The Fuzzy Description Logic G-$\mathcal{FL}_0$ with Greatest Fixed-Point Semantics*

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Abstract. We study the fuzzy extension of the Description Logic $\mathcal{FL}_0$ with semantics based on the Gödel t-norm. We show that subsumption w.r.t. a finite set of primitive definitions, using greatest fixed-point semantics, can be characterized by a relation on weighted automata. We use this result to provide tight complexity bounds for reasoning in this logic, showing that it is PSpace-complete. If the definitions do not contain cycles, subsumption becomes co-NP-complete.

1 Introduction

Description logics (DLs) are used to describe the knowledge of an application domain in a formally well-defined manner [3]. The basic building blocks are *concepts* that intuitively describe a set of elements of the domain, and *roles*, which model binary relations over the domain. The expressivity of DLs is given by a set of *constructors* that are used to build complex concepts from so-called *concept names*, and is usually chosen to end up in decidable fragments of first-order predicate logic.

Knowledge about domain-specific terminology can be expressed by different kinds of axioms. For example, the *concept definition*

$$\mathsf{Father} \doteq \mathsf{Human} \sqcap \mathsf{Male} \sqcap \exists \mathsf{hasChild}.\top$$

is used to determine the extension of the concept name Father in terms of other concept names (Human, Male) and roles (hasChild). In contrast, a *primitive* concept definition like

$$\mathsf{Human} \sqsubseteq \mathsf{Mammal} \sqcap \mathsf{Biped}$$

only bounds the interpretation of a concept name from above. Sometimes, one restricts (primitive) definitions to be *acyclic*, which means that the definition of a concept name cannot use itself (directly or indirectly via other definitions). In *general concept inclusions (GCIs)* such as

$$\forall \mathsf{hasParent}.\mathsf{Human} \sqsubseteq \mathsf{Human}$$

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one can relate arbitrary complex expressions. These axioms are collected into so-called *TBoxes*, which can be either acyclic (containing acyclic definitions), cyclic (containing possibly cyclic definitions), or general (containing GCIs). To interpret cyclic TBoxes, several competing semantics have been proposed [19].

Different DLs vary in the choice of constructors allowed for building complex concepts. For example, the small DL $\mathcal{EL}$ uses the constructors *top* ($\top$), *conjunction* ($\sqcap$), and *existential restriction* ($\exists r.C$ for a role r and a concept C). We consider here mainly $\mathcal{FL}_0$, which has top, conjunctions, and *value restrictions* ($\forall r.C$). The DL $\mathcal{ALC}$ combines all the above constructors with *negation* ($\neg C$).

Fuzzy description logics have been introduced as extensions of classical DLs capable of representing and reasoning with vague or imprecise knowledge. The main idea behind these logics is to allow for a set of truth degrees, beyond the standard true and false; usually, the real interval $[0, 1]$ is considered. In this way, one can allow fuzzy concepts like Tall to assign an arbitrary degree of tallness to each individual, instead of simply classifying them into *tall* and *not tall.* Based on Mathematical Fuzzy Logic [13], a so-called *t-norm* defines the interpretation of conjunctions, and determines the semantics of the other constructors as well. The three main continuous t-norms are *Gödel* (G), *Łukasiewicz* (Ł), and *Product* (Π). The *Zadeh* semantics is another popular choice that is based on fuzzy set theory [25].

The area of fuzzy DLs recently experienced a shift, when it was shown that reasoning with GCIs easily becomes undecidable [4,7,9]. To guarantee decidability in fuzzy DLs, one can (i) restrict the semantics to consider finitely many truth degrees [8]; (ii) allow only acyclic or unfoldable TBoxes [5,22]; or (iii) restrict to Zadeh or Gödel semantics [6,17,20,21].

In the cases where the Gödel t-norm is used, the complexity of reasoning is typically the same as for its classical version, as shown for subsumption w.r.t. GCIs in G-$\mathcal{EL}$, which is polynomial [17,20], and G-$\mathcal{ALC}$, ExpTime-complete [6]. This latter result implies that subsumption in G-$\mathcal{FL}_0$ with general TBoxes is also ExpTime-complete since it is ExpTime-hard already in classical $\mathcal{FL}_0$ [2]. On the other hand, if TBoxes are restricted to contain only (cyclic) definitions, then deciding subsumption in classical $\mathcal{FL}_0$ under the greatest fixed-point semantics is known to be PSpace-complete [1]. For acyclic TBoxes, the complexity reduces to co-NP-complete [18]. In this paper, we analyze reasoning in the Gödel extension of this logic.

Consider the cyclic definition of a *tall person with only tall offspring* (Toto):

$$\mathsf{Toto} \sqsubseteq \mathsf{Person} \sqcap \mathsf{Tall} \sqcap \forall \mathsf{hasChild}.\mathsf{Toto}$$

Choosing greatest fixed-point semantics is very natural in this setting, as it requires to always assign the largest possible degree for an individual to belong to Toto. Otherwise, Toto could simply assign degree 0 to all individuals, which is clearly not the intended meaning.

We show that the PSpace-upper bound for reasoning in the classical case also applies to this fuzzy DL. To prove this, we characterize the greatest fixed-point semantics of G-$\mathcal{FL}_0$ by means of $[0, 1]$-weighted automata. We then show that reasoning with these automata can be reduced to a linear number of inclusion tests between unweighted automata, which can be solved using only polynomial space [11]. For the case of acyclic TBoxes, our reduction yields acyclic automata and thus implies a co-NP upper bound, again matching the complexity of reasoning in classical $\mathcal{FL}_0$.

2 Preliminaries

We first introduce some basic notions of lattice theory, which we use later to define the greatest fixed-point semantics in our fuzzy DL. For a more comprehensive overview on the topic, refer to [12]. Afterwards, we introduce fuzzy logics based on Gödel semantics, which are studied in more detail in [10,13,16].

2.1 Lattices, Operators, and Fixed-Points

A *lattice* is an algebraic structure $(L, \vee, \wedge)$ with two commutative, associative and idempotent binary operations $\vee$ (supremum) and $\wedge$ (infimum) that distribute over each other. It is *complete* if suprema and infima of arbitrary subsets $S \subseteq L$, denoted by $\bigvee_{x \in S} x$ and $\bigwedge_{x \in S} x$ respectively, exist. In this case, the lattice is *bounded* by the greatest element $\mathbf{1} := \bigvee_{x \in L} x$ and the least element $\mathbf{0} := \bigwedge_{x \in L} x$. Lattices induce a natural partial ordering on the elements of L where $x \leq y$ iff $x \wedge y = x$.

One common complete lattice used in fuzzy logics (see e.g. [10,13]) is the interval $[0,1]$ with the usual order on the real numbers. Other complete lattices can be constructed as follows. Given a complete lattice L and a set S, the set L^S of all functions $f\colon S \to L$ is also a complete lattice, if infimum and supremum are defined component-wise. More precisely, for any two $f_1, f_2 \in L^S$, we define $f_1 \vee f_2$ for all $x \in S$ as $(f_1 \vee f_2)(x) := f_1(x) \vee f_2(x)$. If we similarly define the infimum, we obtain a lattice with the order $f_1 \leq f_2$ iff $f_1(x) \leq f_2(x)$ holds for all $x \in S$. It is easy to verify that infinite infima and suprema can then also be computed component-wise. We are particularly interested in operators on complete lattices L and their properties.

Definition 1 (fixed-point). *Let L be a complete lattice. A* fixed-point *of an operator $T\colon L \to L$ is an element $x \in L$ such that $T(x) = x$. It is the* greatest fixed-point *of T if for any fixed-point y of T we have $y \leq x$.*

The operator T is monotone *if for all $x, y \in L$, $x \leq y$ implies $T(x) \leq T(y)$. It is* downward ω-continuous *if for every decreasing chain $x_0 \geq x_1 \geq x_2 \geq \dots$ in L we have $T(\bigwedge_{i \geq 0} x_i) = \bigwedge_{i \geq 0} T(x_i)$.*

If it exists, the greatest fixed-point of T is unique and denoted by $\mathsf{gfp}(T)$.

It is easy to verify that every downward ω-continuous operator is also monotone. By a fundamental result from [24], every monotone operator T has a greatest fixed-point. If T is downward ω-continuous, then $\mathsf{gfp}(T)$ corresponds to the infimum of the decreasing chain $\mathbf{1} \geq T(\mathbf{1}) \geq T(T(\mathbf{1})) \geq \dots \geq T^i(\mathbf{1}) \geq \dots$ [15].

Proposition 2. *If L is a complete lattice and T a downward ω-continuous operator on L, then $\mathsf{gfp}(T) = \bigwedge_{i \geq 0} T^i(\mathbf{1})$.*

2.2 Gödel Fuzzy Logic

Our fuzzy DL is based on the well-known Gödel semantics for fuzzy logics, which is one of the main t-norm-based semantics used in Mathematical Fuzzy Logic [10,13] over the standard interval $[0,1]$. The *Gödel t-norm* is the binary minimum operator on $[0,1]$. For consistency, we use the lattice-theoretic notation $\wedge$ instead of min. An important property of this operator is that it preserves arbitrary infima and suprema on $[0,1]$, i.e. $\bigwedge_{i\in I}(x_i \wedge x) = \big(\bigwedge_{i\in I} x_i\big) \wedge x$ and $\bigvee_{i\in I}(x_i \wedge x) = \big(\bigvee_{i\in I} x_i\big) \wedge x$ for any index set I and elements $x, x_i \in [0,1]$ for all $i \in I$. In particular, this means that the Gödel t-norm is monotone in both arguments. The *residuum* of the Gödel t-norm is the binary operator $\Rightarrow_{\mathsf{G}}$ on $[0,1]$ defined for all $x, y \in [0,1]$ by

$$x \Rightarrow_{\mathsf{G}} y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

It is a fundamental property of a t-norm and its residuum that for all values $x, y, z \in [0,1]$, $x \wedge y \leq z$ iff $y \leq x \Rightarrow_{\mathsf{G}} z$. As with the Gödel t-norm, its residuum preserves arbitrary infima in its second component. However, in the first component the order on $[0,1]$ is reversed.

Proposition 3. *For any index set I and values $x, x_i \in [0,1]$, $i \in I$, we have*

$$x \Rightarrow_{\mathsf{G}} \Big(\bigwedge_{i\in I} x_i\Big) = \bigwedge_{i\in I}(x \Rightarrow_{\mathsf{G}} x_i) \quad \textit{and} \quad \Big(\bigvee_{i\in I} x_i\Big) \Rightarrow_{\mathsf{G}} x = \bigwedge_{i\in I}(x_i \Rightarrow_{\mathsf{G}} x).$$

This shows that the residuum is monotone in the second argument and antitone in the first argument. The following reformulation of nested residua in terms of infima will also prove useful.

Proposition 4. *For all values $x, x_1, \dots, x_n \in [0,1]$, we have*

$$\big((x_1 \wedge \dots \wedge x_n) \Rightarrow_{\mathsf{G}} x\big) = \big(x_1 \Rightarrow_{\mathsf{G}} \dots (x_n \Rightarrow_{\mathsf{G}} x) \dots\big).$$

Proof. Both values are either x or 1, and they are 1 iff one of the operands x_i, $1 \leq i \leq n$, is smaller than or equal to x. □

3 Fuzzy $\mathcal{FL}_0$

The fuzzy description logic G-$\mathcal{FL}_0$ has the same syntax as classical $\mathcal{FL}_0$. The difference lies in the interpretation of G-$\mathcal{FL}_0$-concepts.

Definition 5 (syntax). *Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be two non-empty, disjoint sets of* concept names *and* role names*, respectively.* Concepts *are built from concept names using the constructors $\top$ (top), $C \sqcap D$ (conjunction), and $\forall r.C$ (value restriction for $r \in \mathsf{N_R}$).*

A (primitive concept) definition *is of the form $\langle A \sqsubseteq C \geq p\rangle$, where $A \in \mathsf{N_C}$, C is a concept, and $p \in [0,1]$. A* (cyclic) TBox *is a finite set of definitions. Given a TBox $\mathcal{T}$, a concept name is* defined *if it appears on the left-hand side of a definition in $\mathcal{T}$, and* primitive *otherwise.*

In contrast to the treatment of classical $\mathcal{FL}_0$ in [1], we permit several primitive definitions instead of only one (full) definition of the form $\langle A \doteq C_1 \sqcap \cdots \sqcap C_n \geq p \rangle$ for each concept name. This allows us to specify fuzzy degrees p_i for each of the conjuncts C_i independently. An *acyclic TBox* is a finite set of definitions without cyclic dependencies between the defined concept names.

We use the expression $\forall w.C$ with $w = r_1 r_2 \dots r_n \in \mathsf{N}_\mathsf{R}^*$ to abbreviate the concept $\forall r_1.\forall r_2.\dots\forall r_n.C$. We also allow $w = \varepsilon$, in which case $\forall w.C$ is simply C. We denote the set of concept names occurring in the TBox $\mathcal{T}$ by $\mathsf{N}_\mathsf{C}^\mathcal{T}$, the set of defined concept names in $\mathsf{N}_\mathsf{C}^\mathcal{T}$ by $\mathsf{N}_\mathsf{D}^\mathcal{T}$, and the set of primitive concept names in $\mathsf{N}_\mathsf{C}^\mathcal{T}$ by $\mathsf{N}_\mathsf{P}^\mathcal{T}$. Likewise, we collect all role names occurring in $\mathcal{T}$ into the set $\mathsf{N}_\mathsf{R}^\mathcal{T}$.

Definition 6 (semantics). *An* interpretation *is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where $\Delta^\mathcal{I}$ is a non-empty set, called the* domain *of $\mathcal{I}$, and the* interpretation function *$\cdot^\mathcal{I}$ maps every concept name A to a fuzzy set $A^\mathcal{I} \colon \Delta^\mathcal{I} \to [0,1]$ and every role name r to a fuzzy binary relation $r^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$. This function is extended to concepts by setting $\top^\mathcal{I}(x) := 1$, $(C \sqcap D)^\mathcal{I}(x) := C^\mathcal{I}(x) \wedge D^\mathcal{I}(x)$, and $(\forall r.C)^\mathcal{I}(x) := \bigwedge_{y \in \Delta^\mathcal{I}} (r^\mathcal{I}(x,y) \Rightarrow_\mathsf{G} C^\mathcal{I}(y))$ for all $x \in \Delta^\mathcal{I}$.*

The interpretation $\mathcal{I}$ satisfies *(or is a* model *of) the definition $\langle A \sqsubseteq C \geq p \rangle$ if $A^\mathcal{I}(x) \Rightarrow_\mathsf{G} C^\mathcal{I}(x) \geq p$ holds for all $x \in \Delta^\mathcal{I}$. It* satisfies *(or is a* model *of) a TBox if it satisfies all its definitions.*

For an interpretation $\mathcal{I} = (\Delta, \cdot^\mathcal{I})$, $w = r_1 r_2 \dots r_n \in \mathsf{N}_\mathsf{R}^*$, and elements $x_0, x_n \in \Delta$, we set $w^\mathcal{I}(x_0, x_n) := \bigvee_{x_1,\dots,x_{n-1} \in \Delta} (r_1^\mathcal{I}(x_0,x_1) \wedge \cdots \wedge r_n^\mathcal{I}(x_{n-1},x_n))$, and can thus treat $\forall w.C$ like an ordinary value restriction with

$$
\begin{aligned}
(\forall w.C)^\mathcal{I}(x_0) &:= \bigwedge_{x_n \in \Delta} (w^\mathcal{I}(x_0,x_n) \Rightarrow_\mathsf{G} C^\mathcal{I}(x_n)) \\
&= \bigwedge_{x_1,\dots,x_n \in \Delta} \big(\big(r_1^\mathcal{I}(x_0,x_1) \wedge \cdots \wedge r_n^\mathcal{I}(x_{n-1},x_n)\big) \Rightarrow_\mathsf{G} C^\mathcal{I}(x_n)\big) \\
&= \bigwedge_{x_1,\dots,x_n \in \Delta} \big(r_1^\mathcal{I}(x_0,x_1) \Rightarrow_\mathsf{G} \dots (r_n^\mathcal{I}(x_{n-1},x_n) \Rightarrow_\mathsf{G} C^\mathcal{I}(x_n)) \dots\big) \\
&= (\forall r_1.\dots\forall r_n.C)^\mathcal{I}(x_0)
\end{aligned}
$$

for all $x_0 \in \Delta$ (see Propositions 3 and 4).

It is convenient to consider TBoxes in normal form. The TBox $\mathcal{T}$ is in *normal form* if all definitions in $\mathcal{T}$ are of the form $\langle A \sqsubseteq \forall w.B \geq p \rangle$, where $A, B \in \mathsf{N}_\mathsf{C}$, $w \in \mathsf{N}_\mathsf{R}^*$, and $p \in [0,1]$, and there are no two definitions $\langle A \sqsubseteq \forall w.B \geq p \rangle$, $\langle A \sqsubseteq \forall w.B \geq p' \rangle$ with $p \neq p'$. Every TBox can be transformed into an equivalent TBox in normal form, as follows. First, we distribute the value restrictions over the conjunctions.

Lemma 7. *For every $r \in \mathsf{N}_\mathsf{R}$, concepts C, D, and interpretation $\mathcal{I} = (\Delta, \cdot^\mathcal{I})$, it holds that $(\forall r.(C \sqcap D))^\mathcal{I} = (\forall r.C \sqcap \forall r.D)^\mathcal{I}$.*

Proof. For every $x \in \Delta$, we have

$$\begin{aligned}
(\forall r.(C \sqcap D))^{\mathcal{I}}(x) &= \bigwedge_{y \in \Delta} \left(r^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} (C^{\mathcal{I}}(y) \wedge D^{\mathcal{I}}(y)) \right) \\
&= \bigwedge_{y \in \Delta} \left((r^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} C^{\mathcal{I}}(y)) \wedge (r^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} D^{\mathcal{I}}(y)) \right) \\
&= \Big(\bigwedge_{y \in \Delta} (r^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} C^{\mathcal{I}}(y)) \Big) \wedge \Big(\bigwedge_{y \in \Delta} (r^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} D^{\mathcal{I}}(y)) \Big) \\
&= (\forall r.C \sqcap \forall r.D)^{\mathcal{I}}(x)
\end{aligned}$$

by Proposition 3. □

Thus, we can equivalently write the right-hand sides of the definitions in $\mathcal{T}$ in the form $\forall w_1.B_1 \sqcap \cdots \sqcap \forall w_n.B_n$, where $w_i \in \mathsf{N}_\mathsf{R}^*$ and $B_i \in \mathsf{N}_\mathsf{C} \cup \{\top\}, 1 \leq i \leq n$. Since $\forall r.\top$ is equivalent to $\top$, we can remove all conjuncts of the form $\forall w.\top$ from this representation. After this transformation, all the definitions in the TBox are of the form $\langle A \sqsubseteq \forall w_1.B_1 \sqcap \cdots \sqcap \forall w_n.B_n \geq p \rangle$ with $B_i \in \mathsf{N}_\mathsf{C}, 1 \leq i \leq n$, or $\langle A \sqsubseteq \top \geq p \rangle$. The latter axioms are tautologies, and can hence be removed from the TBox without affecting the semantics.

It follows from Proposition 3 that an interpretation $\mathcal{I}$ satisfies the definition $\langle A \sqsubseteq \forall w_1.B_1 \sqcap \cdots \sqcap \forall w_n.B_n \geq p \rangle$ iff it satisfies $\langle A \sqsubseteq \forall w_i.B_i \geq p \rangle$, $1 \leq i \leq n$. Thus, the former axiom can be equivalently replaced by the latter set of axioms.

After these steps, the TBox contains only axioms of the form $\langle A \sqsubseteq \forall w.B \geq p \rangle$ with $A, B \in \mathsf{N}_\mathsf{C}$, satisfying the first condition of the definition of normal form. Suppose now that $\mathcal{T}$ contains the axioms $\langle A \sqsubseteq \forall w.B \geq p \rangle$ and $\langle A \sqsubseteq \forall w.B \geq p' \rangle$ with $p > p'$. Then $\mathcal{T}$ is equivalent to the TBox $\mathcal{T} \setminus \{\langle A \sqsubseteq \forall w.B \geq p' \rangle\}$, i.e. the weaker axiom can be removed. It is clear that all of these transformations can be done in polynomial time in the size of the original TBox.

Concept definitions can be seen as a restriction of the interpretation of the defined concepts, depending on the interpretation of the primitive concepts. We use this intuition and consider *greatest fixed-point* semantics. The following construction is based on the classical notions from [1].

A *primitive interpretation* is a pair $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ as in Definition 6, except that $\cdot^{\mathcal{J}}$ is only defined on N_R and $\mathsf{N}_\mathsf{P}^{\mathcal{T}}$. Given such a $\mathcal{J}$, we use functions $f \in ([0,1]^{\Delta})^{\mathsf{N}_\mathsf{D}^{\mathcal{T}}}$ to describe the interpretation of the remaining (defined) concept names. Recall that these functions form a complete lattice. In the following, we use the abbreviation $L_{\mathcal{J}}^{\mathcal{T}} := ([0,1]^{\Delta})^{\mathsf{N}_\mathsf{D}^{\mathcal{T}}}$ for this lattice. Given a primitive interpretation $\mathcal{J}$ and a function $f \in L_{\mathcal{J}}^{\mathcal{T}}$, the *induced interpretation* $\mathcal{I}_{\mathcal{J},f}$ has the same domain as $\mathcal{J}$ and extends the interpretation function of $\mathcal{J}$ to the defined concepts names $A \in \mathsf{N}_\mathsf{D}^{\mathcal{T}}$ by taking $A^{\mathcal{I}_{\mathcal{J},f}} := f(A)$. The interpretation of the remaining concept names, i.e. those that do not occur in $\mathcal{T}$, is fixed to $\mathbf{0}$.

We can describe the effect that the axioms in $\mathcal{T}$ have on $L_{\mathcal{J}}^{\mathcal{T}}$ by the operator $T_{\mathcal{J}}^{\mathcal{T}} \colon L_{\mathcal{J}}^{\mathcal{T}} \to L_{\mathcal{J}}^{\mathcal{T}}$, which is defined as follows for all $f \in L_{\mathcal{J}}^{\mathcal{T}}$, $A \in \mathsf{N}_\mathsf{D}^{\mathcal{T}}$, and $x \in \Delta$:

$$T_{\mathcal{J}}^{\mathcal{T}}(f)(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p \rangle \in \mathcal{T}} (p \Rightarrow_{\mathsf{G}} C^{\mathcal{I}_{\mathcal{J},f}}(x)).$$

This operator computes new values of the defined concept names according to the old interpretation $\mathcal{I}_{\mathcal{J},f}$ and their definitions in $\mathcal{T}$.

We are interested in using the greatest fixed-point of $T_{\mathcal{J}}^{\mathcal{T}}$, for some primitive interpretation $\mathcal{J}$, to define a new semantics for TBoxes $\mathcal{T}$ in G-$\mathcal{FL}_0$. Before being able to do this, we have to ensure that such a fixed-point exists.

Lemma 8. *Given a TBox $\mathcal{T}$ and a primitive interpretation $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$, the operator $T_{\mathcal{J}}^{\mathcal{T}}$ on $L_{\mathcal{J}}^{\mathcal{T}}$ is downward ω-continuous.*

Proof. Consider a decreasing chain $f_0 \geq f_1 \geq f_2 \geq \dots$ of functions in $L_{\mathcal{J}}^{\mathcal{T}}$. We use the abbreviations $f := \bigwedge_{i\geq 0} f_i$, $\mathcal{I} := \mathcal{I}_{\mathcal{J},f}$, and $\mathcal{I}_i := \mathcal{I}_{\mathcal{J},f_i}$ for all $i \geq 0$, and have to show that $T_{\mathcal{J}}^{\mathcal{T}}(f) = \bigwedge_{i\geq 0} T_{\mathcal{J}}^{\mathcal{T}}(f_i)$ holds.

First, we prove by induction on the structure of C that $C^{\mathcal{I}} = \bigwedge_{i\geq 0} C^{\mathcal{I}_i}$ holds for all concepts C built from $\mathsf{N}_{\mathsf{R}}^{\mathcal{T}}$ and $\mathsf{N}_{\mathsf{C}}^{\mathcal{T}}$, where $\bigwedge$ is defined as usual over the complete lattice $[0,1]^{\Delta}$.

For $A \in \mathsf{N}_{\mathsf{P}}^{\mathcal{T}}$, by the definition of $\mathcal{I}_{\mathcal{J},f}$ and $\mathcal{I}_{\mathcal{J},f_i}$ we have $A^{\mathcal{I}} = A^{\mathcal{J}} = A^{\mathcal{I}_i}$ for all $i \geq 0$, and thus $A^{\mathcal{I}} = A^{\mathcal{J}} = \bigwedge_{i\geq 0} A^{\mathcal{I}_i}$. For $A \in \mathsf{N}_{\mathsf{D}}^{\mathcal{T}}$, we have

$$A^{\mathcal{I}} = f(A) = \Big(\bigwedge_{i\geq 0} f_i\Big)(A) = \bigwedge_{i\geq 0} f_i(A) = \bigwedge_{i\geq 0} A^{\mathcal{I}_i}$$

by the definition of $\mathcal{I}_{\mathcal{J},f}$ and $\mathcal{I}_{\mathcal{J},f_i}$ and the component-wise ordering on the complete lattice $L_{\mathcal{J}}^{\mathcal{T}}$.

For concepts of the form $C \sqcap D$, by the induction hypothesis and associativity of $\wedge$ we have

$$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \wedge D^{\mathcal{I}} = \Big(\bigwedge_{i\geq 0} C^{\mathcal{I}_i}\Big) \wedge \Big(\bigwedge_{i\geq 0} D^{\mathcal{I}_i}\Big) = \bigwedge_{i\geq 0} (C^{\mathcal{I}_i} \wedge D^{\mathcal{I}_i}) = \bigwedge_{i\geq 0} (C \sqcap D)^{\mathcal{I}_i}.$$

Consider now a value restriction $\forall r.C$. Using Proposition 3 we get for all $x \in \Delta$,

$$\begin{aligned}
(\forall r.C)^{\mathcal{I}}(x) &= \bigwedge_{y\in\Delta} (r^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} C^{\mathcal{I}}(y)) \\
&= \bigwedge_{y\in\Delta} \Big(r^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} \Big(\bigwedge_{i\geq 0} C^{\mathcal{I}_i}(y)\Big)\Big) \\
&= \bigwedge_{y\in\Delta} \bigwedge_{i\geq 0} (r^{\mathcal{I}_i}(x,y) \Rightarrow_{\mathsf{G}} C^{\mathcal{I}_i}(y)) = \Big(\bigwedge_{i\geq 0} (\forall r.C)^{\mathcal{I}_i}\Big)(x)
\end{aligned}$$

by the induction hypothesis and the component-wise ordering on $[0,1]^{\Delta}$.

Using this, we can now prove the actual claim of the lemma. For all $A \in \mathsf{N}_{\mathsf{D}}^{\mathcal{T}}$ and all $x \in \Delta$, we get, using again Proposition 3 and the previous claim,

$$\begin{aligned}
T_{\mathcal{J}}^{\mathcal{T}}(f)(A)(x) &= \bigwedge_{\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}} (p \Rightarrow_{\mathsf{G}} C^{\mathcal{I}}(x)) \\
&= \bigwedge_{\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}} \Big(p \Rightarrow_{\mathsf{G}} \Big(\bigwedge_{i\geq 0} C^{\mathcal{I}_i}(x)\Big)\Big)
\end{aligned}$$

$$= \bigwedge_{\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}} \bigwedge_{i \geq 0} (p \Rightarrow_{\mathsf{G}} C^{\mathcal{I}_i}(x)) = \Big(\bigwedge_{i \geq 0} T_{\mathcal{J}}^{\mathcal{T}}(f_i) \Big)(A)(x)$$

by the definition of $T_{\mathcal{J}}^{\mathcal{T}}$ and the component-wise ordering on $L_{\mathcal{J}}^{\mathcal{T}}$. □

By Proposition 2, we know that $\mathsf{gfp}(T_{\mathcal{J}}^{\mathcal{T}})$ exists and is equal to $\bigwedge_{i \geq 0} (T_{\mathcal{J}}^{\mathcal{T}})^i(\mathbf{1})$, where $\mathbf{1}$ is the greatest element of the lattice $L_{\mathcal{J}}^{\mathcal{T}}$ that maps all defined concept names to $\top^{\mathcal{J}}$. In the following, we denote by $\mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$ the interpretation $\mathcal{I}_{\mathcal{J},f}$ for $f := \mathsf{gfp}(T_{\mathcal{J}}^{\mathcal{T}})$. Note that $\mathcal{I} := \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$ is actually a model of $\mathcal{T}$ since for every $\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}$ and every $x \in \Delta$ we have

$$A^{\mathcal{I}}(x) = f(A)(x) = T_{\mathcal{J}}^{\mathcal{T}}(f)(A)(x) = \bigwedge_{\langle A \sqsubseteq C' \geq p'\rangle \in \mathcal{T}} (p' \Rightarrow_{\mathsf{G}} C'^{\mathcal{I}}(x)) \leq p \Rightarrow_{\mathsf{G}} C^{\mathcal{I}}(x),$$

and thus $p \wedge A^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x)$, which is equivalent to $p \leq A^{\mathcal{I}}(x) \Rightarrow_{\mathsf{G}} C^{\mathcal{I}}(x)$.

We can now define the reasoning problem in $\mathsf{G}\text{-}\mathcal{FL}_0$ that we want to solve.

Definition 9 (gfp-subsumption). *An interpretation $\mathcal{I}$ is a* gfp-model *of a TBox $\mathcal{T}$ if there is a primitive interpretation $\mathcal{J}$ such that $\mathcal{I} = \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$. Given $A, B \in \mathsf{N_C}$ and $p \in [0,1]$, A is* gfp-subsumed *by B to degree p w.r.t. $\mathcal{T}$ (written $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$), if for every gfp-model $\mathcal{I}$ of $\mathcal{T}$ and every $x \in \Delta^{\mathcal{I}}$ we have $A^{\mathcal{I}}(x) \Rightarrow_{\mathsf{G}} B^{\mathcal{I}}(x) \geq p$. The* best gfp-subsumption degree *of A and B w.r.t. $\mathcal{T}$ is the supremum over all p such that $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$.*

Let now $\mathcal{T}$ be a TBox and $\mathcal{T}'$ the result of transforming $\mathcal{T}$ into normal form as described before. It is easy to verify that the operators $T_{\mathcal{J}}^{\mathcal{T}}$ and $T_{\mathcal{J}}^{\mathcal{T}'}$ coincide, and therefore the gfp-models of $\mathcal{T}$ are the same as those of $\mathcal{T}'$. To solve the problem of deciding gfp-subsumptions, it thus suffices to consider TBoxes in normal form.

4 Characterizing Subsumption Using Finite Automata

To decide gfp-subsumption between concept names, we employ an automata-based approach following [1]. However, here we use *weighted* automata.

Definition 10 (WWA). *A* weighted automaton with word transitions (WWA) *is a tuple $\mathcal{A} = (\Sigma, Q, q_0, \mathsf{wt}, q_f)$, where Σ is a finite alphabet of* input symbols, *Q is a finite set of* states, *$q_0 \in Q$ is the* initial state, *$\mathsf{wt}\colon Q \times \Sigma^* \times Q \to [0,1]$ is the* transition weight function *with the property that its* support

$$\mathsf{supp}(\mathsf{wt}) := \{(q, w, q') \in Q \times \Sigma^* \times Q \mid \mathsf{wt}(q, w, q') > 0\}$$

is finite, and $q_f \in Q$ is the final state.

A finite path *in $\mathcal{A}$ is a sequence $\pi = q_0 w_1 q_1 w_2 \ldots w_n q_n$, where $q_i \in Q$ and $w_i \in \Sigma^*$ for all $i \in \{1, \ldots, n\}$, and $q_n = q_f$. Its* label *is the finite word $\ell(\pi) := w_1 w_2 \ldots w_n$. The* weight *of π is $\mathsf{wt}(\pi) := \bigwedge_{i=1}^n \mathsf{wt}(q_{i-1}, w_i, q_i)$. The set of all finite paths with label w in $\mathcal{A}$ is denoted $\mathsf{paths}(\mathcal{A}, w)$. The* behavior *$\|\mathcal{A}\|\colon \Sigma^* \to [0,1]$ of $\mathcal{A}$ is defined by $\|\mathcal{A}\|(w) := \bigvee_{\pi \in \mathsf{paths}(\mathcal{A}, w)} \mathsf{wt}(\pi)$ for $w \in \Sigma^*$.*

If the image of the transition weight function is included in $\{0,1\}$, then we have a classical finite automaton with word transitions (WA). In this case, wt is usually described as a subset of $Q \times \Sigma^* \times Q$ and the behavior is characterized by the set $L(\mathcal{A})$, called the *language* of $\mathcal{A}$, of all words whose behavior is 1. The *inclusion problem* for WA is to decide, given two such automata $\mathcal{A}$ and $\mathcal{A}'$, whether $L(\mathcal{A}) \subseteq L(\mathcal{A}')$. This problem is known to be PSPACE-complete [11].

Our goal is to describe the restrictions imposed by a G-$\mathcal{FL}_0$ TBox $\mathcal{T}$ using a WWA. For the rest of this paper, we assume w.l.o.g. that $\mathcal{T}$ is in normal form.

Definition 11 (automata $\mathcal{A}^{\mathcal{T}}_{A,B}$). *For concept names $A, B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$, the WWA $\mathcal{A}^{\mathcal{T}}_{A,B} = (\mathsf{N}_\mathsf{R}, \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}, A, \mathsf{wt}_{\mathcal{T}}, B)$ is defined by the transition weight function*

$$\mathsf{wt}_{\mathcal{T}}(A', w, B') := \begin{cases} p & \textit{if } \langle A' \sqsubseteq \forall w.B' \geq p\rangle \in \mathcal{T}, \\ 0 & \textit{otherwise.} \end{cases}$$

For a TBox $\mathcal{T}$ and $A, A', B, B' \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$, the automata $\mathcal{A}^{\mathcal{T}}_{A,B}$ and $\mathcal{A}^{\mathcal{T}}_{A',B'}$ differ only in their initial and final states; their states and transition weight function are identical. Since $\mathcal{T}$ is in normal form, for any $A', B' \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$ and $w \in \mathsf{N}^*_{\mathsf{R}}$, there is at most one axiom $\langle A' \sqsubseteq \forall w.B' \geq p\rangle$ in $\mathcal{T}$, and hence the transition weight function is well-defined. This function has finite support since $\mathcal{T}$ is finite.

We now characterize the gfp-models of $\mathcal{T}$ by properties of the automata $\mathcal{A}^{\mathcal{T}}_{A,B}$.

Lemma 12. *For every gfp-model $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ of $\mathcal{T}$, $x \in \Delta$, and $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$,*

$$A^{\mathcal{I}}(x) = \bigwedge_{B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w \in \mathsf{N}^*_{\mathsf{R}}} \big(\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) \Rightarrow_{\mathsf{G}} (\forall w.B)^{\mathcal{I}}(x)\big).$$

Proof. If A is primitive, then the empty path $\pi = A \in \mathsf{paths}(\mathcal{A}^{\mathcal{T}}_{A,A}, \varepsilon)$ has weight $\mathsf{wt}_{\mathcal{T}}(\pi) = 1$, and hence $\|\mathcal{A}^{\mathcal{T}}_{A,A}\|(\varepsilon) = 1$. We also have $(\forall \varepsilon.A)^{\mathcal{I}}(x) = A^{\mathcal{I}}(x)$; thus, $A^{\mathcal{I}}(x) = (1 \Rightarrow_{\mathsf{G}} A^{\mathcal{I}}(x)) \geq \bigwedge_{B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w \in \mathsf{N}^*_{\mathsf{R}}} \big(\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) \Rightarrow_{\mathsf{G}} (\forall w.B)^{\mathcal{I}}(x)\big)$. Let now $B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $w \in \mathsf{N}^*_{\mathsf{R}}$ such that $A \neq B$ or $w \neq \varepsilon$. Since A is primitive, by Definition 11 any finite path π in $\mathcal{A}^{\mathcal{T}}_{A,B}$ with $\ell(\pi) = w$ must have weight 0; i.e. $\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) = 0$, and thus $0 \Rightarrow_{\mathsf{G}} (\forall w.B)^{\mathcal{I}}(x) = 1 \geq A^{\mathcal{I}}(x)$. This shows that the whole infimum is equal to $A^{\mathcal{I}}(x)$.

Consider now the case that $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$. Since $\mathcal{I}$ is a gfp-model of $\mathcal{T}$, there is a primitive interpretation $\mathcal{J}$ such that $\mathcal{I} = \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$; let $f := \mathsf{gfp}(T^{\mathcal{T}}_{\mathcal{J}})$. Thus, we have $A^{\mathcal{I}} = f(A) = T^{\mathcal{T}}_{\mathcal{J}}(f)(A) = \bigwedge_{i \geq 0} (T^{\mathcal{T}}_{\mathcal{J}})^i(\mathbf{1})(A)$ for all $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$.
[$\leq$] By Proposition 3 it suffices to show that for all $x \in \Delta$, $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$, $B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$, and all finite non-empty paths π in $\mathcal{A}^{\mathcal{T}}_{A,B}$ it holds that

$$A^{\mathcal{I}}(x) \leq \mathsf{wt}_{\mathcal{T}}(\pi) \Rightarrow_{\mathsf{G}} (\forall w.B)^{\mathcal{I}}(x), \tag{1}$$

where $w := \ell(\pi)$. This obviously holds for $\mathsf{wt}_{\mathcal{T}}(\pi) = 0$, and thus it remains to show this for paths with positive weight. Let $\pi = Aw_1A_1w_2\dots w_nA_n$, where $A_i \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$ and $w_i \in \mathsf{N}^*_{\mathsf{R}}$ for all $i \in \{1, \dots, n\}$ and $A_n = B$ is the only primitive concept name in this path. We prove (1) by induction on n. For $n = 1$, we have $\pi = Aw_1B$ and $\mathsf{wt}_{\mathcal{T}}(A, w_1, B) = \mathsf{wt}_{\mathcal{T}}(\pi) > 0$, and thus $\mathcal{T}$ contains the definition $\langle A \sqsubseteq \forall w_1.B \geq p\rangle$, with $p := \mathsf{wt}_{\mathcal{T}}(A, w_1, B)$. By the definition of $T^{\mathcal{T}}_{\mathcal{J}}$, we obtain

$$A^{\mathcal{I}}(x) = T^{\mathcal{T}}_{\mathcal{J}}(f)(A)(x) \leq p \Rightarrow_{\mathsf{G}} (\forall w_1.B)^{\mathcal{I}}(x) = \mathsf{wt}_{\mathcal{T}}(\pi) \Rightarrow_{\mathsf{G}} (\forall w.B)^{\mathcal{I}}(x).$$

For $n > 1$, consider the subpath $\pi' = A_1w_2 \dots w_nB$ in $\mathcal{A}^{\mathcal{T}}_{A_1,B}$ with the label $\ell(\pi') = w' := w_2 \dots w_n$. For all $y \in \Delta$, the induction hypothesis yields that $A_1^{\mathcal{I}}(y) \leq \mathsf{wt}_{\mathcal{T}}(\pi') \Rightarrow_{\mathsf{G}} (\forall w'.B)^{\mathcal{I}}(y)$. Again, $p := \mathsf{wt}_{\mathcal{T}}(A, w_1, A_1) \geq \mathsf{wt}_{\mathcal{T}}(\pi) > 0$, and thus $\mathcal{T}$ contains the definition $\langle A \sqsubseteq \forall w_1.A_1 \geq p\rangle$. By the definitions of $T^{\mathcal{T}}_{\mathcal{J}}$, $\mathsf{wt}_{\mathcal{T}}(\pi)$, $w^{\mathcal{I}}$, and Propositions 3 and 4, we have

$$\begin{aligned} A^{\mathcal{I}}(x) &= T^{\mathcal{T}}_{\mathcal{J}}(f)(A)(x) \\ &\leq p \Rightarrow_{\mathsf{G}} (\forall w_1.A_1)^{\mathcal{I}}(x) \\ &= \bigwedge_{y\in\Delta} \big(p \Rightarrow_{\mathsf{G}} (w_1^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} A_1^{\mathcal{I}}(y))\big) \\ &\leq \bigwedge_{y\in\Delta} \Big(p \Rightarrow_{\mathsf{G}} \Big(w_1^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} \big(\mathsf{wt}_{\mathcal{T}}(\pi') \Rightarrow_{\mathsf{G}} (\forall w'.B)^{\mathcal{I}}(y)\big)\Big)\Big) \\ &= \big(p \wedge \mathsf{wt}_{\mathcal{T}}(\pi')\big) \Rightarrow_{\mathsf{G}} \Big(\bigwedge_{y\in\Delta} \big(w_1^{\mathcal{I}}(x,y) \Rightarrow_{\mathsf{G}} (\forall w'.B)^{\mathcal{I}}(y)\big)\Big) \\ &= \mathsf{wt}_{\mathcal{T}}(\pi) \Rightarrow_{\mathsf{G}} (\forall w.B)^{\mathcal{I}}(x). \end{aligned}$$

[≥] We show by induction on i that for all $x \in \Delta$, $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$, and $i \geq 0$, it holds

$$(T^{\mathcal{T}}_{\mathcal{J}})^i(\mathbf{1})(A)(x) \geq \bigwedge_{B\in\mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w\in\mathsf{N}^*_{\mathsf{R}}} \big(\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) \Rightarrow_{\mathsf{G}} (\forall w.B)^{\mathcal{I}}(x)\big). \tag{2}$$

For $i = 0$, we have $(T^{\mathcal{T}}_{\mathcal{J}})^0(\mathbf{1})(A)(x) = \mathbf{1}(A)(x) = 1$, which obviously satisfies (2). For $i > 0$, by Proposition 3 we obtain

$$\begin{aligned} (T^{\mathcal{T}}_{\mathcal{J}})^i(\mathbf{1})(A)(x) &= T^{\mathcal{T}}_{\mathcal{J}}((T^{\mathcal{T}}_{\mathcal{J}})^{i-1}(\mathbf{1}))(A)(x) \\ &= \bigwedge_{\langle A\sqsubseteq\forall w'.A'\geq p\rangle\in\mathcal{T}} (p \Rightarrow_{\mathsf{G}} (\forall w'.A')^{\mathcal{I}_{i-1}}(x)), \end{aligned} \tag{3}$$

where $\mathcal{I}_{i-1} := \mathcal{I}_{\mathcal{J},(T^{\mathcal{T}}_{\mathcal{J}})^{i-1}(\mathbf{1})}$. Consider now any definition $\langle A \sqsubseteq \forall w'.A' \geq p\rangle \in \mathcal{T}$. Then $\pi' = Aw'A'$ is a finite path in $\mathcal{A}^{\mathcal{T}}_{A,A'}$ with label w' and weight p.

If A' is a primitive concept name, then we have

$$p \Rightarrow_{\mathsf{G}} (\forall w'.A')^{\mathcal{I}_{i-1}}(x) \geq \|\mathcal{A}^{\mathcal{T}}_{A,A'}\|(w') \Rightarrow_{\mathsf{G}} (\forall w'.A')^{\mathcal{I}}(x)$$

by the definition of $\|\mathcal{A}^{\mathcal{T}}_{A,A'}\|(w')$ and the fact that the interpretation of $\forall w'.A'$ under $\mathcal{I}_{i-1}$ and $\mathcal{I}$ only depends on $\mathcal{J}$. If A' is defined, then we similarly get

$$\begin{aligned} &p \Rightarrow_{\mathsf{G}} (\forall w'.A')^{\mathcal{I}_{i-1}}(x) \\ &\quad = \bigwedge_{y\in\Delta} \Big(p \Rightarrow_{\mathsf{G}} \big(w'^{\mathcal{J}}(x,y) \Rightarrow_{\mathsf{G}} A'^{\mathcal{I}_{i-1}}(y)\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&\geq \bigwedge_{y\in\Delta} \bigwedge_{B\in\mathsf{N}_\mathsf{P}^\mathcal{T}} \bigwedge_{w\in\mathsf{N}_\mathsf{R}^*} \Big(p \Rightarrow_\mathsf{G} \big(w'^\mathcal{I}(x,y) \Rightarrow_\mathsf{G} (\|\mathcal{A}^\mathcal{T}_{A',B}\|(w) \Rightarrow_\mathsf{G} (\forall w.B)^\mathcal{I}(y))\big)\Big)\\
&= \bigwedge_{B\in\mathsf{N}_\mathsf{P}^\mathcal{T}} \bigwedge_{w\in\mathsf{N}_\mathsf{R}^*} \Big(\big(p \wedge \|\mathcal{A}^\mathcal{T}_{A',B}\|(w)\big) \Rightarrow_\mathsf{G} \Big(\bigwedge_{y\in\Delta} \big(w'^\mathcal{I}(x,y) \Rightarrow_\mathsf{G} (\forall w.B)^\mathcal{I}(y)\big)\Big)\Big)\\
&= \bigwedge_{B\in\mathsf{N}_\mathsf{P}^\mathcal{T}} \bigwedge_{w\in\mathsf{N}_\mathsf{R}^*} \Big(\Big(\bigvee_{\pi\in\mathsf{paths}(\mathcal{A}^\mathcal{T}_{A',B},w)} (\mathsf{wt}_\mathcal{T}(\pi') \wedge \mathsf{wt}_\mathcal{T}(\pi))\Big) \Rightarrow_\mathsf{G} (\forall w'w.B)^\mathcal{I}(x)\Big)\\
&\geq \bigwedge_{B\in\mathsf{N}_\mathsf{P}^\mathcal{T}} \bigwedge_{w\in\mathsf{N}_\mathsf{R}^*} \big(\|\mathcal{A}^\mathcal{T}_{A,B}\|(w'w) \Rightarrow_\mathsf{G} (\forall w'w.B)^\mathcal{I}(x)\big)
\end{aligned}
$$

by the induction hypothesis, Propositions 3 and 4, and the definition of $\|\mathcal{A}^\mathcal{T}_{A,B}\|$.

In both cases, $p \Rightarrow_\mathsf{G} (\forall w'.A')^{\mathcal{I}_{i-1}}(x)$ is an upper bound for the infimum in (2), and thus by (3) the same is true for $(T^\mathcal{T}_\mathcal{J})^i(\mathbf{1})(A)(x)$. □

This allows us to prove gfp-subsumptions by comparing the behavior of WWA.

Lemma 13. *Let $A, B \in \mathsf{N}_\mathsf{C}^\mathcal{T}$ and $p \in [0,1]$. Then $\mathcal{T} \models_\mathsf{gfp} \langle A \sqsubseteq B \geq p\rangle$ iff for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w \in \mathsf{N}_\mathsf{R}^*$ it holds that $p \wedge \|\mathcal{A}^\mathcal{T}_{B,C}\|(w) \leq \|\mathcal{A}^\mathcal{T}_{A,C}\|(w)$.*

Proof. Assume that there exist $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w = r_1 \dots r_n \in \mathsf{N}_\mathsf{R}^*$ such that $p\wedge\|\mathcal{A}^\mathcal{T}_{B,C}\|(w) > \|\mathcal{A}^\mathcal{T}_{A,C}\|(w)$. We define the primitive interpretation $\mathcal{J} = (\Delta, \cdot^\mathcal{J})$ where $\Delta := \{x_0, \dots, x_n\}$, and for all $D \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $r \in \mathsf{N}_\mathsf{R}$, the interpretation function is given by

$$
D^\mathcal{J}(x) := \begin{cases} \|\mathcal{A}^\mathcal{T}_{A,C}\|(w) & \text{if } D = C \text{ and } x = x_n,\\ 1 & \text{otherwise; and}\end{cases}
$$

$$
r^\mathcal{J}(x,y) := \begin{cases} 1 & \text{if } x = x_{i-1},\ y = x_i, \text{ and } r = r_i \text{ for some } i \in \{1,\dots,n\},\\ 0 & \text{otherwise.}\end{cases}
$$

Consider now the gfp-model $\mathcal{I} := \mathsf{gfp}_\mathcal{T}(\mathcal{J})$ of $\mathcal{T}$. By construction, for all pairs $(w', D) \in \mathsf{N}_\mathsf{R}^* \times \mathsf{N}_\mathsf{P}^\mathcal{T} \setminus \{(w, C)\}$ we have $(\forall w'.D)^\mathcal{I}(x_0) = 1$. Moreover, we know that $(\forall w.C)^\mathcal{I}(x_0)$ is equal to $\|\mathcal{A}^\mathcal{T}_{A,C}\|(w)$, and thus strictly smaller than p and $\|\mathcal{A}^\mathcal{T}_{B,C}\|(w)$. By Lemma 12, all this implies that

$$
\begin{aligned}
A^\mathcal{I}(x_0) &= \|\mathcal{A}^\mathcal{T}_{A,C}\|(w) \Rightarrow_\mathsf{G} (\forall w.C)^\mathcal{I}(x_0) = 1 \text{ and}\\
B^\mathcal{I}(x_0) &= \|\mathcal{A}^\mathcal{T}_{B,C}\|(w) \Rightarrow_\mathsf{G} (\forall w.C)^\mathcal{I}(x_0) = (\forall w.C)^\mathcal{I}(x_0).
\end{aligned}
$$

Thus $A^\mathcal{I}(x_0) \Rightarrow_\mathsf{G} B^\mathcal{I}(x_0) = (\forall w.C)^\mathcal{I}(x_0) < p$, and $\mathcal{T} \not\models_\mathsf{gfp} \langle A \sqsubseteq B \geq p\rangle$.

Conversely, assume that there are a primitive interpretation $\mathcal{J} = (\Delta, \cdot^\mathcal{J})$ and an element $x \in \Delta$ such that $A^\mathcal{I}(x) \Rightarrow_\mathsf{G} B^\mathcal{I}(x) < p$, where $\mathcal{I} := \mathsf{gfp}_\mathcal{T}(\mathcal{J})$. Thus, we have $p \wedge A^\mathcal{I}(x) > B^\mathcal{I}(x)$, which implies by Lemma 12 the existence of a $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and a $w \in \mathsf{N}_\mathsf{R}^*$ with $p \wedge A^\mathcal{I}(x) > \|\mathcal{A}^\mathcal{T}_{B,C}\|(w) \Rightarrow_\mathsf{G} (\forall w.C)^\mathcal{I}(x)$. Again by Lemma 12, this shows that

$$
\begin{aligned}
p \wedge \|\mathcal{A}^\mathcal{T}_{B,C}\|(w) &> A^\mathcal{I}(x) \Rightarrow_\mathsf{G} (\forall w.C)^\mathcal{I}(x)\\
&\geq \big(\|\mathcal{A}^\mathcal{T}_{A,C}\|(w) \Rightarrow_\mathsf{G} (\forall w.C)^\mathcal{I}(x)\big) \Rightarrow_\mathsf{G} (\forall w.C)^\mathcal{I}(x).
\end{aligned}
$$

In particular, the latter value cannot be 1, and thus it is equal to $(\forall w.C)^{\mathcal{I}}(x)$. But this can only be the case if $\|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w) \leq (\forall w.C)^{\mathcal{I}}(x)$. To summarize, we obtain $p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > (\forall w.C)^{\mathcal{I}}(x) \geq \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$, as desired. □

Denote by $\mathcal{V}_{\mathcal{T}} := \{0,1\} \cup \{p \in [0,1] \mid \langle A \sqsubseteq \forall w.B \geq p\rangle \in \mathcal{T}\}$ the set of all values appearing in $\mathcal{T}$, together with 0 and 1. Since $\mathsf{wt}_{\mathcal{T}}$ has finite support and takes only values from $\mathcal{V}_{\mathcal{T}}$, $p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$ holds iff $p' \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$, where p' is the smallest element of $\mathcal{V}_{\mathcal{T}}$ such that $p' \geq p$. This shows that it suffices to be able to check gfp-subsumptions for the values in $\mathcal{V}_{\mathcal{T}}$. We now show how to do this by simulating $\mathcal{A}^{\mathcal{T}}_{B,C}$ and $\mathcal{A}^{\mathcal{T}}_{A,C}$ by polynomially many *unweighted* automata.

Definition 14 (automata $\mathcal{A}_{\geq p}$). *Given a WWA $\mathcal{A} = (\Sigma, Q, q_0, \mathsf{wt}, q_f)$ and a value $p \in [0,1]$, the WA $\mathcal{A}_{\geq p} = (\Sigma, Q, q_0, \mathsf{wt}_{\geq p}, q_f)$ is given by the transition relation $\mathsf{wt}_{\geq p} := \{(q,w,q') \in Q \times \Sigma^* \times Q \mid \mathsf{wt}(q,w,q') \geq p\}$.*

The language of this automaton has an obvious relation to the behavior of the original WWA.

Lemma 15. *Let $\mathcal{A}$ be a WWA over the alphabet Σ and $p \in [0,1]$. Then we have $L(\mathcal{A}_{\geq p}) = \{w \in \Sigma^* \mid \|\mathcal{A}\|(w) \geq p\}$.*

Proof. We have $w \in L(\mathcal{A}_{\geq p})$ iff there is a finite path $\pi = q_0 w_1 q_1 \dots w_n q_n$ in $\mathcal{A}$ with label w such that $\mathsf{wt}(q_{i-1}, w_i, q_i) \geq p$ holds for all $i \in \{1, \dots, n\}$. The latter condition is equivalent to the fact that $\mathsf{wt}(\pi) \geq p$. Thus, $w \in L(\mathcal{A}_{\geq p})$ implies that $\|\mathcal{A}\|(w) \geq p$. Conversely, since wt has finite support, there are only finitely many possible weights for any finite path in $\mathcal{A}$, and thus $\|\mathcal{A}\|(w) \geq p$ also implies that there exists a $\pi \in \mathsf{paths}(\mathcal{A}, w)$ with $\mathsf{wt}(\pi) \geq p$, and thus $w \in L(\mathcal{A}_{\geq p})$. □

We thus obtain the following characterization of gfp-subsumption.

Lemma 16. *Let $A, B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$ and $p \in \mathcal{V}_{\mathcal{T}}$. Then $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$ iff for all $C \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $p' \in \mathcal{V}_{\mathcal{T}}$ with $p' \leq p$ it holds that $L((\mathcal{A}^{\mathcal{T}}_{B,C})_{\geq p'}) \subseteq L((\mathcal{A}^{\mathcal{T}}_{A,C})_{\geq p'})$.*

Proof. Assume that we have $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$ and consider any $C \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$, $w \in \mathsf{N}^*_{\mathsf{R}}$, and $p' \in \mathcal{V}_{\mathcal{T}} \cap [0,p]$ with $w \in L((\mathcal{A}^{\mathcal{T}}_{B,C})_{\geq p'})$. By Lemma 15, we obtain $\|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) \geq p'$, and by Lemma 13 we know that $\|\mathcal{A}^{\mathcal{T}}_{A,C}\| \geq p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) \geq p'$. Thus, $w \in L((\mathcal{A}^{\mathcal{T}}_{A,C})_{\geq p'})$.

Conversely, assume that $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$ does not hold. Then by Lemma 13 there are $C \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $w \in \mathsf{N}^*_{\mathsf{R}}$ such that $p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$. For the value $p' := p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) \in \mathcal{V}_{\mathcal{T}} \cap [0,p]$, we have $\|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) \geq p'$, but $\|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w) < p'$, and thus $L((\mathcal{A}^{\mathcal{T}}_{B,C})_{\geq p'}) \not\subseteq L((\mathcal{A}^{\mathcal{T}}_{A,C})_{\geq p'})$ by Lemma 15. □

Since the automata $(\mathcal{A}^{\mathcal{T}}_{A,C})_{\geq p'}$ correspond to those from [1] simulating subsumption in the (classical) TBoxes $\mathcal{T}_{\geq p'} := \{A' \sqsubseteq C' \mid \langle A' \sqsubseteq C' \geq q\rangle \in \mathcal{T}, q \geq p'\}$, we have shown that gfp-subsumption in $\mathsf{G}\text{-}\mathcal{FL}_0$ can be reduced to polynomially many subsumption tests in $\mathcal{FL}_0$. The detour through WWA was necessary to account for the differences between the gfp-models of $\mathcal{T}$ and those of $\mathcal{T}_{\geq p'}$.

A direct consequence of this reduction is that gfp-subsumption between concept names in $\mathsf{G}\text{-}\mathcal{FL}_0$ remains in the same complexity class as for classical $\mathcal{FL}_0$.

Theorem 17. *In* G-$\mathcal{FL}_0$ *with cyclic TBoxes, deciding gfp-subsumption between concept names is* PSPACE*-complete.*

Proof. By the reductions above, it suffices to decide the language inclusions $L((\mathcal{A}^{\mathcal{T}}_{B,C})_{\geq p}) \subseteq L((\mathcal{A}^{\mathcal{T}}_{A,C})_{\geq p})$ for all $C \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $p \in \mathcal{V}_{\mathcal{T}}$. These polynomially many inclusion tests for WA can be done in polynomial space [11]. The problem is PSPACE-hard since gfp-subsumption in classical $\mathcal{FL}_0$ is PSPACE-hard [1]. □

To compute the *best* gfp-subsumption degree between A and B, we have to check the above inclusions for increasing values $p \in \mathcal{V}_{\mathcal{T}}$. The largest p for which these checks succeed is the requested degree.

In the case of an acyclic TBox $\mathcal{T}$, it is easy to verify that the automata $(\mathcal{A}^{\mathcal{T}}_{B,C})_{\geq p}$ constructed above are in fact acyclic. Since inclusion between acyclic automata can be decided in co-NP [11], we again obtain the same complexity as in the classical case.

Corollary 18. *In* G-$\mathcal{FL}_0$ *with acyclic TBoxes, deciding gfp-subsumption between concept names is* co-NP*-complete.*

5 Conclusions

We have studied the complexity of reasoning in G-$\mathcal{FL}_0$ w.r.t. primitive concept definitions under greatest fixed-point semantics. Specifically, we have shown that gfp-subsumption between concept names can be reduced to a comparison of the behavior of weighted automata with word transitions. The latter can be solved by a polynomial number of inclusion tests on *unweighted* automata, and thus gfp-subsumption is PSPACE-complete for this logic, just as in the classical case. The same reduction yields co-NP-completeness in the case of acyclic TBoxes.

In fuzzy DLs, reasoning is often restricted to so-called *witnessed* models [14]. Intuitively, they guarantee that the semantics of value restrictions can be computed as minima instead of possibly infinite infima. As our reduction does not make use of this property and the model constructed in the proof of Lemma 13 is witnessed, our results hold under both witnessed and general semantics.

These complexity results are consistent with previous work on extensions of description logics with Gödel semantics. Indeed, such extensions of $\mathcal{EL}$ [17,20] and $\mathcal{ALC}$ [6] have been shown to preserve the complexity of their classical counterpart. Since reasoning in both $\mathcal{FL}_0$ and in G-$\mathcal{ALC}$ w.r.t. general TBoxes is ExpTime-complete, so is deciding subsumption in G-$\mathcal{FL}_0$ w.r.t. general TBoxes.

We expect our results to generalize easily to any other set of truth degrees that form a total order. However, the arguments used in this paper fail for arbitrary lattices, where incomparable truth degrees might exist [8,23]. Studying these two cases in detail is a task for future work. We also plan to consider fuzzy extensions of $\mathcal{FL}_0$ with semantics based on non-idempotent t-norms, such as the Łukasiewicz or product t-norms [13].

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