathsf{N_R}$ (w.r.t. $\mathcal{R}$) if every model of $\mathcal{R}$ is a model of $r \sqsubseteq s$. Finally, r is *simple w.r.t.* $\mathcal{R}$ if it has no transitive subrole. It is not hard to see that $r \in \mathsf{N_R}$ is simple w.r.t. $\mathcal{R}$ iff $\mathsf{trans}(r) \notin \mathcal{R}$ and there do not exist roles $s_1, \dots, s_k \in \mathsf{N_R}$ such that $\{\mathsf{trans}(s_1),\ s_1 \sqsubseteq s_2,\ \dots,\ s_{k-1} \sqsubseteq s_k,\ s_k \sqsubseteq r\} \subseteq \mathcal{R}$. Thus deciding whether $r \in \mathsf{N_R}$ is simple can be decided in time polynomial in the size of $\mathcal{R}$ by simple syntactic checks.

It follows from a result in [HST00] that the problem of checking whether a given $\mathcal{SHQ}$-BKB $\mathfrak{B} = (\mathcal{B}, \mathcal{R})$ over N is consistent is undecidable in general. One regains decidability with a syntactic restriction as follows: if $\leqslant_n r.C$ occurs in $\mathcal{B}$, r must be simple w.r.t. $\mathcal{R}$. In the following, we make this restriction to the syntax of $\mathcal{SHQ}$ and all its extensions.

This restriction is also the reason why there are no Boolean combinations of role inclusions and transitivity axioms allowed in the RBox $\mathcal{R}$ over N in the above definition. Otherwise the notion of a simple role w.r.t. $\mathcal{R}$ involves reasoning. Consider, for instance, the Boolean combination of axioms $(\mathsf{trans}(r) \vee \mathsf{trans}(s)) \wedge r \sqsubseteq s$. It should be clear that s is not simple, but this is no longer a pure syntactic check.

We are now ready to define the syntax of $\mathcal{L}_M[\![\mathcal{L}_O]\!]$. Throughout the paper, let $\mathsf{O_C}$, $\mathsf{O_R}$, and $\mathsf{O_I}$ be respectively sets of concept names, role names, and individual names for the object logic $\mathcal{L}_O$. Analogously, we define the sets $\mathsf{M_C}$, $\mathsf{M_R}$, and $\mathsf{M_I}$ for the meta logic $\mathcal{L}_M$. Without loss of generality, we assume that all those sets are pairwise disjoint. Moreover, let $\mathsf{O} := (\mathsf{O_C}, \mathsf{O_R}, \mathsf{O_I})$ and $\mathsf{M} := (\mathsf{M_C}, \mathsf{M_R}, \mathsf{M_I})$.

Definition 3 [*Syntax of* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$]
A *concept of the object logic* $\mathcal{L}_O$ *(o-concepts)* is an $\mathcal{L}_O$-concept over O. An *o-axiom* is an $\mathcal{L}_O$-GCI over O, an $\mathcal{L}_O$-concept assertion over O, or an $\mathcal{L}_O$-role assertion over O.

The set of *concepts of the meta logic* $\mathcal{L}_M$ *(m-concepts)* is the smallest set such that

- every $\mathcal{L}_M$-concept over M is an m-concept and
- $[\![\alpha]\!]$ is an m-concept if α is an o-axiom.

The notion of an *m-axiom* is defined analogously.

A *Boolean m-axiom formula* is defined inductively as follows:

- every m-axiom is a Boolean m-axiom formula,
- if $\mathcal{B}_1, \mathcal{B}_2$ are Boolean m-axiom formulas, then so are $\neg\mathcal{B}_1$ and $\mathcal{B}_1 \wedge \mathcal{B}_2$, and
- nothing else is a Boolean m-axiom formula.

Finally, a *Boolean* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$*-knowledge base* $(\mathcal{L}_M[\![\mathcal{L}_O]\!]$*-BKB)* is a triple $\mathfrak{B} = (\mathcal{B}, \mathcal{R}_\mathsf{O}, \mathcal{R}_\mathsf{M})$ where $\mathcal{R}_\mathsf{O}$ is an RBox over O, $\mathcal{R}_\mathsf{M}$ an RBox over M, and $\mathcal{B}$ is a Boolean m-axiom formula. $\diamond$

For the reasons above, role inclusions over O and transitivity axioms over O are not allowed to constitute m-concepts. However, we fix an RBox $\mathcal{R}_\mathsf{O}$ over O that contains such o-axioms and holds in *all* contexts. The same applies to role inclusions over M and transitivity axioms over M, which are only allowed to occur in a RBox $\mathcal{R}_\mathsf{M}$ over M.

Again, we use the usual abbreviations (for disjunctions etc.) for m-concepts and Boolean m-axiom formulas.

The semantics of $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ is defined by the notion of *nested interpretations*. These consist of O-interpretations for the specific contexts and an M-interpretation for the relational structure between them. We assume that all contexts speak about the same non-empty domain (*constant domain assumption*).

As argued in the introduction, sometimes it is desired that concepts or roles in the object logic are interpreted the same in all contexts. Let $\mathsf{O}_{\mathsf{Crig}} \subseteq \mathsf{O}_{\mathsf{C}}$ denote the set of *rigid concepts*, and let $\mathsf{O}_{\mathsf{Rrig}} \subseteq \mathsf{O}_{\mathsf{R}}$ denote the set of *rigid roles*. We call concept names and role names in $\mathsf{O}_{\mathsf{C}} \setminus \mathsf{O}_{\mathsf{Crig}}$ and $\mathsf{O}_{\mathsf{R}} \setminus \mathsf{O}_{\mathsf{Rrig}}$ *flexible*. Moreover, we assume that individuals of the object logic are always interpreted the same in all contexts (*rigid individual assumption*).

Definition 4 [*Nested interpretation*]
A *nested interpretation* is a tuple $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c\in\mathbb{C}})$, where $\mathbb{C}$ is a non-empty set (called *contexts*) and $(\mathbb{C}, \cdot^{\mathcal{J}})$ is an M-interpretation.

Moreover, for every $c \in \mathbb{C}$, $\mathcal{I}_c := (\Delta, \cdot^{\mathcal{I}_c})$ is an O-interpretation such that we have for all $c, c' \in \mathbb{C}$ that $x^{\mathcal{I}_c} = x^{\mathcal{I}_{c'}}$ for every $x \in \mathsf{O}_{\mathsf{I}} \cup \mathsf{O}_{\mathsf{Crig}} \cup \mathsf{O}_{\mathsf{Rrig}}$. $\diamond$

We are now ready to define the semantics of $\mathcal{L}_M[\![\mathcal{L}_O]\!]$.

Definition 5 [*Semantics of* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$]
Let $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c\in\mathbb{C}})$ be a nested interpretation. The mapping $\cdot^{\mathcal{J}}$ is extended to o-axioms as follows: $[\![\alpha]\!]^{\mathcal{J}} := \{c \in \mathbb{C} \mid \mathcal{I}_c \models \alpha\}$.

Moreover, $\mathcal{J}$ is a model of the m-axiom β if $(\mathbb{C}, \cdot^{\mathcal{J}})$ is a model of β. This is extended to Boolean m-axiom formulas inductively as follows:

- $\mathcal{J}$ is a model of $\neg\mathcal{B}_1$ if it is not a model of $\mathcal{B}_1$, and
- $\mathcal{J}$ is a model of $\mathcal{B}_1 \wedge \mathcal{B}_2$ if it is a model of both $\mathcal{B}_1$ and $\mathcal{B}_2$.

We write $\mathcal{J} \models \mathcal{B}$ if $\mathcal{J}$ is a model of the Boolean m-axiom formula $\mathcal{B}$. Furthermore, $\mathcal{J}$ is a model of $\mathcal{R}_{\mathsf{M}}$ (written $\mathcal{J} \models \mathcal{R}_{\mathsf{M}}$) if $(\mathbb{C}, \cdot^{\mathcal{J}})$ is a model of $\mathcal{R}_{\mathsf{M}}$, and $\mathcal{J}$ is a model of $\mathcal{R}_{\mathsf{O}}$ (written $\mathcal{J} \models \mathcal{R}_{\mathsf{O}}$) if $\mathcal{I}_c$ is a model of $\mathcal{R}_{\mathsf{O}}$ for all $c \in \mathbb{C}$.

Finally, $\mathcal{J}$ is a model of the $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-BKB $\mathfrak{B} = (\mathcal{B}, \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ (written $\mathcal{J} \models \mathfrak{B}$) if $\mathcal{J}$ is a model of $\mathcal{B}$, $\mathcal{R}_{\mathsf{O}}$, and $\mathcal{R}_{\mathsf{M}}$. We call $\mathfrak{B}$ *consistent* if it has a model.

The *consistency problem in* $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ is the problem of deciding whether a given $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-BKB is consistent. $\diamond$

Note that besides the consistency problem there are several other reasoning tasks for $\mathcal{L}_M[\![\mathcal{L}_O]\!]$. The entailment problem, for instance, is the problem of deciding, given a BKB $\mathfrak{B}$ and an m-axiom β, whether $\mathfrak{B}$ *entails* β, i.e. whether all models of $\mathfrak{B}$ are also models of β. The consistency problem, however, is fundamental in the sense that most other standard decision problems (reasoning tasks) can be polynomially reduced to it (in the presence of negation). For the entailment problem, note that it can be reduced to the *in*consistency problem: $\mathfrak{B} = (\mathcal{B}, \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ entails β iff $(\mathcal{B} \wedge \neg\beta, \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ is inconsistent. Hence, we focus in the present paper only on the consistency problem.

3 Complexity of the Consistency Problem

Our results for the computational complexity of the consistency problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ are listed in Table 2. In this section, we focus on the cases where $\mathcal{L}_M$ and $\mathcal{L}_O$ are DLs between $\mathcal{ALC}$ and $\mathcal{SHOQ}$. In Section 4, we treat the cases where $\mathcal{L}_M$ or $\mathcal{L}_O$ are $\mathcal{EL}$.

Since the lower bounds of context DLs treated in this section already hold for the fragment $\mathcal{EL}[\![\mathcal{ALC}]\!]$, they are shown in Section 4.

Table 2: Complexity results for consistency in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ (where "c." is short for "complete")

$\mathcal{L}_M$ \ $\mathcal{L}_O$	no rigid names			only rigid concepts			rigid roles		
	$\mathcal{EL}$	$\mathcal{ALC}$	$\mathcal{SHOQ}$	$\mathcal{EL}$	$\mathcal{ALC}$	$\mathcal{SHOQ}$	$\mathcal{EL}$	$\mathcal{ALC}$	$\mathcal{SHOQ}$
$\mathcal{EL}$	const.	Exp-c.	Exp-c.	const.	NExp-c.	NExp-c.	const.	2Exp-c.	2Exp-c.
$\mathcal{ALC}$	Exp-c.	Exp-c.	Exp-c.	NExp-c.	NExp-c.	NExp-c.	NExp-c.	2Exp-c.	2Exp-c.
$\mathcal{SHOQ}$	Exp-c.	Exp-c.	Exp-c.	NExp-c.	NExp-c.	NExp-c.	NExp-c.	2Exp-c.	2Exp-c.

For the upper bounds, let in the following $\mathfrak{B} = (\mathcal{B}, \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ be a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-BKB. We proceed similar to what was done for $\mathcal{ALC}$-LTL in [BGL08; BGL12] (and $\mathcal{SHOQ}$-LTL in [Lip14]) and reduce the consistency problem to two separate decision problems.

For the first problem, we consider the so-called *outer abstraction*, which is the $\mathcal{SHOQ}$-BKB over M obtained by replacing each m-concept of the form $[\![\alpha]\!]$ occurring in $\mathcal{B}$ by a fresh concept name such that there is a 1–1 relationship between them.

Definition 6 [*Outer abstraction*]
Let $\mathfrak{B} = (\mathcal{B}, \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ be a $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-BKB. Let b be the bijection mapping every m-concept of the form $[\![\alpha]\!]$ occurring in $\mathcal{B}$ to the concept name $A_{[\![\alpha]\!]} \in \mathsf{M}_{\mathsf{C}}$, where we assume w.l.o.g. that $A_{[\![\alpha]\!]}$ does not occur in $\mathcal{B}$.

1. The Boolean $\mathcal{L}_M$-axiom formula $\mathcal{B}^{\mathsf{b}}$ over M is obtained from $\mathcal{B}$ by replacing every occurrence of an m-concept of the form $[\![\alpha]\!]$ by $\mathsf{b}([\![\alpha]\!])$. We call the $\mathcal{L}_M$-BKB $\mathfrak{B}^{\mathsf{b}} = (\mathcal{B}^{\mathsf{b}}, \mathcal{R}_{\mathsf{M}})$ the *outer abstraction of* $\mathfrak{B}$.
2. Given $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c\in\mathbb{C}})$, its *outer abstraction* is the M-interpretation $\mathcal{J}^{\mathsf{b}} = (\mathbb{C}, \cdot^{\mathcal{J}^{\mathsf{b}}})$ where
   - for every $x \in \mathsf{M}_{\mathsf{R}} \cup \mathsf{M}_{\mathsf{I}} \cup (\mathsf{M}_{\mathsf{C}} \setminus \mathsf{Im}(\mathsf{b}))$, we have $x^{\mathcal{J}^{\mathsf{b}}} = x^{\mathcal{J}}$, and
   - for every $A \in \mathsf{Im}(\mathsf{b})$, we have $A^{\mathcal{J}^{\mathsf{b}}} = (\mathsf{b}^{-1}(A))^{\mathcal{J}}$,

   where $\mathsf{Im}(\mathsf{b})$ denotes the image of b. ◇

For simplicity, for $\mathfrak{B}' = (\mathcal{B}', \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ where $\mathcal{B}'$ is a subformula of $\mathcal{B}$, we denote by $(\mathfrak{B}')^{\mathsf{b}}$ the outer abstraction of $\mathfrak{B}'$ that is obtained by restricting b to the m-concepts occurring in $\mathcal{B}'$.

Example 7. Let $\mathfrak{B}_{\text{ex}} = (\mathcal{B}_{\text{ex}}, \varnothing, \varnothing)$ with $\mathcal{B}_{\text{ex}} := C \sqsubseteq ([\![A \sqsubseteq \bot]\!]) \ \wedge \ (C \sqcap [\![A(a)]\!])(c)$ be a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-BKB. Then, b maps $[\![A \sqsubseteq \bot]\!]$ to $A_{[\![A\sqsubseteq\bot]\!]}$ and $[\![A(a)]\!]$ to $A_{[\![A(a)]\!]}$. Thus, we have that

$$\mathfrak{B}^{\mathsf{b}}_{\text{ex}} := \Big(C \sqsubseteq (A_{[\![A\sqsubseteq\bot]\!]}) \ \wedge \ (C \sqcap A_{[\![A(a)]\!]})(c), \ \varnothing\Big)$$

is the outer abstraction of $\mathfrak{B}_{\text{ex}}$. △

The following lemma makes the relationship between $\mathfrak{B}$ and its outer abstraction $\mathfrak{B}^{\mathsf{b}}$ explicit. It is proved by induction on the structure of $\mathcal{B}$.

Lemma 8. *Let $\mathcal{J}$ be a nested interpretation such that $\mathcal{J}$ is a model of $\mathcal{R}_{\mathsf{O}}$. Then, $\mathcal{J}$ is a model of $\mathfrak{B}$ iff $\mathcal{J}^{\mathsf{b}}$ is a model of $\mathfrak{B}^{\mathsf{b}}$.*

Proof. Since $r^{\mathcal{J}} = r^{\mathcal{J}^{\mathsf{b}}}$ for all $r \in \mathsf{M}_{\mathsf{R}}$, we have that $\mathcal{J}$ is a model of $\mathcal{R}_{\mathsf{M}}$ iff $\mathcal{J}^{\mathsf{b}}$ is a model of $\mathcal{R}_{\mathsf{M}}$. Thus, it is only left to show that for any m-axiom γ occurring in $\mathcal{B}$, it holds that $\mathcal{J} \models \gamma$ iff $\mathcal{J}^{\mathsf{b}} \models \gamma^{\mathsf{b}}$.

<u>*Claim:*</u> Let C^{b} be the m-concept obtained from the m-concept C by replacing every occurrence of $[\![\alpha]\!]$ by $\mathsf{b}([\![\alpha]\!])$. Then, for any $x \in \mathbb{C}$ it holds that $x \in C^{\mathcal{J}}$ iff $x \in (C^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$.

Proof: We prove the claim by induction on the structure of C:

$C = A \in \mathsf{M_C} \setminus \mathsf{Im}(\mathsf{b})$: $x \in A^{\mathcal{J}}$ iff $x \in (A^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ by definition of $\mathcal{J}^{\mathsf{b}}$ and since $A = A^{\mathsf{b}}$

$C = [\![\alpha]\!]$: $x \in [\![\alpha]\!]^{\mathcal{J}}$ iff $x \in A_{[\![\alpha]\!]}^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in ([\![\alpha]\!]^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$

$C = \neg D$: $x \in (\neg D)^{\mathcal{J}}$ iff $x \notin D^{\mathcal{J}}$ iff, by induction hypothesis, $x \notin (D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in (\neg D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in ((\neg D)^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$

$C = D \sqcap E$: $x \in (D \sqcap E)^{\mathcal{J}}$ iff $x \in D^{\mathcal{J}}$ and $x \in E^{\mathcal{J}}$ iff, by induction hypothesis, $x \in (D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ and $x \in (E^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in (D^{\mathsf{b}} \sqcap E^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in ((D \sqcap E)^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$

$C = \exists r.D$: $x \in (\exists r.D)^{\mathcal{J}}$ iff there exists $y \in \mathbb{C}$ s. t. $(x, y) \in r^{\mathcal{J}}$ and $y \in D^{\mathcal{J}}$ iff there exists $y \in \mathbb{C}$ s. t. $(x, y) \in r^{\mathcal{J}^{\mathsf{b}}}$ and $y \in (D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in (\exists r.D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in ((\exists r.D)^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$

$C = \{a\}$: $x \in \{a\}^{\mathcal{J}}$ iff $x \in (\{a\}^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ by definition of $\mathcal{J}^{\mathsf{b}}$ and since $\{a\} = \{a\}^{\mathsf{b}}$

$C = \leqslant_n r.D$: $x \in (\leqslant_n r.D)^{\mathcal{J}}$ iff there are at most n elements $y \in \mathbb{C}$ s.t. $(x, y) \in r^{\mathcal{J}}$ and $y \in D^{\mathcal{J}}$ iff there are at most n elements $y \in \mathbb{C}$ s.t. $(x, y) \in r^{\mathcal{J}^{\mathsf{b}}}$ and $y \in (D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in (\leqslant_n r.D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $x \in ((\leqslant_n r.D)^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ ∕

If γ of the form $C \sqsubseteq D$, we have that $\mathcal{J} \models C \sqsubseteq D$ iff $x \in C^{\mathcal{J}}$ implies $x \in D^{\mathcal{J}}$ iff (by claim) $x \in (C^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ implies $x \in (D^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $\mathcal{J}^{\mathsf{b}} \models C^{\mathsf{b}} \sqsubseteq D^{\mathsf{b}}$.

If γ is of the form $C(a)$, we have that $\mathcal{J} \models C(a)$ iff $a^{\mathcal{J}} \in C^{\mathcal{J}}$ iff (by claim) $a^{\mathcal{J}^{\mathsf{b}}} \in (C^{\mathsf{b}})^{\mathcal{J}^{\mathsf{b}}}$ iff $\mathcal{J}^{\mathsf{b}} \models C^{\mathsf{b}}(a)$.

If γ is of the form $r(a, b)$, we have that $\mathcal{J} \models r(a, b)$ iff $(a^{\mathcal{J}}, b^{\mathcal{J}}) \in r^{\mathcal{J}}$ iff $(a^{\mathcal{J}^{\mathsf{b}}}, b^{\mathcal{J}^{\mathsf{b}}}) \in r^{\mathcal{J}^{\mathsf{b}}}$ iff $\mathcal{J}^{\mathsf{b}} \models r(a, b)$.

If $\mathcal{B}$ is of the form $\neg\mathcal{B}_1$, we have that $\mathcal{J} \models \mathcal{B}$ iff not $\mathcal{J} \models \mathcal{B}_1$ iff not $\mathcal{J}^{\mathsf{b}} \models \mathcal{B}^{\mathsf{b}}{}_1$ iff $\mathcal{J}^{\mathsf{b}} \models \mathcal{B}^{\mathsf{b}}$.

If $\mathcal{B}$ is of the form $\mathcal{B}_1 \wedge \mathcal{B}_2$, we have that $\mathcal{J} \models \mathcal{B}$ iff $\mathcal{J} \models \mathcal{B}_1$ and $\mathcal{B}_2$ iff $\mathcal{J}^{\mathsf{b}} \models \mathcal{B}^{\mathsf{b}}{}_1$ and $\mathcal{J}^{\mathsf{b}} \models \mathcal{B}^{\mathsf{b}}{}_2$ iff $\mathcal{J}^{\mathsf{b}} \models \mathcal{B}^{\mathsf{b}}$.

Since $\mathcal{J} \models \mathcal{R}_{\mathsf{O}}$, $\mathcal{J} \models \mathcal{R}_{\mathsf{M}}$ iff $\mathcal{J}^{\mathsf{b}} \models \mathcal{R}_{\mathsf{M}}$ and $\mathcal{J} \models \mathcal{B}$ iff $\mathcal{J}^{\mathsf{b}} \models \mathcal{B}^{\mathsf{b}}$, we have $\mathcal{J} \models \mathfrak{B}$ iff $\mathcal{J}^{\mathsf{b}} \models \mathfrak{B}^{\mathsf{b}}$. □

Note that this lemma yields that consistency of $\mathfrak{B}$ implies consistency of $\mathfrak{B}^{\mathsf{b}}$. However, the converse does not hold as the following example shows.

Example 9. Consider again $\mathfrak{B}_{\text{ex}}$ of Example 7. Take any M-interpretation $\mathcal{H} = (\Gamma, \cdot^{\mathcal{H}})$ with $\Gamma = \{e\}$, $d^{\mathcal{H}} = e$, and $C^{\mathcal{H}} = A^{\mathcal{H}}_{[\![A \sqsubseteq \bot]\!]} = A^{\mathcal{H}}_{[\![A(a)]\!]} = \{e\}$.

Clearly, $\mathcal{H}$ is a model of $\mathfrak{B}^{\mathsf{b}}_{\text{ex}}$. But there is no nested interpretation $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c \in \mathbb{C}})$ with $\mathcal{J} \models \mathfrak{B}_{\text{ex}}$ since this would imply $\mathbb{C} = \Gamma$, and that $\mathcal{I}_e$ is a model of both $A \sqsubseteq \bot$ and $A(a)$, which is not possible. △

Therefore, we need to ensure that the concept names in $\mathsf{Im}(\mathsf{b})$ are not treated independently. For expressing such a restriction on the model $\mathcal{I}$ of $\mathfrak{B}^{\mathsf{b}}$, we adapt a notion of [BGL08; BGL12]. Here it is worth noting that this problem occurs also in much less expressive DLs as $\mathcal{ALC}$ or $\mathcal{EL}^{\bot}$ (i.e. $\mathcal{EL}$ extended with the bottom concept).

Definition 10 [N-*interpretation (weakly) respects* $(\mathcal{U}, \mathcal{Y})$]
Let $\mathcal{U} \subseteq \mathsf{N_C}$ and let $\mathcal{Y} \subseteq \mathcal{P}(\mathcal{U})$. The N-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *respects* $(\mathcal{U}, \mathcal{Y})$ if $\mathcal{Z} = \mathcal{Y}$ where

$$\mathcal{Z} := \{Y \subseteq \mathcal{U} \mid \text{there is some } d \in \Delta^{\mathcal{I}} \text{ with } d \in (C_{\mathcal{U},Y})^{\mathcal{I}}\}$$

and

$$C_{\mathcal{U},Y} := \bigsqcap_{A \in Y} A \sqcap \bigsqcap_{A \in \mathcal{U} \setminus Y} \neg A.$$

It *weakly respects* $(\mathcal{U}, \mathcal{Y})$ if $\mathcal{Z} \subseteq \mathcal{Y}$. ◇

The second decision problem that we use for deciding consistency is needed to make sure that such a set of concept names is admissible in the following sense.

Definition 11 [*Admissibility*]
Let $\mathcal{X} = \{X_1, \ldots, X_k\} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$. We call $\mathcal{X}$ *admissible* if there exist O-interpretations $\mathcal{I}_1 = (\Delta, \cdot^{\mathcal{I}_1}), \ldots, \mathcal{I}_k = (\Delta, \cdot^{\mathcal{I}_k})$ such that

- $x^{\mathcal{I}_i} = x^{\mathcal{I}_j}$ for all $x \in \mathsf{O_I} \cup \mathsf{O_{Crig}} \cup \mathsf{O_{Rrig}}$ and all $i, j \in \{1, \ldots, k\}$, and
- every $\mathcal{I}_i$, $1 \le i \le k$, is a model of the $\mathcal{L}_O$-BKB $\mathfrak{B}_{X_i} = (\mathcal{B}_{X_i}, \mathcal{R}_{\mathsf{O}})$ over O where

$$\mathcal{B}_{X_i} := \bigwedge_{\mathsf{b}(\llbracket\alpha\rrbracket) \in X_i} \alpha \;\wedge \bigwedge_{\mathsf{b}(\llbracket\alpha\rrbracket) \in \mathsf{Im}(\mathsf{b}) \setminus X_i} \neg\alpha.$$

◇

Note that any subset $\mathcal{X}' \subseteq \mathcal{X}$ is admissible if $\mathcal{X}$ is admissible.

Intuitively, the sets X_i in an admissible set $\mathcal{X}$ consist of concept names such that the corresponding o-axioms "fit together". Consider again Example 9. Clearly, the set $\{A_{\llbracket A \sqsubseteq \bot \rrbracket}, A_{\llbracket A(a) \rrbracket}\} \in \mathcal{P}(\mathsf{Im}(\mathsf{b}))$ *cannot* be contained in any admissible set $\mathcal{X}$.

The next definition captures the above mentioned restriction on the model $\mathcal{I}$ of $\mathfrak{B}^{\mathsf{b}}$.

Definition 12 [*Outer consistency*]
Let $\mathcal{X} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$. We call the $\mathcal{L}_M$-BKB $\mathfrak{B}^{\mathsf{b}}$ over M *outer consistent w.r.t.* $\mathcal{X}$ if there exists a model of $\mathfrak{B}^{\mathsf{b}}$ that weakly respects $(\mathsf{Im}(\mathsf{b}), \mathcal{X})$. ◇

The next two lemmas show that the consistency problem in $\mathcal{L}_M\llbracket\mathcal{L}_O\rrbracket$ can be decided by checking whether there is an admissible set $\mathcal{X}$ and the outer abstraction of the given $\mathcal{L}_M\llbracket\mathcal{L}_O\rrbracket$-BKB is outer consistent w.r.t. $\mathcal{X}$.

Lemma 13. *For every* M*-interpretation* $\mathcal{H} = (\Gamma, \cdot^{\mathcal{H}})$*, the following two statements are equivalent:*

1. *There exists a model* $\mathcal{J}$ *of* $\mathfrak{B}$ *with* $\mathcal{J}^{\mathsf{b}} = \mathcal{H}$*.*
2. $\mathcal{H}$ *is a model of* $\mathfrak{B}^{\mathsf{b}}$ *and the set* $\{X_d \mid d \in \Gamma\}$ *is admissible, where* $X_d := \{A \in \mathsf{Im}(\mathsf{b}) \mid d \in A^{\mathcal{H}}\}$*.*

Proof. ($1 \Rightarrow 2$): Let $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c \in \mathbb{C}})$ be a model of $\mathfrak{B}$ with $\mathcal{J}^{\mathsf{b}} = \mathcal{H}$. Since $\mathcal{J}^{\mathsf{b}} = \mathcal{H}$, we have that $\mathbb{C} = \Gamma$. By Lemma 8, we have that $\mathcal{H}$ is a model of $\mathfrak{B}^{\mathsf{b}}$. Moreover, since b is a bijection between m-concepts of the form $\llbracket\alpha\rrbracket$ occurring in $\mathfrak{B}$ and concept names of $\mathsf{M_C}$, we have that $\mathsf{Im}(\mathsf{b})$ is finite, and thus also the set $\mathcal{X} := \{X_d \mid d \in \Gamma\} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$ is finite. Let $\mathcal{X} = \{Y_1, \ldots, Y_k\}$. Since $\mathbb{C} = \Gamma$, there exists an index function $\nu \colon \mathbb{C} \to \{1, \ldots, k\}$ such that $X_c = Y_{\nu(c)}$ for every $c \in \mathbb{C}$, i.e.

$$Y_{\nu(c)} = \big\{\mathsf{b}(\llbracket\alpha\rrbracket) \mid \llbracket\alpha\rrbracket \text{ occurs in } \mathfrak{B} \text{ and } c \in \llbracket\alpha\rrbracket^{\mathcal{H}}\big\} = \big\{\mathsf{b}(\llbracket\alpha\rrbracket) \mid \llbracket\alpha\rrbracket \text{ occurs in } \mathfrak{B} \text{ and } \mathcal{I}_c \models \alpha\big\}.$$

Conversely, for every $\mu \in \{1, \ldots, k\}$, there is an element $c \in \mathbb{C}$ such that $\nu(c) = \mu$. The O-interpretations for showing admissibility of $\mathcal{X}$ are obtained as follows. Take $c_1, \ldots, c_k \in \mathbb{C}$ such that $\nu(c_1) = 1, \ldots, \nu(c_k) = k$. Now, for every i, $1 \le i \le k$, we define the O-interpretation $\mathcal{G}_i := (\Delta, \cdot^{\mathcal{I}_{c_i}})$. Clearly, we have that $G_i \models \mathcal{B}_{Y_i}$ and since $\mathcal{J} \models \mathcal{R}_{\mathsf{O}}$, we have that $G_i \models \mathfrak{B}_{Y_i}$. Moreover, the definition of a nested interpretation yields that $x^{\mathcal{G}_i} = x^{\mathcal{G}_j}$ for all $x \in \mathsf{O_I} \cup \mathsf{O_{Crig}} \cup \mathsf{O_{Rrig}}$ and all $i, j \in \{1, \ldots, k\}$. Hence, the O-interpretations $\mathcal{G}_1, \ldots, \mathcal{G}_k$ attest admissibility of $\mathcal{X}$.

(2 ⇒ 1): Assume that $\mathcal{H} = (\Gamma, \cdot^{\mathcal{H}})$ is a model of $\mathfrak{B}^{\mathsf{b}}$ and that the set $\mathcal{X} := \{X_d \mid d \in \Gamma\}$ is admissible. Again, since $\mathsf{Im}(\mathsf{b})$ is finite, we have that $\mathcal{X} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$ is finite. Let $\mathcal{X} = \{Y_1, \dots, Y_k\}$. Since $\mathcal{X}$ is admissible, there are O-interpretations $\mathcal{G}_1 = (\Delta, \cdot^{\mathcal{G}_1}), \dots, \mathcal{G}_k = (\Delta, \cdot^{\mathcal{G}_k})$ such that $\mathcal{G}_i \models \mathfrak{B}_{Y_i}$ and $x^{\mathcal{G}_i} = x^{\mathcal{G}_j}$ for all $x \in \mathsf{O_I} \cup \mathsf{O_{Crig}} \cup \mathsf{O_{Rrig}}$ and all $i, j \in \{1, \dots, k\}$. Furthermore, there exists an index function $\nu\colon \Gamma \to \{1, \dots, k\}$ such that $Y_{\nu(d)} = X_d$ for every $d \in \Gamma$. We define a nested interpretation $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c \in \mathbb{C}})$ as follows:

- $\mathbb{C} := \Gamma$;
- $x^{\mathcal{J}} := x^{\mathcal{H}}$ for every $x \in \mathsf{M_C} \cup \mathsf{M_R} \cup \mathsf{M_I}$; and
- $x^{\mathcal{I}_c} := x^{\mathcal{G}_{\nu(c)}}$ for every $x \in \mathsf{O_C} \cup \mathsf{O_R} \cup \mathsf{O_I}$ and every $c \in \mathbb{C}$.

By construction of $\mathcal{J}$, we have that $x^{\mathcal{J}^{\mathsf{b}}} = x^{\mathcal{H}}$ for every $x \in \mathsf{M_R} \cup \mathsf{M_I} \cup (\mathsf{M_C} \setminus \mathsf{Im}(\mathsf{b}))$. Let $A \in \mathsf{Im}(\mathsf{b})$, and let $\mathsf{b}^{-1}(A) = [\![\alpha]\!]$. We have for every $d \in \Gamma = \mathbb{C}$ that $d \in A^{\mathcal{J}^{\mathsf{b}}}$ iff $d \in (\mathsf{b}^{-1}(A))^{\mathcal{J}}$ iff $d \in [\![\alpha]\!]^{\mathcal{J}}$ iff $\mathcal{I}_d \models \alpha$ iff $\mathcal{G}_{\nu(d)} \models \alpha$ iff $\mathsf{b}([\![\alpha]\!]) = A \in Y_{\nu(d)}$ (since $\mathcal{G}_{\nu(d)} \models \mathcal{B}_{Y_{\nu(d)}}$) iff $A \in X_d$ iff $d \in A^{\mathcal{H}}$. Hence, we have $\mathcal{J}^{\mathsf{b}} = \mathcal{H}$. Since $\mathcal{H}$ is a model of $\mathfrak{B}^{\mathsf{b}}$ and, by construction of $\mathcal{J}$, $\mathcal{J}$ is a model of $\mathcal{R}_{\mathsf{O}}$, we have by Lemma 8 that $\mathcal{J}$ is a model of $\mathfrak{B}$. ☐

The following lemma is a consequence of the previous one.

Lemma 14. *The $\mathcal{L}_M[\![\mathcal{L}_O]\!]$-BKB $\mathfrak{B}$ is consistent iff there is a set $\mathcal{X} = \{X_1, \dots, X_k\} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$ such that*

1. *$\mathcal{X}$ is admissible, and*
2. *$\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$.*

Proof. ($\Longrightarrow$): Let $\mathcal{J}$ be a model of $\mathfrak{B}$, and let $\mathcal{J}^{\mathsf{b}} = (\mathbb{C}, \cdot^{\mathcal{J}^{\mathsf{b}}})$. By Lemma 13, we have that $\mathcal{J}^{\mathsf{b}}$ is a model of $\mathfrak{B}^{\mathsf{b}}$, and the set $\mathcal{X} := \{X_c \mid c \in \mathbb{C}\}$ is admissible. By construction, $\mathcal{J}^{\mathsf{b}}$ weakly respects $(\mathsf{Im}(\mathsf{b}), \mathcal{X})$, and hence $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$.

($\Longleftarrow$): Let $\mathcal{X} = \{X_1, \dots, X_k\} \subseteq \mathcal{P}(\mathsf{Im}(b))$ such that $\mathcal{X}$ is admissible and $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$. Hence there is a model $\mathcal{G} = (\mathbb{C}, \cdot^{\mathcal{G}})$ of $\mathfrak{B}^{\mathsf{b}}$ that weakly respects $(\mathsf{Im}(\mathsf{b}), \mathcal{X})$. We define $\mathcal{X}' := \{Y_c \mid c \in \mathbb{C}\}$, where $Y_c := \{A \in \mathsf{Im}(b) \mid c \in A^{\mathcal{G}}\}$. Since $\mathcal{G}$ weakly respects $(\mathsf{Im}(\mathsf{b}), \mathcal{X})$ and $c \in (C_{\mathsf{Im}(b), Y_c})^{\mathcal{G}}$ for every $c \in \mathbb{C}$, we have that $\mathcal{X}' \subseteq \mathcal{X}$. Since $\mathcal{X}$ is admissible, this yields admissibility of $\mathcal{X}'$. Lemma 13 yields now consistency of $\mathfrak{B}$. ☐

To obtain a decision procedure for $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ consistency, we have to non-deterministically guess or construct the set $\mathcal{X}$, and then check the two conditions of Lemma 14. Beforehand, we focus on how to decide the second condition. For that, assume that a set $\mathcal{X} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$ is given.

Lemma 15. *Deciding whether $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$ can be done in time exponential in the size of $\mathfrak{B}^{\mathsf{b}}$ and linear in size of $\mathcal{X}$.*

Proof. It is enough to show that deciding whether $\mathfrak{B}^{\mathsf{b}}$ has a model that weakly respects $(\mathsf{Im}(\mathsf{b}), \mathcal{X})$ can be done in time exponential in the size of $\mathfrak{B}^{\mathsf{b}}$ and linear in the size of $\mathcal{X}$. It is not hard to see that we can adapt the notion of a quasimodel respecting a pair $(\mathcal{U}, \mathcal{Y})$ of [Lip14] to a quasimodel *weakly* respecting $(\mathcal{U}, \mathcal{Y})$. Indeed, one just has to drop Condition (i) in Definition 3.25 of [Lip14]. Then, the proof of Lemma 3.26 there can be adapted such that our claim follows. This is done by dropping one check in Step 4 of the algorithm of [Lip14]. ☐

Using this lemma, we provide decision procedures for $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ consistency. However, these depend also on the first condition of Lemma 14. We take care of this differently depending on which names are allowed to be rigid.

3.1 Consistency in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ without rigid names

In this section, we consider the case where no rigid concept names or role names are allowed. So we fix $\mathsf{O}_{\mathsf{Crig}} = \mathsf{O}_{\mathsf{Rrig}} = \varnothing$.

Theorem 16. *The consistency problem in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ is in* Exp *if $\mathsf{O}_{\mathsf{Crig}} = \mathsf{O}_{\mathsf{Rrig}} = \varnothing$.*

Proof. Let $\mathfrak{B}$ be a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-BKB and $\mathfrak{B}^{\mathsf{b}}$ its outer abstraction. We can decide consistency of $\mathfrak{B}$ using Lemma 14. We define $\mathcal{X} := \{X \subseteq \mathsf{Im}(\mathsf{b}) \mid \mathfrak{B}_X \text{ is consistent}\}$ where $\mathfrak{B}_X$ is defined as in Definition 11. We first show that $\mathcal{X} = \{X_1, \dots, X_k\}$ is admissible. Let $\mathcal{I}_i$ be a model of $\mathfrak{B}_{X_i}$, which exists since $\mathfrak{B}_{X_i}$ is consistent. Due to the Löwenheim-Skolem theorem, we can assume that all models $\mathcal{I}_i$, $1 \leq i \leq k$, have a countably infinite domain. Thus, w.l.o.g. we can assume that all models have the same domain Δ. Furthermore, we can assume that individual names are interpreted the same. Since $\mathsf{O}_{\mathsf{Crig}} = \mathsf{O}_{\mathsf{Rrig}} = \varnothing$, the set $\mathcal{X}$ fulfills all conditions of Definition 11 for admissibility.

Thus, if $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$, then we have by Lemma 14 that $\mathfrak{B}$ is consistent. Conversely, assume that $\mathfrak{B}$ is consistent. Then, by Lemma 14, there is an admissible set $\mathcal{X}' \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$ and $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}'$. Since $\mathcal{X}$ is the maximal admissible subset of $\mathcal{P}(\mathsf{Im}(\mathsf{b}))$, we have $\mathcal{X}' \subseteq \mathcal{X}$. If $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}'$, it is also outer consistent w.r.t. $\mathcal{X}$. Hence, $\mathfrak{B}$ is consistent iff $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$, which yields a decision procedure for the consistency problem in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$.

It remains to analyze the complexity. There are exponentially many $X \in \mathcal{P}(\mathsf{Im}(\mathsf{b}))$, but each $\mathcal{SHOQ}$-BKB $\mathfrak{B}_X$ can be constructed in time polynomial in the size of $\mathfrak{B}$. We can decide consistency of $\mathfrak{B}_X$ in time exponential [Lip14]. Thus, the set $\mathcal{X}$ can be constructed in time exponential in the size of $\mathfrak{B}$ and it is of exponential size. Due to Lemma 15, deciding whether $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$ can be done in time exponential in the size of $\mathfrak{B}^{\mathsf{b}}$ and linear in the size of $\mathcal{X}$. Thus, overall we can decide the consistency problem in exponential time. $\square$

Together with the lower bounds shown in Section 4, we obtain Exp-completeness for the consistency problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ for $\mathcal{L}_M$ and $\mathcal{L}_O$ being DLs between $\mathcal{ALC}$ and $\mathcal{SHOQ}$ if $\mathsf{O}_{\mathsf{Crig}} = \mathsf{O}_{\mathsf{Rrig}} = \varnothing$.

3.2 Consistency in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ with rigid concept and role names

In this section, we consider the case where rigid concept and role names are present. So we fix $\mathsf{O}_{\mathsf{Crig}} \neq \varnothing$ and $\mathsf{O}_{\mathsf{Rrig}} \neq \varnothing$.

Theorem 17. *The consistency problem in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ is in* 2Exp *if $\mathsf{O}_{\mathsf{Crig}} \neq \varnothing$ and $\mathsf{O}_{\mathsf{Rrig}} \neq \varnothing$.*

Proof. Let $\mathfrak{B} = (\mathcal{B}, \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ be a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-BKB and $\mathfrak{B}^{\mathsf{b}} = (\mathcal{B}^{\mathsf{b}}, \mathcal{R}_{\mathsf{M}})$ its outer abstraction. We can decide consistency of $\mathfrak{B}$ using Lemma 14. For that, we enumerate all sets $\mathcal{X} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$, which can be done in time doubly exponential in $\mathfrak{B}$. For each of these sets $\mathcal{X} = \{X_1, \dots, X_k\}$, we check whether $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$, which can be done in time exponential in the size of $\mathfrak{B}^{\mathsf{b}}$ and linear in the size of $\mathcal{X}$. Then, we check $\mathcal{X}$ for admissibility using the renaming technique of [BGL08; BGL12]. For every i, $1 \leq i \leq k$, every flexible concept name A occurring in $\mathcal{B}^{\mathsf{b}}$, and every flexible role name r occurring in $\mathcal{B}^{\mathsf{b}}$ or $\mathcal{R}_{\mathsf{O}}$, we introduce copies $A^{(i)}$ and $r^{(i)}$. The $\mathcal{SHOQ}$-BKB $\mathfrak{B}_{X_i}^{(i)} = (\mathcal{B}_{X_i}^{(i)}, {\mathcal{R}_{\mathsf{O}}}^{(i)})$ over O is obtained from $\mathfrak{B}_{X_i}$ (see Definition 11) by replacing every occurrence of a flexible name x by $x^{(i)}$. We define

$$\mathfrak{B}_{\mathcal{X}} := \left(\bigwedge\nolimits_{1 \leq i \leq k} \mathcal{B}_{X_i}^{(i)}, \bigcup\nolimits_{1 \leq i \leq k} {\mathcal{R}_{\mathsf{O}}}^{(i)}\right).$$

It is not hard to verify (using arguments of [Lip14]) that $\mathcal{X}$ is admissible iff $\mathfrak{B}_{\mathcal{X}}$ is consistent. Note that $\mathfrak{B}_{\mathcal{X}}$ is of size at most exponential in $\mathfrak{B}$ and can be constructed in exponential time. Moreover, consistency of $\mathfrak{B}_{\mathcal{X}}$ can be decided in time exponential in the size of $\mathfrak{B}_{\mathcal{X}}$ [Lip14], and thus in time doubly exponential in the size of $\mathfrak{B}$. ☐

Together with the lower bounds shown in Section 4, we obtain 2Exp-completeness for the consistency problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ for $\mathcal{L}_M$ and $\mathcal{L}_O$ being DLs between $\mathcal{ALC}$ and $\mathcal{SHOQ}$ if $\mathsf{O}_{\mathsf{Crig}} \neq \emptyset$ and $\mathsf{O}_{\mathsf{Rrig}} \neq \emptyset$.

3.3 Consistency in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ with only rigid concept names

In this section, we consider the case where rigid concept are present, but rigid role names are not allowed. So we fix $\mathsf{O}_{\mathsf{Crig}} \neq \emptyset$ but $\mathsf{O}_{\mathsf{Rrig}} = \emptyset$.

Theorem 18. *The consistency problem in $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$ is in* NExp *if $\mathsf{O}_{\mathsf{Crig}} \neq \emptyset$ and $\mathsf{O}_{\mathsf{Rrig}} = \emptyset$.*

Proof. Let $\mathfrak{B} = (\mathcal{B}, \mathcal{R}_{\mathsf{O}}, \mathcal{R}_{\mathsf{M}})$ be a $\mathcal{SHOQ}[\![\mathcal{SHOQ}]\!]$-BKB and $\mathfrak{B}^{\mathsf{b}} = (\mathcal{B}^{\mathsf{b}}, \mathcal{R}_{\mathsf{M}})$ its outer abstraction. We can decide consistency of $\mathfrak{B}$ using Lemma 14. We first non-deterministically guess the set $\mathcal{X} = \{X_1, \dots, X_k\} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$, which is of size at most exponential in $\mathfrak{B}$. Due to Lemma 15 we can check whether $\mathfrak{B}^{\mathsf{b}}$ is outer consistent w.r.t. $\mathcal{X}$ in time exponential in the size of $\mathfrak{B}^{\mathsf{b}}$ and linear in the size of $\mathcal{X}$. It remains to check $\mathcal{X}$ for admissibility. For that let $\mathsf{O}_{\mathsf{Crig}}(\mathcal{B}) \subseteq \mathsf{O}_{\mathsf{Crig}}$ and $\mathsf{O}_{\mathsf{I}}(\mathcal{B}) \subseteq \mathsf{O}_{\mathsf{I}}$ be the sets of all rigid concept names and individual names occurring in $\mathcal{B}$, respectively. As done in [BGL08; BGL12] we non-deterministically guess a set $\mathcal{Y} \subseteq \mathcal{P}(\mathsf{O}_{\mathsf{Crig}}(\mathcal{B}))$ and a mapping $\kappa\colon \mathsf{O}_{\mathsf{I}}(\mathcal{B}) \to \mathcal{Y}$ which also can be done in time exponential in the size of $\mathfrak{B}$. Using the same arguments as in [BGL08; BGL12] we can show that $\mathcal{X}$ is admissible iff

$$\widehat{\mathfrak{B}}_{X_i} := \left(\mathcal{B}_{X_i} \wedge \bigwedge_{a \in \mathsf{O}_{\mathsf{I}}(\mathcal{B})} \left(\bigsqcap_{A \in \kappa(a)} A \sqcap \bigsqcap_{A \in \mathsf{O}_{\mathsf{Crig}}(\mathcal{B}) \setminus \kappa(a)} \neg A\right)(a),\ \mathcal{R}_{\mathsf{O}}\right)$$

has a model that respects $(\mathsf{O}_{\mathsf{Crig}}(\mathcal{B}), \mathcal{Y})$, for all $1 \leq i \leq k$. The $\mathcal{SHOQ}$-BKB $\widehat{\mathfrak{B}}_{X_i}$ is of size polynomial in the size of $\mathcal{B}$ and can be constructed in time exponential in the size of $\mathcal{B}$. We can check if $\widehat{\mathfrak{B}}_{X_i}$ has a model that respects $(\mathsf{O}_{\mathsf{Crig}}(\mathcal{B}), \mathcal{Y})$ in time exponential in the size of $\widehat{\mathfrak{B}}_{X_i}$ [BGL08; BGL12], and thus exponential in the size of $\mathfrak{B}$. ☐

Together with the lower bounds shown in Section 4, we obtain NExp-completeness for the consistency problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ for $\mathcal{L}_M$ and $\mathcal{L}_O$ being DLs between $\mathcal{ALC}$ and $\mathcal{SHOQ}$ if $\mathsf{O}_{\mathsf{Crig}} \neq \emptyset$ and $\mathsf{O}_{\mathsf{Rrig}} = \emptyset$.

Summing up the results, we obtain the following corollary.

Corollary 19. *For all $\mathcal{L}_M$, $\mathcal{L}_O$ between $\mathcal{ALC}$ and $\mathcal{SHOQ}$, the consistency problem in $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ is*

- Exp-*complete if $\mathsf{O}_{\mathsf{Crig}} = \emptyset$ and $\mathsf{O}_{\mathsf{Rrig}} = \emptyset$,*
- NExp-*complete if $\mathsf{O}_{\mathsf{Crig}} \neq \emptyset$ and $\mathsf{O}_{\mathsf{Rrig}} = \emptyset$, and*
- 2Exp-*complete if $\mathsf{O}_{\mathsf{Crig}} \neq \emptyset$ and $\mathsf{O}_{\mathsf{Rrig}} \neq \emptyset$.*

4 The Case of $\mathcal{EL}$: $\mathcal{L}_M[\![\mathcal{EL}]\!]$ and $\mathcal{EL}[\![\mathcal{L}_O]\!]$

In this section, we give some complexity results for context DLs $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ where $\mathcal{L}_M$ or $\mathcal{L}_O$ are $\mathcal{EL}$. In Section 4.1, we consider $\mathcal{L}_M[\![\mathcal{EL}]\!]$ where $\mathcal{L}_M$ is between $\mathcal{ALC}$ and $\mathcal{SHOQ}$. Then, in Section 4.2, we consider the remaining context DLs $\mathcal{EL}[\![\mathcal{L}_O]\!]$ where $\mathcal{L}_O$ is either $\mathcal{EL}$ or between $\mathcal{ALC}$ and $\mathcal{SHOQ}$.

4.1 The Context DLs $\mathcal{L}_M[\![\mathcal{EL}]\!]$

In this section, we consider $\mathcal{L}_M[\![\mathcal{EL}]\!]$ where $\mathcal{L}_M$ is between $\mathcal{ALC}$ and $\mathcal{SHOQ}$. The lower bounds already hold for $\mathcal{ALC}[\![\mathcal{EL}]\!]$.

Theorem 20. *The consistency problem in* $\mathcal{ALC}[\![\mathcal{EL}]\!]$ *is* Exp*-hard if* $\mathsf{O}_{\mathsf{Crig}} = \mathsf{O}_{\mathsf{Rrig}} = \varnothing$.

Proof. Deciding whether a given conjunction of $\mathcal{ALC}$-axioms $\mathcal{B}$ is consistent is Exp-hard [Sch91]. Obviously, $\mathcal{B}$ is also an $\mathcal{ALC}[\![\mathcal{EL}]\!]$-BKB. □

For the cases of rigid names, the lower bounds of NExp are obtained by a careful reduction of the satisfiability problem in the temporalized DL $\mathcal{EL}$-LTL [BT15b; BT15a], which is a fragment of $\mathcal{ALC}$-LTL introduced in [BGL08; BGL12]. For the sake of completeness, we recall the basic definitions of $\mathcal{L}$-LTL here, where $\mathcal{L}$ is a DL.

Definition 21 [*Syntax of $\mathcal{L}$-LTL*]
$\mathcal{L}$-LTL-formulas over O are defined by induction:

- if α is an $\mathcal{L}$-axiom over O, then α is an $\mathcal{L}$-LTL-formula, and
- if ϕ, ψ are $\mathcal{L}$-LTL-formulas over O, then so are $\phi \wedge \psi$, $\neg\phi$, $\phi \,\mathsf{U}\, \psi$, $\mathsf{X}\phi$, and
- nothing else is an $\mathcal{L}$-LTL-formula. ◇

The usual abbreviations are used:

- $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$,
- true for $A(a) \vee \neg A(a)$,
- $\Diamond\phi$ for $\mathsf{true}\,\mathsf{U}\,\phi$, and
- $\Box\phi$ for $\neg\Diamond\neg\phi$.

The semantics of $\mathcal{L}$-LTL is based on DL-LTL-structures. These are sequences of O-interpretations over the same non-empty domain that additionally respect rigid names and the rigid individual assumption.

Definition 22 [*DL-LTL-structure*]
A *DL-LTL-structure over* O is a sequence $\mathfrak{I} = (\mathcal{I}_i)_{i\geq 0}$ of O-interpretations $(\Delta, \cdot^{\mathcal{I}_i})$ such that $x^{\mathcal{I}_i} = x^{\mathcal{I}_j}$ holds for all $x \in \mathsf{O}_{\mathsf{Crig}} \cup \mathsf{O}_{\mathsf{Rrig}} \cup \mathsf{O}_{\mathsf{I}}$, $i, j > 0$. ◇

We are now ready to define the semantics of $\mathcal{L}$-LTL.

Definition 23 [*Semantics of $\mathcal{L}$-LTL*]
The validity of an $\mathcal{L}$-LTL-formula ϕ in a DL-LTL-structure $\mathfrak{I} = (\mathcal{I}_i)_{i\geq 0}$ at time $i \geq 0$, denoted by $\mathfrak{I}, i \models \varphi$, is defined inductively:

$\mathfrak{I}, i \models \alpha$ iff $\mathcal{I}_i \models \alpha$ where α is an $\mathcal{ALC}$-axiom over O,
$\mathfrak{I}, i \models \phi \wedge \psi$ iff $\mathfrak{I}, i \models \phi$ and $\mathfrak{I}, i \models \psi$,
$\mathfrak{I}, i \models \neg\phi$ iff not $\mathfrak{I}, i \models \phi$,
$\mathfrak{I}, i \models \mathsf{X}\phi$ iff $\mathfrak{I}, i+1 \models \phi$,
$\mathfrak{I}, i \models \phi \,\mathsf{U}\, \psi$ iff there is $k \geq i$ such that $\mathfrak{I}, k \models \psi$ and $\mathfrak{I}, j \models \phi$ for all j with $i \leq j < k$.

We call an $\mathcal{L}$-LTL-structure $\mathfrak{I}$ a *model of* ϕ if $\mathfrak{I}, 0 \models \phi$. The *satisfiability problem in $\mathcal{L}$-LTL* is the question whether a given $\mathcal{L}$-LTL-formula ϕ has a model. ◇

In [BT15b; BT15a], it is shown that the satisfiability problem in $\mathcal{EL}$-LTL is NExp-hard as soon as rigid concept names are available. We reduce the satisfiability problem in $\mathcal{EL}$-LTL to the consistency problem in $\mathcal{ALC}[\![\mathcal{EL}]\!]$ to obtain the lower bounds of NExp, where we use the fact that the lower bounds of [BT15b; BT15a] hold already for a syntactically restricted fragment of $\mathcal{EL}$-LTL.

Theorem 24. *The consistency problem in $\mathcal{ALC}[\![\mathcal{EL}]\!]$ is* NEXP*-hard if* $\mathsf{O}_{\mathsf{Crig}} \neq \varnothing$ *and* $\mathsf{O}_{\mathsf{Rrig}} = \varnothing$.

Proof. In fact, the lower bounds hold for $\mathcal{EL}$-LTL-formulas of the form $\Box\phi$ where ϕ is an $\mathcal{EL}$-LTL-formula that contains only X as temporal operator [BT15a].

Let $\Box\phi$ be such an $\mathcal{EL}$-LTL-formula over O. We obtain now an m-concept C_ϕ from ϕ by replacing $\mathcal{EL}$-axioms α by $[\![\alpha]\!]$, $\wedge$ by $\sqcap$, and subformulas of the form $\mathsf{X}\psi$ by $\forall t.\psi \sqcap \exists t.\psi$, where $t \in \mathsf{M_R}$ is arbitrary but fixed.

Claim: $\Box\phi$ is satisfiable iff $\mathcal{B} = \top \sqsubseteq C_\phi \sqcap \exists t.\top$ is consistent.

Proof: ($\Longrightarrow$): Take any DL-LTL-structure $\mathfrak{I} = (\Delta, \cdot^{\mathcal{I}_i})_{i \geq 0}$ with $\mathfrak{I}, 0 \models \Box\phi$. We define the nested interpretation $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c \in \mathbb{C}})$ as follows:

$$\begin{aligned} \mathbb{C} &:= \{c_i \mid i \geq 0\}, \\ \cdot^{\mathcal{I}_{c_i}} &:= \cdot^{\mathcal{I}_i}, \\ t^{\mathcal{J}} &:= \{(c_i, c_{i+1})\}. \end{aligned}$$

We show now that for every $i \geq 0$, we have $\mathfrak{I}, i \models \phi$ iff $c_i \in C_\phi^{\mathcal{J}}$ by induction on the structure of ϕ.

If ϕ is an $\mathcal{EL}$-axiom over O, then we have $\mathfrak{I}, i \models \phi$ iff $\mathcal{I}_i \models \phi$ iff $\mathcal{I}_{c_i} \models \phi$ iff $c_i \in [\![\phi]\!]^{\mathcal{J}}$.

If ϕ is of the form $\neg\psi$, then we have $\mathfrak{I}, i \models \phi$ iff $\mathfrak{I}, i \not\models \psi$ iff $c_i \notin C_\psi^{\mathcal{J}}$ iff $c_i \in (\neg C_\psi)^{\mathcal{J}} = C_\phi^{\mathcal{J}}$.

If ϕ is of the form $\psi_1 \wedge \psi_2$, the claim follows by similar arguments.

If ϕ is of the form $\mathsf{X}\psi$, we have that $\mathfrak{I}, i \models \phi$ iff $\mathfrak{I}, i+1 \models \psi$ iff $c_{i+1} \in C_\psi^{\mathcal{J}}$ iff $c_i \in (\forall t.C_\psi \sqcap \exists t.C_\psi)^{\mathcal{J}}$ iff $c_i \in C_\phi^{\mathcal{J}}$.

It follows that $\mathfrak{I}, 0 \models \Box\phi$ iff $\mathcal{J} \models \top \sqsubseteq C_\phi$.

Furthermore, since $(c_i, c_{i+1}) \in t^{\mathcal{J}}$, we have $c_i \in (\exists t.\top)^{\mathcal{J}}$. Thus, $\mathcal{J} \models \top \sqsubseteq \exists t.\top$.

($\Longleftarrow$): Take any nested interpretation $\mathcal{J} = (\mathbb{C}, \cdot^{\mathcal{J}}, \Delta, (\cdot^{\mathcal{I}_c})_{c \in \mathbb{C}})$ that is a model of $\top \sqsubseteq C_\phi \sqcap \exists t.\top$. Let P be an infinite path $P = c_0 c_1 \ldots$ with $c_i \in \mathbb{C}$ and $(c_i, c_{i+1}) \in t^{\mathcal{J}}$ for every $i \geq 0$. Such a path exists, because $\mathcal{J} \models \top \sqsubseteq \exists t.\top$. We define the nested interpretation $\mathcal{J}_P := (\{c_i \mid i \geq 0\}, \cdot^{\mathcal{J}_P}, \Delta, (\cdot^{\mathcal{I}_{c_i}})_{i \geq 0})$ where $\cdot^{\mathcal{J}_P}$ is the restriction of $\cdot^{\mathcal{J}}$ to the domain $\{c_i \mid i \geq 0\}$.

By construction we have that $\mathcal{J}_P \models \top \sqsubseteq \exists t.\top$. We show by a simple case distinction that $\mathcal{J}_P \models \top \sqsubseteq C_\phi$.

If C_ϕ does not contain any role name $r \in \mathsf{M_R}$, the restriction on the set of worlds preserves the entailment relation. Otherwise, C_ϕ is of the form $\forall t.C_\psi \sqcap \exists t.C_\psi$. Since $\mathcal{J}_P \models \top \sqsubseteq \exists r.\top$, $\mathcal{J}_P \models \top \sqsubseteq C_\psi$, and there is only one t-successor, we have $\mathcal{J}_P \models \top \sqsubseteq C_\phi$.

Hence, $\mathcal{J}_P \models \top \sqsubseteq C_\phi \sqcap \exists t.\top$.

We define the DL-LTL-structure $\mathfrak{I}$ over O as $\mathfrak{I} := (\Delta, \cdot^{\mathcal{I}_i})_{i \geq 0}$ where $\cdot^{\mathcal{I}_i} := \cdot^{\mathcal{I}_{c_i}}$.

Again we show that for every $i \geq 0$, that we have $c_i \in C_\phi^{\mathcal{J}_P}$ iff $\mathfrak{I}, i \models \phi$ by induction on the structure of ϕ.

If ϕ is an $\mathcal{EL}$-axiom over O, we have $c_i \in [\![\phi]\!]^{\mathcal{J}_P}$ iff $\mathcal{I}_{c_i} \models \phi$ iff $\mathcal{I}_i \models \phi$ iff $\mathfrak{I}, i \models \phi$.

If ϕ is of the form $\neg\psi$, then we have $c_i \in C_\phi^{\mathcal{J}_P}$ iff $c_i \notin C_\psi^{\mathcal{J}_P}$ iff $\mathfrak{I}, i \not\models \psi$ iff $\mathfrak{I}, i \models \phi$.

If ϕ is of the form $\psi_1 \wedge \psi_2$, the claim follows by similar arguments.

If ϕ is of the form $\mathsf{X}\psi$, we have that $c_i \in C_\phi^{\mathcal{J}_P}$ iff $c_i \in (\forall t.C_\psi \sqcap \exists t.C_\psi)^{\mathcal{J}_P}$ iff $c_{i+1} \in C_\psi^{\mathcal{J}_P}$ iff $\mathfrak{I}, i+1 \models \psi$ iff $\mathfrak{I}, i \models \phi$.

It follows that $\mathcal{J}_P \models \top \sqsubseteq C_\phi$ iff $\mathfrak{I}, 0 \models \Box\phi$. ⁄

This claim yields the lower bound of NExp for the consistency problem in $\mathcal{ALC}[\![\mathcal{EL}]\!]$ if $\mathsf{O}_{\mathsf{Crig}} \neq \varnothing$. □

Next, we prove the upper bound of NExp for the consistency problem in the case of rigid names.

Theorem 25. *The consistency problem in $\mathcal{SHOQ}[\![\mathcal{EL}]\!]$ is in* NExp *if $\mathsf{O}_{\mathsf{Crig}} \neq \varnothing$ and $\mathsf{O}_{\mathsf{Rrig}} \neq \varnothing$.*

Proof. We again use Lemma 14. First, we non-deterministically guess a set $\mathcal{X} \subseteq \mathcal{P}(\mathsf{Im}(\mathsf{b}))$ and construct the $\mathcal{EL}$-BKB $\mathcal{B}_\mathcal{X}$ over O as in the proof of Theorem 17, which is actually a conjunction of $\mathcal{EL}$-literals over O, i.e. of (negated) $\mathcal{EL}$-axioms over O. The following claim shows that consistency of$\mathcal{B}_\mathcal{X}$ can be reduced to consistency of a conjunction of $\mathcal{ELO}^\perp$-axioms over O, where $\mathcal{ELO}^\perp$ is the extension of $\mathcal{EL}$ with nominals and the bottom concept.

Claim: For every conjunction of $\mathcal{EL}$-literals $\mathcal{B}$ over O, there exists an equisatisfiable conjunction $\mathcal{B}'$ of $\mathcal{ELO}^\perp$-axioms over O.

Proof: Let $\mathcal{B}$ be a conjunction of $\mathcal{EL}$-literals over O, i.e.

$$\mathcal{B} = \alpha_1 \wedge \cdots \wedge \alpha_n \wedge \neg\beta_1 \wedge \cdots \wedge \neg\beta_m$$

where α_i, $1 \leq i \leq n$, β_j, $1 \leq j \leq m$ are $\mathcal{EL}$-axioms over O. We define $\mathcal{B}'$ as follows:

$$\mathcal{B}' = \alpha_1 \wedge \cdots \wedge \alpha_n \wedge \gamma_1 \wedge \cdots \wedge \gamma_m,$$

where

$$\gamma_i := \begin{cases} C(a_i) \wedge D'(a_i) \wedge D \sqcap D' \sqsubseteq \bot & \text{if } \beta_i = C \sqsubseteq D, \\ A'(a) \wedge A \sqcap A' \sqsubseteq \bot & \text{if } \beta_i = A(a), \text{ and} \\ \{a\} \sqcap \exists r.\{b\} \sqsubseteq \bot & \text{if } \beta_i = r(a,b) \end{cases}$$

with A', D' being fresh concept names and a_i being fresh individual names. It is easy to see that if an O-interpretation $\mathcal{I}$ is a model of $\neg\beta_1 \wedge \cdots \wedge \neg\beta_m$, there exists an extension of $\mathcal{I}$ that is a model of $\gamma_1 \wedge \cdots \wedge \gamma_m$. Conversely, if an O-interpretation $\mathcal{I}'$ is a model of $\gamma_1 \wedge \cdots \wedge \gamma_m$, it is also a model of $\neg\beta_1 \wedge \cdots \wedge \neg\beta_m$. Hence $\mathcal{B}$ and $\mathcal{B}'$ are equisatisfiable. ⁄

By this claim and the fact that consistency of conjunctions of $\mathcal{ELO}^\perp$-axioms can be decided in polynomial time [BBL05], we obtain our claimed upper bound. □

Summming up the results of this section, we obtain the following corollary.

Corollary 26. *For all $\mathcal{L}_M$ between $\mathcal{ALC}$ and $\mathcal{SHOQ}$, the consistency problem in $\mathcal{L}_M[\![\mathcal{EL}]\!]$ is*

- Exp*-complete if $\mathsf{O}_{\mathsf{Crig}} = \varnothing$ and $\mathsf{O}_{\mathsf{Rrig}} = \varnothing$, and*
- NExp*-complete otherwise.*

Proof. The lower bounds follow from Theorems 20 and 24. The upper bound of Exp in the case $\mathsf{O}_{\mathsf{Crig}} = \mathsf{O}_{\mathsf{Rrig}} = \varnothing$ follows immediately from Theorem 16, whereas the upper bound of NExp follows from Theorem 25. □

4.2 The Context DLs $\mathcal{EL}[\![\mathcal{L}_O]\!]$

In this section, we consider $\mathcal{EL}[\![\mathcal{L}_O]\!]$ where $\mathcal{L}_M$ is either $\mathcal{EL}$ or between $\mathcal{ALC}$ and $\mathcal{SHOQ}$. Instead of considering $\mathcal{EL}[\![\mathcal{L}_O]\!]$-BKBs, we allow only *conjunctions* of m-axioms. Then the consistency problem becomes trivial in the case of $\mathcal{EL}[\![\mathcal{EL}]\!]$ since all $\mathcal{EL}[\![\mathcal{EL}]\!]$-BKBs are consistent, as $\mathcal{EL}$ lacks to express contradictions. This restriction, however, does not yield a better complexity in the cases of $\mathcal{EL}[\![\mathcal{L}_O]\!]$, where $\mathcal{L}_O$ is between $\mathcal{ALC}$ and $\mathcal{SHOQ}$.

Next, we show the lower bounds for the consistency problem in $\mathcal{EL}[\![\mathcal{ALC}]\!]$. We again distinguish the three cases of which names are allowed to be rigid.

Theorem 27. *The consistency problem in $\mathcal{EL}[\![\mathcal{ALC}]\!]$ is* Exp*-hard if $\mathsf{O_{Crig}} = \mathsf{O_{Rrig}} = \emptyset$.*

Proof. Deciding whether a given conjunction $\mathcal{B} = \alpha_1 \wedge \cdots \wedge \alpha_n$ of $\mathcal{ALC}$-axioms is consistent is Exp-hard [Sch91]. Obviously, $\mathcal{B}$ is consistent iff the $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKB $([\![\alpha_1]\!] \sqcap \cdots \sqcap [\![\alpha_n]\!])(a)$ is consistent, where $a \in \mathsf{M_I}$. $\square$

For the case of rigid role names, we have lower bounds of 2Exp.

Theorem 28. *The consistency problem in $\mathcal{EL}[\![\mathcal{ALC}]\!]$ is* 2Exp*-hard if $\mathsf{O_{Crig}} \neq \emptyset$ and $\mathsf{O_{Rrig}} \neq \emptyset$.*

Proof. To show the lower bound, we adapt the proof ideas of [BGL08; BGL12], and reduce the word problem for exponentially space-bounded alternating Turing machines (i.e. is a given word w accepted by the machine M) to the consistency problem in $\mathcal{EL}[\![\mathcal{ALC}]\!]$ with rigid roles, i.e. $\mathsf{O_{Rrig}} \neq \emptyset$. In [BGL08; BGL12], a reduction was provided to show 2Exp-hardness for the temporalized DL $\mathcal{ALC}$-LTL in the presence of rigid roles. Here, we mimic the properties of the time dimension that are important for the reduction using a role name $t \in \mathsf{M_R}$.

Our $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKB is the conjunction of the $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKBs introduced below. First, we ensure that we never have a "last" time point:

$$\top \sqsubseteq \exists t.\top$$

Note that in the corresponding model, we do not enforce a t-chain since cycles are not prohibited. This, however, is not important in the reduction.

The $\mathcal{ALC}$-LTL-formula obtained in the reduction of [BGL08; BGL12] is a conjunction of $\mathcal{ALC}$-LTL-formulas of the form $\Box\phi$, where ϕ is an $\mathcal{ALC}$-LTL-formula. This makes sure that ϕ holds in all (temporal) worlds. For the cases where ϕ is an $\mathcal{ALC}$-axiom, we can simply express this by:

$$\top \sqsubseteq [\![\phi]\!]$$

This captures all except for two conjuncts of the $\mathcal{ALC}$-LTL-formula of the reduction of [BGL08; BGL12]. There, a k-bit binary counter using concept names $A'_0, \ldots, A'_{k-1}$ was attached to the individual name a, which is incremented along the temporal dimension. We can express something similar in $\mathcal{EL}[\![\mathcal{ALC}]\!]$, but instead of incrementing the counter values along a sequence of t-successors, we have to go backwards since $\mathcal{EL}$ does allow for branching but does not allow for values restrictions, i.e. we cannot make sure that all t-successors behave the same. More precisely, if the counter value n is attached to a in context c, the value $n+1$ (modulo $2^k - 1$) must be attached to a in *all* of c's t-predecessors.

First, we ensure which bits must be flipped:

$$\bigwedge_{i<k}\Big(\exists\, t.\big([\![A'_0(a)]\!] \sqcap \ldots \sqcap [\![A'_{i-1}(a)]\!] \sqcap [\![A'_i(a)]\!]\big) \ \sqsubseteq\ [\![(\neg A'_i)(a)]\!]\Big)$$
$$\bigwedge_{i<k}\Big(\exists\, t.\big([\![A'_0(a)]\!] \sqcap \ldots \sqcap [\![A'_{i-1}(a)]\!] \sqcap [\![(\neg A'_i)(a)]\!]\big) \ \sqsubseteq\ [\![A'_i(a)]\!]\Big)$$

Next, we ensure that all other bits stay the same:

$$\bigwedge_{0<i<k}\ \bigwedge_{j<i}\Big(\exists\, t.\big([\![(\neg A'_j)(a)]\!] \sqcap [\![A'_i(a)]\!]\big) \ \sqsubseteq\ [\![A'_i(a)]\!]\Big)$$
$$\bigwedge_{0<i<k}\ \bigwedge_{j<i}\Big(\exists\, t.\big([\![(\neg A'_j)(a)]\!] \sqcap [\![(\neg A'_i)(a)]\!]\big) \ \sqsubseteq\ [\![(\neg A'_i)(a)]\!]\Big)$$

Note that due to the first m-axiom above, we enforce that every context has a t-successor. By the other m-axioms, we make sure that we enforce a t-chain of length 2^k. As in [BGL08; BGL12], it is not necessary to initialize the counter. Since we decrement the counter along the t-chain (modulo $2^k - 1$), every value between 0 and $2^k - 1$ is reached.

The conjunction of all the $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKBs above yields an $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKB $\mathcal{B}$ that is consistent iff w is accepted by M. □

Finally, we obtain a lower bound of NExp in the case of rigid concept names only.

Theorem 29. *The consistency problem in $\mathcal{EL}[\![\mathcal{ALC}]\!]$ is* NExp*-hard if $\mathsf{O}_{\mathsf{Crig}} \neq \varnothing$ and $\mathsf{O}_{\mathsf{Rrig}} = \varnothing$.*

Proof. To show the lower bound, we again adapt the proof ideas of [BGL08; BGL12], and reduce an exponentially bounded version of the domino problem to the consistency problem in $\mathcal{EL}[\![\mathcal{ALC}]\!]$ with rigid concepts, i.e. $\mathsf{O}_{\mathsf{Crig}} \neq \varnothing$ and $\mathsf{O}_{\mathsf{Rrig}} = \varnothing$. In [BGL08; BGL12], a reduction was provided to show NExp-hardness for the temporalized DL $\mathcal{ALC}$-LTL in the presence of rigid concepts. As in the proof of Theorem 28, we mimic the properties of the time dimension that are important for the reduction using a role name $t \in \mathsf{M}_{\mathsf{R}}$.

Our $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKB is the conjunction of the $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKBs introduced below. We proceed in a similar way as in the proof of Theorem 28. First, we ensure that we never have a “last” time point:

$$\top \sqsubseteq \exists t.\top$$

Note that in the corresponding model, we do not enforce a t-chain since cycles are not prohibited. As in the reduction in the proof of Theorem 28, this is not important in the reduction here.

Next, note that since the $\Box$-operator distributes over conjunction, most of the conjuncts of the $\mathcal{ALC}$-LTL-formula of the reduction of [BGL08; BGL12] can be rewritten as conjunctions of $\mathcal{ALC}$-LTL-formulas of the form $\Box\alpha$, where α is an $\mathcal{ALC}$-axiom. As already argued in the proof of Theorem 28, this can equivalently be expressed by $\top \sqsubseteq [\![\alpha]\!]$.

In [BGL08; BGL12], a $(2n+2)$-bit binary counter is employed using concept names $Z_0, \ldots, Z_{2n+1}$. This counter is attached to an individual name a, which is incremented along the temporal dimension.

This can be expressed in $\mathcal{EL}[\![\mathcal{ALC}]\!]$ as shown in the proof of Theorem 28:

$$\bigwedge_{i<2n+2}\Big(\exists\, t.\big([\![Z_0(a)]\!]\sqcap\ldots\sqcap[\![Z_{i-1}(a)]\!]\sqcap[\![Z_i(a)]\!]\big) \;\sqsubseteq\; [\![(\neg Z_i)(a)]\!]\Big)$$

$$\bigwedge_{i<2n+2}\Big(\exists\, t.\big([\![Z_0(a)]\!]\sqcap\ldots\sqcap[\![Z_{i-1}(a)]\!]\sqcap[\![(\neg Z_i)(a)]\!]\big) \;\sqsubseteq\; [\![Z_i(a)]\!]\Big)$$

$$\bigwedge_{0<i<2n+2}\;\bigwedge_{j<i}\Big(\exists\, t.\big([\![(\neg Z_j)(a)]\!]\sqcap[\![Z_i(a)]\!]\big) \;\sqsubseteq\; [\![Z_i(a)]\!]\Big)$$

$$\bigwedge_{0<i<2n+2}\;\bigwedge_{j<i}\Big(\exists\, t.\big([\![(\neg Z_j)(a)]\!]\sqcap[\![(\neg Z_i)(a)]\!]\big) \;\sqsubseteq\; [\![(\neg Z_i)(a)]\!]\Big)$$

Note that due to the first m-axiom above, we enforce that every context has a t-successor. By the other m-axioms, we make sure that we enforce a t-chain of length 2^{2n+2}. As in [BGL08; BGL12], it is not necessary to initialize the counter. Since we decrement the counter along the t-chain (modulo 2^{2n+1}), every value between 0 and 2^{2n+1} is reached.

In [BGL08; BGL12], an $\mathcal{ALC}$-LTL-formula is used to express that the value of the counter in shared by all domain elements belonging to the current (temporal) world. This is expressed using a disjunction, which we can simulate as follows:

$$\bigwedge_{0\le i\le 2n+1}\Big([\![Z_i(a)]\!]\sqsubseteq[\![\top\sqsubseteq Z_i]\!] \;\wedge\; [\![(\neg Z_i)(a)]\!]\sqsubseteq[\![Z_i\sqsubseteq\bot]\!]\Big)$$

Next, there is a concept name N, which is required be non-empty in every (temporal) world. We express this using a role name $r\in\mathsf{O_R}$:

$$\top\sqsubseteq[\![(\exists\, r.N)(a)]\!]$$

It is only left to express the following $\mathcal{ALC}$-LTL-formula of [BGL08; BGL12]:

$$\Box\Big(\bigvee_{d\in D}(\top\sqsubseteq d')\Big)$$

For readability, let $D=\{d_1,\ldots,d_k\}$. We use non-convexity of $\mathcal{ALC}$ as follows to express this:

$$\top\sqsubseteq[\![(d'_1\sqcup\cdots\sqcup d'_k)(a)]\!] \;\wedge\; \bigwedge_{1\le i\le k}\Big([\![d'_i(a)]\!]\sqsubseteq[\![\top\sqsubseteq d'_i]\!]\Big)$$

The conjunction of all the $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKBs above yields an $\mathcal{EL}[\![\mathcal{ALC}]\!]$-BKB $\mathcal{B}$ that is consistent iff the exponentially bounded version of the domino problem has a solution. □

Summing up the results of this section together with the upper bounds of Section 3, we obtain the following corollary.

Corollary 30. *For all $\mathcal{L}_O$ between $\mathcal{ALC}$ and $\mathcal{SHOQ}$, the consistency problem in $\mathcal{EL}[\![\mathcal{L}_O]\!]$ is*

- EXP-*complete if* $\mathsf{O_{Crig}}=\varnothing$ *and* $\mathsf{O_{Rrig}}=\varnothing$,
- NEXP-*complete if* $\mathsf{O_{Crig}}\neq\varnothing$ *and* $\mathsf{O_{Rrig}}=\varnothing$, *and*
- 2EXP-*complete if* $\mathsf{O_{Crig}}\neq\varnothing$ *and* $\mathsf{O_{Rrig}}\neq\varnothing$.

Proof. The lower bounds follow from Theorems 27, 28, and 29. The corresponding upper bounds follow from Theorems 16, 17, and 18. □

5 Conclusions

We have introduced and investigated a family of two-dimensional context DLs $\mathcal{L}_M[\![\mathcal{L}_O]\!]$ capable of representing information on different contexts (using a DL $\mathcal{L}_O$) and the relation between them (using a DL $\mathcal{L}_M$). In these context DLs, the consistency problem is decidable even in the presence of rigid names. We have investigated the complexity of the context DLs built from the classical DLs $\mathcal{EL}$, $\mathcal{ALC}$, and $\mathcal{SHOQ}$, see Table 2.

For future work, we would like to consider DLs admitting inverse roles, which are also useful for representing information about and within contexts. As argued in [McC93], also temporal information is often required to represent information about contexts faithfully. We think that our decision procedures can be adapted to deal with temporalized context DLs such as LTL$[\![\mathcal{L}_M[\![\mathcal{L}_O]\!]]\!]$. Moreover, besides consistency and other standard reasoning tasks, there are also reasoning tasks specific to contexts and rôles that we want to investigate in future, such as to check whether an object is allowed to play two rôles (at the same time).

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References

[BBL05] Franz Baader, Sebastian Brandt, and Carsten Lutz. "Pushing the $\mathcal{EL}$ Envelope". In: *Proceedings of the 19th International Joint Conference on Artificial Intelligence (IJCAI 2005)*. Ed. by Leslie Pack Kaelbling and Alessandro Saffiotti. Morgan Kaufmann, Los Altos, 2005, pp. 364–369.

[BCM+07] Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, eds. *The Description Logic Handbook: Theory, Implementation, and Applications*. 2nd. Cambridge University Press, 2007.

[BGL08] Franz Baader, Silvio Ghilardi, and Carsten Lutz. "LTL over Description Logic Axioms". In: *Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR 2008)*. Ed. by Gerhard Brewka and Jérôme Lang. Sydney, Australia: AAAI Press, 2008, pp. 684–694.

[BGL12] Franz Baader, Silvio Ghilardi, and Carsten Lutz. "LTL over Description Logic Axioms". In: *ACM Transactions on Computational Logic* 13.3 (2012).

[BS03] Alexander Borgida and Luciano Serafini. "Distributed Description Logics: Assimilating Information from Peer Sources". In: *Journal of Data Semantics*. Lecture Notes in Computer Science 2800 (2003). Ed. by Stefano Spaccapietra, Salvatore T. March, and Karl Aberer, pp. 153–184.

[BT15a] Stefan Borgwardt and Veronika Thost. *LTL over $\mathcal{EL}$ Axioms*. LTCS-Report 15-07. See `http://lat.inf.tu-dresden.de/research/reports.html`. Dresden, Germany: Chair for Automata Theory, Institute for Theoretical Computer Science, Technische Universität Dresden, 2015.

[BT15b] Stefan Borgwardt and Veronika Thost. "Temporal Query Answering in the Description Logic $\mathcal{EL}$". In: *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI 2015)*. Ed. by Qiang Yang. To appear. AAAI Press, 2015.

[BVS+09] Jie Bao, George Voutsadakis, Giora Slutzki, and Vasant Honavar. "Package-Based Description Logics". In: *Modular Ontologies: Concepts, Theories and Techniques for Knowledge Modularization*. Ed. by Heiner Stuckenschmidt, Christine Parent, and Stefano Spaccapietra. Vol. 5445. Lecture Notes in Computer Science. Springer, 2009, pp. 349–371.

[HST00] Ian Horrocks, Ulrike Sattler, and Stephan Tobies. "Practical Reasoning for Very Expressive Description Logics". In: *Journal of the Interest Group in Pure and Applied Logic* 8.3 (2000), pp. 239–263.

[KG10] Szymon Klarman and Víctor Gutiérrez-Basulto. "$\mathcal{ALC}_{\mathcal{ALC}}$: A Context Description Logic". In: *Proceedings of the 12th European Conference on Logics in Artificial Intelligence (JELIA 2010)*. Ed. by Tomi Janhunen and Ilkka Niemelä. Vol. 6341. Lecture Notes in Computer Science. Springer, 2010, pp. 208–220.

[KG11a] Szymon Klarman and Víctor Gutiérrez-Basulto. "Two-Dimensional Description Logics for Context-Based Semantic Interoperability". In: *Proceedings of the 25th AAAI Conference on Artificial Intelligence (AAAI 2011)*. Ed. by Wolfram Burgard and Dan Roth. AAAI Press, 2011.

[KG11b] Szymon Klarman and Víctor Gutiérrez-Basulto. "Two-Dimensional Description Logics of Context". In: *Proceedings of the 24th International Workshop on Description Logics (DL 2011)*. Ed. by Riccardo Rosati, Sebastian Rudolph, and Michael Zakharyaschev. Vol. 745. CEUR-WS.org, 2011.

[Lip14] Marcel Lippmann. "Temporalised Description Logics for Monitoring Partially Observable Events". PhD thesis. TU Dresden, Germany, 2014.

[McC87] John McCarthy. "Generality in Artificial Intelligence". In: *Communications of the ACM* 30.12 (1987), pp. 1030–1035.

[McC93] John McCarthy. "Notes on Formalizing Context". In: *Proceedings of the 13th International Joint Conference on Artificial Intelligence (IJCAI 1993)*. Ed. by Ruzena Bajcsy. Chambéry, France: Morgan Kaufmann, Los Altos, 1993, pp. 555–562.

[Sch91] Klaus Schild. "A Correspondence Theory for Terminological Logics: Preliminary Report". In: *Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI 1991)*. Ed. by John Mylopoulos and Raymond Reiter. Morgan Kaufmann, Los Altos, 1991, pp. 466–471.